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*Research article*

# Fracture analysis of multiple axisymmetric interfacial cracks in an FGM coated orthotropic layer

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**Abstract**

Based on the distributed dislocation technique, an analytical solution for the orthotropic layer with functionally graded material (FGM) orthotropic coating containing multiple axisymmetric interfacial cracks subjected to torsional loading is investigated. It is assumed that the material properties of the FGM orthotropic coating vary power-law form along the thickness of the layer. At first, by using the Hankel transform, the solution for Somigliana-type rotational ring dislocation in the layer and its coating is obtained. Then, the dislocation solution is used to derive a set of singular integral equations for a system of coaxial axisymmetric interface cracks, including penny-shaped and annular cracks. cracks with Cauchy type kernel. The integral equations are of Cauchy singular type, which are solved by Erdogan's collocation method for dislocation density on the surface of interfacial crack and the results are used to determine stress intensity factors (SIFs) for axisymmetric interface cracks. Finally, several examples are provided to study the effects of the non-homogeneity constant, orthotropy parameter and thickness of FGM coating on the SIFs for interfacial cracks. The effects of the non-homogeneity constant, orthotropy parameter and thickness of FGM coating as well as the interaction of multiple interfacial cracks on the SIFs are investigated. The results reveal that the value of the SIFs decreases with increasing the non-homogeneity parameter, orthotropy and thickness of FGM coating. The SIFs for inner tips of annular interface crack are larger than the outer tips.

*Keywords:* Stress intensity factor; Torsion; FGM orthotropic coating; Axisymmetric cracks; Dislocation density.

## 1- Introduction

In recent years, functionally graded materials (FGMs) as coating and interfacial zones have been widely applied in structures exposed to harsh environments and severe thermal shocks. FGMs coating and FGMs interlayer help reduce mechanically- and thermally-induced stresses caused by material property mismatch; they also improve bonding strength. Typical applications of FGMs comprise thermal barrier coatings (TBCs) of high temperature components in gas turbines and as inter-layers in microelectronic and optoelectronic components. Therefore, fracture analysis of the FGMs coating-substrate system and the FGMs interlayer is an important design consideration and has attracted the attention of many researchers.

Delale and Erdogan identified the stress intensity factors for a crack situated in the interfacial nonhomogeneous layer between two dissimilar elastic half-planes under tension [1]. Ozturk and Erdogan determined axisymmetric solutions for a penny-shaped crack in an interfacial nonhomogeneous layer between two dissimilar elastic half-spaces under torsion [2] and under tension [3]. Choi et al. investigated the embedded collinear cracks in a layered half plane with a graded nonhomogeneous interfacial zone under static mechanical load [4]. Shbeeb and Binienda analyzed an interface crack in an FGM strip sandwiched between two homogeneous layers of finite thickness and determined the stress intensity factors and strain energy release rate [5]. Itou directed a study of a crack in an FGM layer between two homogeneous half planes. In his paper, the integral equation was solved using the Schmidt method, and a stress intensity factor was calculated [6]. An analytical model for fracture analysis of an FGM interfacial zone with a Griffith crack under the anti-plane shearing load was developed by Wang et al. [7]. Jin and Batra [8] investigated the interface cracking between ceramic and/or FGM coatings and a substrate under anti-plane shear. The interface crack problem between the functionally graded ceramic coating and the homogeneous substrate was studied by Chen and Erdogan [9]. In this study, the mixed mode crack problem was formulated for a crack subjected to surface tractions. Huang et al. developed a brand-new model for the approximate analysis of FGMs, the properties of which may vary arbitrarily, and solved the problem of a crack in an FGM coating bonded to a homogeneous substrate under a static anti-plane shearing load [10]. The crack problem of an FGM coating-substrate structure with an internal or edge crack perpendicular to the interface was investigated under an in-plane load by Guo et al. [11]. A multi-layered model for the crack problem of FGMs under plane stress state deformation with arbitrarily varying Young's modulus was proposed by Huang et al. [12]. A study by Chen and Chue [13] dealt with the anti-plane problems of two bonded FGM strips weakened by an internal crack normal to the interface. Asadi et al. [14] obtained the stress fields for an orthotropic strip with defects and imperfect FGM coating using the Volterra screw dislocation. Cheng et al. [15] studied the plane elasticity problem of two bonded dissimilar FGM strips containing an interface crack in which material properties varied arbitrarily. In relation to the dynamic crack problem, Ueda et al. [16] reported the torsional impact response of a penny-shaped crack on a biomaterial interface. They determined the dynamic stress intensity factor and discussed its dependence on time and material constants. Itou obtained the dynamic stresses of a crack in a nonhomogeneous interfacial layer between two elastic half-planes under tension [17]. Li and Weng considered the elastodynamic response of a penny-shaped crack located in an FGM interlayer between two dissimilar homogeneous half spaces and subjected to a torsional impact loading [18]. The transient response analysis of an FGM coating-substrate system with an internal or edge crack perpendicular to the interface under an in-plane impact load was carried out by

Guo et al. [19]. Li et al. [20, 21] investigated the static and dynamic behavior of an annular interfacial crack between dissimilar magneto-electro-elastic materials. Ren et al. [22] studied the transient response of an annular interfacial crack between dissimilar piezoelectric layers under mechanical and electrical impacts. Singh et al. [23] analyzed the dynamic stress intensity factor of a finite crack with finite depth under impact response and in-plane shear loadings located at the interfacial of two orthotropic semi-infinite strips with different elastic properties. Xu et al. [24] studied a mixed mode interfacial crack in three-dimensional biomaterials by singular integral equation on the basis of the body force method.

According to the above review, the stress analysis of nonhomogeneous medium under dynamic loading was mainly restricted only to a single interface crack. Following introduction in recent years of a powerful semi-analytical method called the distributed dislocation technique, the problem of the multiple interfacial cracks has been studied for out-of-plane and in-plane loadings based on the principle of the superposition. A brief review of relevant articles is given below. The behavior of interface cracks in two bonded dissimilar materials subjected to in-plane loading was studied by Monfared et al. [25]. The dislocation density on the faces of the cracks was obtained numerically and, then, was used to calculate the mixed mode stress intensity factors of multiple interfacial cracks. In another work, the interaction of several interfacial cracks located between a FGM layer and an elastic layer under anti-plane deformation based on the distributed dislocation technique was analyzed by Monfared [26]. Asadi et al. [27] solved the anti-plane shear problem of orthotropic strips with multiple defects and imperfect FGM coating. The dynamic behavior of two bonded elastic and piezoelectric layers with multiple interfacial cracks under time-dependent mechanical load was investigated by Ayatollahi et al. [28]. Fartash et al. [29] investigated the transient problem of multiple interfacial cracks in dissimilar piezoelectric layers under sudden electro-mechanical impacts. Bagheri et al. [30] studied the problem of interaction between multiple cracks with arbitrary patterns in a piezoelectric strip reinforced with FGM coating under anti-plane loading. The aim of the present work is the fracture behavior of multiple axisymmetric interface cracks between an orthotropic layer with FGM orthotropic coating subjected to torsional loading using the distributed dislocation technique. The distributed dislocation technique is an efficient means for treating multiple axisymmetric cracks with smooth geometries. Based on the distributed dislocation technique, the governing equations of the motion and the outer boundary conditions of the domain are satisfied analytically. However, determining stress fields due to a single dislocation in the region has been a major obstacle to the utilization of this method. We take up this task for an FGM coating-substrate system containing an axisymmetric rotational Somigliana ring dislocation.

The paper is organized as follows. By using the Hankel transform the solution of the axisymmetric rotational Somigliana ring dislocation in orthotropic layer with FGM orthotropic coating is given in section 2. Section 3 presents the distributed dislocation to formulate and solve the Cauchy-type singular integral equations for the orthotropic layer with FGM orthotropic coating weakened by several axisymmetric cracks. In section 4 several examples of cracks are solved to illustrate the influence of material non-homogeneity and orthotropy as well as thickness of FGM coating on the SIFs. Concluding remarks are included in Section 5.

**2- Formulation of Problem**

We consider an orthotropic layer with thickness $h$ reinforced by an FGM orthotropic layer having thickness $h_1$. For the problem in question, the only non-zero displacement component is $w(r,z)$ being independent of $\theta$, and two other elastic displacements, $u$ and $v$, oriented in the $r$ and $z$ axes vanish. Consequently, the constitutive equations relationships read as

$$\begin{aligned}&\tau_{r\theta}=\mu_r(z)\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right),\\ &\tau_{\theta z}=\mu_z(z)\frac{\partial u_\theta}{\partial z},\\ &0\leq z\leq h_1\\ &\tau_{r\theta}=\mu_{r0}\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right),\\ &\tau_{\theta z}=\mu_{z0}\frac{\partial u_\theta}{\partial z},\\ &\qquad -h\leq z\leq 0\end{aligned}\tag{1}$$

where $\mu_r(z)$ and $\mu_z(z)$ are material constants of FGM orthotropic strip, and $\mu_{r0}$ and $\mu_{z0}$ are the elastic shear modulus of orthotropic layer in the $r$ and $z$ directions, respectively. By substituting Eq. (1) into equilibrium equation $\tau_{ij,j}=0$ one can obtain:

$$\begin{aligned}&\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\mu_z'(z)}{\mu_r(z)}\frac{\partial u_\theta}{\partial z}\\ &+\frac{\mu_z(z)}{\mu_r(z)}\frac{\partial^2 u_\theta}{\partial z^2}=0,\qquad 0\leq z\leq h_1\\ &\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\\ &+\frac{\mu_{z0}}{\mu_{r0}}\frac{\partial^2 u_\theta}{\partial z^2}=0,\qquad -h\leq z\leq 0\end{aligned}\tag{2}$$

where $\mu_z'(z)$ is the derivative of $\mu_z(z)$. Let a Somigliana-type rotational ring dislocation with the magnitude of Burgers vector $\frac{\varepsilon}{r}b_\theta$ be situated at $z=0, r=\varepsilon$ with dislocation cut in radial direction. The boundary and dislocation conditions of the problem can be expressed as

$$\begin{aligned}&u_\theta(r,0^+)-u_\theta(r,0^-)=\frac{\varepsilon}{r}b_\theta H(r-\varepsilon)\\ &\tau_{\theta z}(r,0^+)=\tau_{\theta z}(r,0^-)\\ &\tau_{\theta z}(r,h_1)=0\\ &\tau_{\theta z}(r,-h)=0\end{aligned}\tag{3}$$

where $H(.)$ is the Heaviside step-function. Applying the Hankel transform of the first order and letting

$$U_\theta(\eta,z)=\int_0^\infty u_\theta(r,z)\,rJ_1(r\eta)dr\tag{4}$$

Application of the Hankel transforms to Eq. (2), assuming that the medium is initially stationary, leads to

$$\begin{aligned}&\frac{\mu_z(z)}{\mu_r(z)}\frac{\partial^2 U_\theta}{\partial z^2}+\frac{\mu_z'(z)}{\mu_r(z)}\frac{\partial U_\theta}{\partial z}\\ &-\eta^2 U_\theta=0,\qquad 0\leq z\leq h_1\\ &\frac{\mu_{z0}}{\mu_{r0}}\frac{\partial^2 U_\theta}{\partial z^2}-\eta^2 U_\theta=0,\\ &\qquad -h\leq z\leq 0\end{aligned}\tag{5}$$

The material properties of FGM orthotropic coating such as, shear moduli $\mu_r$ and $\mu_z$, can be described power-law form of the type

$$\begin{aligned}\mu_r(z)&=\mu_{r0}(1+\alpha z)^2\\ \mu_z(z)&=\mu_{z0}(1+\alpha z)^2\end{aligned}\tag{6}$$

where $\mu_{r0}$, $\mu_{z0}$ and $\alpha$ are the material constants of FGM. Substituting Eq. (6) into Eq. (5), the equation of motion can be rewritten as

$$\frac{\partial^2 U_\theta}{\partial z^2}+\frac{2\alpha}{(1+\alpha z)}\frac{\partial U_\theta}{\partial z}$$

$$-\left(\xi^2+\frac{\rho_0 p^2}{\mu_{z0}}\right)U_\theta=0, \quad 0\le z\le h_1$$

$$\frac{\partial^2 U_\theta}{\partial z^2}-\lambda^2\xi^2 U_\theta=0, \quad -h\le z\le 0 \tag{7}$$

Letting $\xi=\lambda\eta$ and the orthotropy parameter $\lambda=\sqrt{\mu_{r0}/\mu_{z0}}$. The solutions to Eq. (7) are

$$U_\theta(\eta,z)=A_1(\eta)\frac{e^{n_1 z}}{1+\alpha z}+A_2(\eta)\frac{e^{n_2 z}}{1+\alpha z}, \quad 0\le z\le h_1$$
$$U_\theta(\eta,z)=B_1(\eta)e^{n_1 z}+B_2(\eta)e^{n_2 z}, \quad -h\le z\le 0 \tag{8}$$

where $A_1(\eta), A_2(\eta), B_1(\eta)$ and $B_2(\eta)$ are unknowns to be solved. By taking the inverse Hankel transform of Eq. (8) and utilizing Eq. (1), yields the displacement field and stress components

$$u_\theta(r,z)=\int_0^\infty \frac{1}{(1+\alpha z)}\left[A_1(\eta)e^{-\xi z}+A_2(\eta)e^{\xi z}\right]\eta J_1(r\eta)d\eta, \quad 0\le z\le h_1$$

$$\tau_{r\theta}(r,z)=-\mu_{r0}\int_0^\infty (1+\alpha z)\left[A_1(\eta)e^{-\xi z}+A_2(\eta)e^{\xi z}\right]\eta^2 J_2(r\eta)d\eta, \quad 0\le z\le h_1$$

$$\tau_{\theta z}(r,z)=\mu_{z0}\int_0^\infty (1+\alpha z)\begin{Bmatrix}-\left[\xi+\frac{\alpha}{(1+\alpha z)}\right]A_1(\eta)e^{-\xi z}\\ +\left[\xi-\frac{\alpha}{(1+\alpha z)}\right]A_2(\eta)e^{\xi z}\end{Bmatrix}\eta J_1(r\eta)d\eta, \quad 0\le z\le h_1$$

$$u_\theta(r,z)=\int_0^\infty \left[B_1(\eta)e^{-\xi z}+B_2(\eta)e^{\xi z}\right]\eta J_1(r\eta)d\eta, \quad -h\le z\le 0$$

$$\tau_{r\theta}(r,z)=-\mu_{r0}\int_0^\infty \left[B_1(\eta)e^{-\xi z}+B_2(\eta)e^{\xi z}\right]\eta^2 J_2(r\eta)d\eta, \quad -h\le z\le 0$$

$$\tau_{\theta z}(r,z)=\mu_{z0}\int_0^\infty \xi\left[-B_1(\eta)e^{-\xi z}+B_2(\eta)e^{\xi z}\right]\eta J_1(r\eta)d\eta, \quad -h\le z\le 0 \tag{9}$$

Applying the boundary conditions in Eq. (3) to Eq. (9) leads to

$$A_1(\eta)=\frac{\varepsilon b_\theta\xi\left(e^{-2h\xi}-1\right)\left[\alpha-\xi(\alpha h_1+1)\right]J_0(\varepsilon\eta)}{\eta\Lambda}$$
$$A_2(\eta)=-\frac{\varepsilon b_\theta\xi e^{-2h_1\xi}\left(e^{-2h\xi}-1\right)\left[\xi(\alpha h_1+1)+\alpha\right]J_0(\varepsilon\eta)}{\eta\Lambda}$$
$$B_1(\eta)=\frac{-\varepsilon b_\theta\begin{Bmatrix}(\xi+\alpha)\left[\alpha-\xi(\alpha h_1+1)\right]\\ -e^{-2h_1\xi}(\alpha-\xi)\left[\xi(\alpha h_1+1)+\alpha\right]\end{Bmatrix}J_0(\varepsilon\eta)}{\eta\Lambda} \tag{10}$$
$$B_2(\eta)=\frac{-\varepsilon b_\theta\, e^{-2h\xi}\begin{Bmatrix}(\xi+\alpha)\left[\alpha-\xi(\alpha h_1+1)\right]\\ -e^{-2h_1\xi}(\alpha-\xi)\left[\xi(\alpha h_1+1)+\alpha\right]\end{Bmatrix}J_0(\varepsilon\eta)}{\eta\Lambda}$$

wherein

$$\Lambda=-e^{-2h_1\xi}\left[\alpha\left(1+e^{-2h\xi}\right)-2\xi\right]\left[\xi(\alpha h_1+1)+\alpha\right]+\left[\alpha\left(1+e^{-2h\xi}\right)+2\xi e^{-2h\xi}\right]\left[-\xi(\alpha h_1+1)+\alpha\right] \tag{11}$$

The stress components in view of Eqs. (9) and (10) can be achieved as

$$\tau_{r\theta}(r,z) = \varepsilon\mu_{r0}b_\theta \int_0^\infty \frac{\frac{\xi}{\Lambda}(1+\alpha z)\left(e^{-2h\xi}-1\right)}{\left\{\begin{matrix}[\xi(\alpha h_1+1)+\alpha]e^{-\xi z}\\ +[\xi(\alpha h_1+1)-\alpha]\\ e^{-2h_1\xi}e^{\xi z}\end{matrix}\right\}} \eta J_0(\varepsilon\eta)J_2(r\eta)d\eta, \quad 0\le z\le h_1$$

$$\tau_{\theta z}(r,z) = \varepsilon\mu_{z0}b_\theta \int_0^\infty \frac{\frac{\xi}{\Lambda}(1-e^{-2h\xi})}{\left\{\begin{matrix}[\xi(1+\alpha z)+\alpha]\\ [-\xi(\alpha h_1+1)+\alpha]e^{-\xi z}\\ +[\xi(1+\alpha z)-\alpha]\\ [\xi(\alpha h_1+1)+\alpha]e^{-2h_1\xi}e^{\xi z}\end{matrix}\right\}} J_0(\varepsilon\eta)J_1(r\eta)d\eta, \quad 0\le z\le h_1$$

$$\tau_{r\theta}(r,z) = \varepsilon\mu_{r0}b_\theta \int_0^\infty \frac{\frac{1}{\Lambda}\left(e^{-\xi z}+e^{-2h\xi}e^{\xi z}\right)}{\left\{\begin{matrix}(\xi+\alpha)[\alpha-\xi(\alpha h_1+1)]\\ -(\alpha-\xi)\\ [\xi(\alpha h_1+1)+\alpha]e^{-2h_1\xi}\end{matrix}\right\}} \eta J_0(\varepsilon\eta)J_2(r\eta)d\eta, \quad -h\le z\le 0$$

$$\tau_{\theta z}(r,z) = \varepsilon\mu_{z0}b_\theta \int_0^\infty \frac{\frac{\xi}{\Lambda}\left(e^{-\xi z} - e^{-2h\xi}e^{\xi z}\right)}{\left\{\begin{matrix}(\xi+\alpha)\\ [\alpha+-\xi(\alpha h_1+1)]\\ -(\alpha-\xi)\\ [\xi(\alpha h_1+1)+\alpha]e^{-2h_1\xi}\end{matrix}\right\}} J_0(\varepsilon\eta)J_1(r\eta)d\eta, \quad -h\le z\le 0 \tag{12}$$

To study the singular behavior of stress component $\tau_{\theta z}(r,z)$ at the dislocation location, we set $z=0$ in the related equation in Eq. (12), Therefore we have

$$\tau_{\theta z}(r,0) = \varepsilon\mu_{z0}b_\theta \int_0^\infty \frac{\frac{\xi\left(1-e^{-2h\xi}\right)}{\Lambda}}{\left\{\begin{matrix}(\xi+\alpha)\\ [\alpha-\xi(\alpha h_1+1)]\\ -e^{2-h_1\xi}\\ (\alpha-\xi)\\ [\xi(\alpha h_1+1)+\alpha]\end{matrix}\right\}} J_0(\varepsilon\eta)J_1(r\eta)d\eta \tag{13}$$

Since the integrand of integral at Eq. (13) is continuous function of $\eta$ and also finite at $\eta=0$, the singularity must occur as $\eta$ tends to infinity. To circumvent the difficulty an asymptotic analysis of the integrand is carried out as $\eta\to\infty$, arrives to

$$\tau_{\theta z}^{\infty}(r,0) = -\frac{1}{\pi r}\sqrt{\mu_{z0}\mu_{r0}}b_\theta \begin{cases}\dfrac{r\varepsilon}{r^2-\varepsilon^2}\boldsymbol{E}\left(\dfrac{\varepsilon^2}{r^2}\right) & r>\varepsilon\\ \boldsymbol{K}\left(\dfrac{r^2}{\varepsilon^2}\right)+\dfrac{\varepsilon^2}{r^2-\varepsilon^2}\boldsymbol{E}\left(\dfrac{r^2}{\varepsilon^2}\right) & r<\varepsilon\end{cases} \tag{14}$$

where $\boldsymbol{K}(k)=\int_0^{\pi/2} dx/\sqrt{1-k^2 sin^2 x}$ and $\boldsymbol{E}(k)=\int_0^{\pi/2}\sqrt{1-k^2 sin^2 x}\,dx$ are the complete elliptic integrals of the first and second kind, respectively. Finally, the stress component at the dislocation point can be evaluated by addition and subtraction of the asymptotic term of the stress component as follows

$$\tau_{\theta z}(r,0) = \varepsilon\mu_{z0}b_\theta \int_0^\infty \left[\frac{\frac{\xi\left(1-e^{-2h\xi}\right)}{\Lambda}}{\left\{\begin{matrix}(\xi+\alpha)\\ [\alpha-\xi(\alpha h_1+1)]\\ -e^{-2h_1\xi}\\ (\alpha-\xi)[\xi(\alpha h_1+1)+\alpha]\end{matrix}\right\}} + \frac{1}{2}\eta\lambda\right] J_0(\varepsilon\eta)J_1(r\eta)d\eta \tag{15}$$
$$-\frac{1}{\pi r}\sqrt{\mu_{z0}\mu_{r0}}b_\theta \begin{cases}\dfrac{r\varepsilon}{r^2-\varepsilon^2}\boldsymbol{E}\left(\dfrac{\varepsilon^2}{r^2}\right) & r>\varepsilon\\ \boldsymbol{K}\left(\dfrac{r^2}{\varepsilon^2}\right)+\dfrac{\varepsilon^2}{r^2-\varepsilon^2}\boldsymbol{E}\left(\dfrac{r^2}{\varepsilon^2}\right) & r<\varepsilon\end{cases}$$

From Eq. (15), it is observed that the stress component $\tau_{\theta z}(r,0)$ exhibit the familiar Cauchy-type singularity at dislocation location, i.e., $\tau_{\theta z}(r,0)\sim\frac{1}{r-\varepsilon}\ as\ r\to\varepsilon$. This kind of singularity was previously reported by Pourseifi & Faal [31] for an infinite isotropic cylinder with a climb and glide edge dislocations.

**3- Formulation of multiple axisymmetric interface cracks**

In this section, we implement the dislocation solutions accomplished in the prior section to analyze multiple axisymmetric interface crack problems in an orthotropic layer with FGM orthotropic coating. Consider an orthotropic layer bonded to an FGM orthotropic layer weakened by $N = N_1 + 1$ axisymmetric interfacial cracks, including $N_1$ annular and one penny-shaped crack. The inner and outer radii of the annular interface cracks are $a_j$ and $b_j, j = 1,2,\dots,N_1$ and the radii of the penny-shaped interface crack is $c_1$. These cracks can be presented by the following parametric equations

$$\begin{aligned} &r_j(s) = L_j s + 0.5(b_j + a_j), \\ &-1 \le s \le 1 \\ &\qquad for\ annular\ cracks \\ &r_j(s) = L_j(1 - s), \\ &-1 \le s \le 1 \quad (j = 1) \\ &\qquad for\ penny\ shaped\ crack \end{aligned} \tag{16}$$

Noting that,

$$L_j = 0.5\begin{cases} b_j - a_j \\ \quad for \\ annular\ cracks\ \ (j = 1,2,\dots,N_1) \\ c_j \\ \quad for \\ penny - shaped\ crack\ \ (j = 1) \end{cases} \tag{17}$$

By employing the superposition principle, the components of the stress on a given crack surface are obtained. Assume the dynamic rotational ring dislocations with unknown density $b_{\theta j}(q)$ are distributed on the infinitesimal segment at the surfaces of the *j-th* interface crack located at $z = 0$. So, the following integral equation is represented as

$$\begin{aligned} &\tau_{\theta z}(r_i(s), 0) \\ &= \sum_{j=1}^{N} L_j \int_{-1}^{1} k_{ij}^{l}(s,q) b_{\theta j}(q)\, dq\,, \\ &= 1,2,\dots,N \end{aligned} \tag{18}$$

From Eq. (15), the kernel of the integral equation $k_{ij}(s,q)$, respectively, are given by

$$\begin{aligned} &k_{ij}(s,q) = \\ &r_j \mu_{z0} \int_0^\infty \left[ \begin{matrix} \dfrac{\xi\left(1 - e^{-2h\xi}\right)}{\Lambda} \\ \left\{ \begin{matrix} (\xi + \alpha)[\alpha - \xi(\alpha h_1 + 1)] \\ -e^{-2h_1\xi}(\alpha - \xi) \\ [\xi(\alpha h_1 + 1) + \alpha] \end{matrix} \right\} \\ + \dfrac{1}{2}\eta\lambda \end{matrix} \right] \\ &\qquad J_0(r_j\eta) J_1(r_i\eta) d\eta \\ &- \frac{1}{\pi r_i}\sqrt{\mu_{z0}\mu_{r0}} \\ &\begin{cases} \dfrac{r_i r_j}{r_i^2 - r_j^2} \boldsymbol{E}\left(\dfrac{r_j^2}{r_i^2}\right) & r_i > r_j \\ \boldsymbol{K}\left(\dfrac{r_i^2}{r_j^2}\right) + \dfrac{r_j^2}{r_i^2 - r_j^2} \boldsymbol{E}\left(\dfrac{r_i^2}{r_j^2}\right) & r_i < r_j \end{cases} \end{aligned} \tag{19}$$

The left-hand side of Eq. (18), with opposite sign, is stress component caused by applied traction on the presumed crack surfaces in the intact layer. The crack opening displacement across the *j-th* crack is represented using the definition of dislocation as

$$\begin{aligned} &u_{\theta j}(s) - u_{\theta j}(s) \\ &= \frac{L_j}{r_i(s)} \int_{-1}^{s} r_j(q) b_{\theta j}(q)\, dq, \\ &\qquad j = 1,2,\dots,N \end{aligned} \tag{20}$$

For an embedded interfacial crack between two bonded dissimilar materials, Eq. (18) should be complimented with the following well-known closure conditions

$$\begin{aligned} &\int_{-1}^{1} r_j(q) b_{\theta j}(q)\, dq = 0, \\ &\quad j = 1,2,\dots,N \end{aligned} \tag{21}$$

Eq. (18) and Eq. (21) are in terms of dislocation density and must be solved simultaneously. The stress fields at the tips of the annular and penny-shaped interface cracks behave like $1/\sqrt{r}$, where $r$ is the distance from the crack tips. Consequently, the dislocation densities for each kind of the interfacial cracks are taken to be as

$$b_{\theta j}(q)=\frac{g_{aj}(q)}{\sqrt{1-q^2}},\quad for\ annular\ cracks$$
$$b_{\theta j}(q)=\sqrt{\frac{1-q}{1+q}}\,g_{\wp j}(q),\quad for\ penny-shaped\ cracks \tag{22}$$

Substituting Eq. (22) into Eq. (18) and Eq. (21) and applying the numerical technique devised by Erdogan et al. [32] result in $g_{xj}(q), x=\{a,\wp\}$. The SIFs for an annular and penny-shaped interface cracks are defined, respectively, as

$$k_{III,L}=\lim_{r\to a^-}\sqrt{2(a-r)}\,\tau_{\theta z}(r,0)$$
$$k_{III,R}=\lim_{r\to b^+}\sqrt{2(r-b)}\,\tau_{\theta z}(r,0) \tag{23}$$

and

$$k_{III}=\lim_{r\to c^+}\sqrt{2(r-c)}\,\tau_{\theta z}(r,0) \tag{24}$$

where subscripts *L* and *R* designate as inner and outer tips of annular crack, respectively. Substitution of Eq. (22) into Eq. (18) and then into Eq. (23) yields the SIFs at the annular interface crack tips $s=\pm1$ for the *j-th* crack as

$$k_{III,Lj}=\frac{1}{2}\sqrt{\mu_{r0}\mu_{z0}L_j}\,g_{aj}^*(-1)$$
$$k_{III,Rj}=-\frac{1}{2}\sqrt{\mu_{r0}\mu_{z0}L_j}\,g_{aj}^*(+1) \tag{25}$$

For the penny-shaped interface crack, from Eq. (18), Eq. (22) and Eq. (24), the SIFs become

$$k_{III,j}=\frac{1}{2}\sqrt{\mu_{r0}\mu_{z0}L_j}\,g_{\wp j}^*(-1) \tag{26}$$

**4- Numerical examples**

In this section some examples are solved to show the capabilities of the distributed dislocation technique to treat the problem of the orthotropic strip with perfect orthotropic FGM coating with multiple axisymmetric interfacial cracks under torsional impact loading. These defects may include penny-shaped and annular interfacial cracks. Variations of non-homogeneity constant, orthotropy parameter and thickness of FGM coating on the SIFs investigated, and also the interactions between several cracks which are located at interface. In all the proceeding examples, the thickness of the orthotropic layer is assumed as $h_1=10cm$. Also, the dynamic shear stress $\tau_{\theta z}=\tau_0$ is applied on the surface of the interfacial crack , in which $\tau_0$ is constant twisting loads. In addition, the SIFs are normalized as $k_{III}/k_0$, where we take $k_0=\tau_0\sqrt{l}$, in which $l$ is the length of the crack.

***Example 1: An infinite medium with a penny-shaped crack***

In order to demonstrate and verify the solution of the dislocation method given in this study with the published results, for our first example we consider the problem of an infinite domain $(h,h_1\to\infty,\lambda=1\ and\ \alpha=0)$ containing a penny-shaped crack with radius $c_1$ under loading conditions that mentioned above. The dimensionless SIFs of the singular crack tip are tabulated in Table 1.

**Table 1:** Normalized $k_{III}/\tau_0\sqrt{c_1}$ for a penny-shaped crack at the infinite FGM medium ($h, h_1 \to \infty, \lambda = 1$)

| $\alpha c_1$ | $\frac{k_{III}}{\tau_0\sqrt{c_1}}$ | |
|---|---|---|
| | Present work | Ref. [2] |
| 0.0 | 0.4981 | 0.5 |

The numerical results are compared with those of Ozturk & Erdogan [2] to establish their accuracy. A good agreement is seen in the results.

***Example 2: An orthotropic layer with orthotropic FGM coating weakened by a penny-shaped interfacial crack***

The next example deals with the orthotropic layer with orthotropic FGM coating weakened by a penny-shaped interfacial crack with length $l = c_1$. The effects of the non-homogeneity constant, orthotropy coefficient and thickness of FGM coating on the SIFs are listed in Tables 2 -4. From these Tables, it can be seen that normalized SIFs decrease with increasing the non-homogeneity constant, orthotropy coefficient and thickness of FGM coating.

***Example 3: An orthotropic layer with orthotropic FGM coating weakened by an annular interfacial crack***

In the third example, an orthotropic layer with orthotropic FGM coating containing annular interfacial crack with length $2l = (b_1 - a_1)$, where $a_1$ and $b_1$ are the inner and outer radii of the annular crack and $b_1 = 2a_1$, is analyzed. In this example, variations of the non-homogeneity parameters, orthotropy coefficient and the FGM coating thickness are investigated on the normalized SIFs. From Tables 5-7, as it was expected, the normalized SIFs of the crack tips decrease by increasing the non-homogeneity parameters, orthotropy coefficient and the FGM coating thickness. In addition, it is bserved from these Tables that normalized mode III SIFs for inner tips of annular cracks are larger than the outer tips.

**Table 2:** Variation of normalized SIFs of penny-shaped interface crack with non-homogeneity constant

| $\alpha l$ | $h_1/h = 0.1$ $\lambda = 1.5$ |
|---|---|
| | $k_{III}/\tau_0\sqrt{l}$ |
| 0.1 | 0.5555 |
| 0.3 | 0.5530 |
| 0.5 | 0.5505 |
| 1.5 | 0.5383 |
| 2.0 | 0.5324 |
| 2.5 | 0.5266 |
| 3.0 | 0.5208 |

**Table 3:** Variation of normalized SIFs of penny-shaped interface crack with FGM coating thickness

| $h_1/h$ | $\alpha l = 1$ $\lambda = 2$ |
|---|---|
| | $k_{III}/\tau_0\sqrt{l}$ |
| 1.0 | 0.4763 |
| 0.9 | 0.4795 |
| 0.8 | 0.4810 |
| 0.7 | 0.4829 |
| 0.6 | 0.4842 |
| 0.5 | 0.4860 |
| 0.4 | 0.4871 |
| 0.3 | 0.4890 |
| 0.2 | 0.4938 |
| 0.1 | 0.5228 |

***Example 4: An orthotropic layer with orthotropic FGM coating weakened by two concentric interfacial cracks***

The final example deals with the interaction of two coplanar concentric interfacial cracks. The two coplanar concentric interacting cracks including an annular crack and a penny-shaped crack, where crack numbers 1 and 2, respectively designates them.

**Table 4:** Variation of normalized SIFs of penny-shaped interface crack with orthotropic coefficient

| $\lambda$ | $\alpha l = 0.5$ $h_1/h = 0.1$ |
|---|---|
| | $k_{III}/\tau_0\sqrt{l}$ |
| 0.1 | 1.4230 |
| 0.5 | 0.7295 |
| 1.5 | 0.5505 |
| 2.5 | 0.5155 |
| 3.0 | 0.5085 |

The cracks are assumed to be on the same plane $(z_1 = z_2 = 0)$ and have radii $a_1$, $b_1$ and $c_2$, respectively. The annular crack length is $2l = (b_1 - a_1)$ and the penny-shaped crack length is $l = c_2$.

**Table 5:** Variation of normalized SIFs the tips of annular interface crack with non-homogeneity constant, when $b_1 = 2a_1$

| $\alpha l$ | $h_1/h = 0.1$ | $\lambda = 1.5$ |
|---|---|---|
| | $k_{III,L}/\tau_0\sqrt{l}$ | $k_{III,R}/\tau_0\sqrt{l}$ |
| 0.1 | 1.0169 | 0.8784 |
| 0.3 | 0.9956 | 0.8629 |
| 0.5 | 0.9758 | 0.8484 |
| 1.5 | 0.8983 | 0.7923 |
| 2.0 | 0.8711 | 0.7728 |
| 2.5 | 0.8494 | 0.7573 |
| 3.0 | 0.8319 | 0.7447 |

The variations of dimensionless SIFs for two distances between the centers of cracks are given in Tables 8 and 9. In comparison with the previous examples, we observe that interaction between two cracks increases the SIFs. In addition, it is clear that the SIFs overall amount decrease with the increasing amount of crack distances.

**Table 6:** Variation of normalized SIFs the tips of annular interface crack with FGM coating thickness, when $b_1 = 2a_1$

| $h_1/h$ | $\alpha l = 1$ | $\lambda = 2$ |
|---|---|---|
| | $k_{III,L}/\tau_0\sqrt{l}$ | $k_{III,R}/$ $\tau_0\sqrt{l}$ |
| 1.0 | 0.9440 | 0.8235 |
| 0.9 | 0.9449 | 08243 |
| 0.8 | 0.9455 | 0.8251 |
| 0.7 | 0.9462 | 0.8260 |
| 0.6 | 0.9470 | 0.8269 |
| 0.5 | 0.9481 | 0.8280 |
| 0.4 | 0.9493 | 0.8291 |
| 0.3 | 0.9505 | 0.8299 |
| 0.2 | 0.9520 | 0.8306 |
| 0.1 | 0.9527 | 0.8314 |

**Table 7:** Variation of normalized SIFs the tips of annular interface crack with orthotropic coefficient, when $b_1 = 2a_1$

| $\lambda$ | $\alpha l = 0.5$ | $h_1/h = 0.1$ |
|---|---|---|
| | $k_{III,L}/\tau_0\sqrt{l}$ | $k_{III,R}/\tau_0\sqrt{l}$ |
| 0.1 | 1.0002 | 0.8660 |
| 0.5 | 0.9950 | 0.8622 |
| 1.5 | 0.9758 | 0.8484 |
| 2.5 | 0.8982 | 0.7922 |
| 3.0 | 0.7640 | 0.6960 |

**Table 8:** Variation of normalized SIFs the tips of concentric interface cracks with $\alpha l = 1$, $h_1/h = 0.1$, $\lambda = 2$

| $a_1 = 1.5c_2$ | $b_1 = 3.5c_2$ |
|---|---|
| $k_{III,L1}/\tau_0\sqrt{l}$ | 1.1905 |
| $k_{III,R1}/\tau_0\sqrt{l}$ | 0.9477 |
| $k_{III,2}/\tau_0\sqrt{l}$ | 0.5523 |

**Table 9:** Variation of normalized SIFs the tips of concentric interface cracks with $\alpha l = 1$ , $h_1/h = 0.1$, $\lambda = 2$

| $a_1 = 2c_2 \quad b_1 = 4c_2$ | |
|---|---|
| $k_{III,L1}/\tau_0\sqrt{l}$ | 1.0786 |
| $k_{III,R1}/\tau_0\sqrt{l}$ | 0.9169 |
| $k_{III,2}/\tau_0\sqrt{l}$ | 0.5335 |

**5- Conclusion**

In this paper, the fracture behavior of multiple axisymmetric interface cracks between an orthotropic layer with FGM orthotropic coating subjected to torsional loading using the distributed dislocation technique was studied. A solution for the stress field caused by the rotational Somigliana ring dislocation in an orthotropic layer with FGM orthotropic coating is first obtained. Next, using the distributed dislocation technique, the problem was formulated for an orthotropic layer with FGM orthotropic coating with a system of coaxial axisymmetric interface cracks. The unknown dislocation density on the surfaces of the cracks was obtainable by solving a set of integral equations of Cauchy singular type. To study the effect of the non-homogeneity constant, orthotropy parameter and thickness of FGM coating, SIFs are obtained for some examples. The following key points are observed:

1) The non-homogeneity parameter, orthotropy constant and thickness of FGM coating have quite a considerable influence on the SIFs.
2) The value of the SIFs decreases with increasing the non-homogeneity parameter, orthotropy and thickness of FGM coating.
3) For the single annular interface crack, the modes III SIFs for inner tips are larger than the outer tips.
4) For the case of a penny-shaped interface crack surrounded by an annular interface crack, the values of the SIFs for the annular crack are higher than the SIFs of the penny-shaped crack.
5) The interaction between the cracks is an important factor affecting SIFs.

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