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Spline Collocation for Fredholm and Volterra Integro-Differential Equations

N. Ebrahimi[a,*] and J. Rashidinia[a]

Department of Mathematics, Islamic Azad University, Central Tehran Branch, Iran.

Abstract. A collocation procedure is developed for the linear and nonlinear Fredholm and Volterra integro-differential equations, using the globally defined B-spline and auxiliary basis functions.The solution is collocated by cubic B-spline and the integrand is approximated by the Newton-Cotes formula. The error analysis of proposed numerical method is studied theoretically. Numerical results are given to illustrate the efficiency of the proposed method which shows that our method can be applied for large values of N. The results are compared with the results obtained by other methods to illustrate the accuracy and the implementation of our method.

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1. Introduction

Consider the linear and nonlinear integro-differential equations of the form

$$\sum_{r=0}^{m} p_r(t)y^{(r)}(t) = g(t) + \int_a^b K(t,x,y(x))dx, \quad t \in [a,b], m = 1,2, \tag{1}$$

with the boundary conditions,

[*] Corresponding author. Email: Ebrahimi_Nehzat@yahoo.com

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$$\sum_{r=0}^{m-1}[\alpha_{i,r}y^{(r)}(a)+\beta_{i,r}y^{(r)}(b)]=\gamma_i, \qquad 0\leqslant i\leqslant m-1,$$

where $\alpha_{i,r},\beta_{i,r}$ and $\gamma_i$are given real constants. The given kernel K is continuous on [a, b] and satisfie a uniform Lipschitz, and $g(t)$ and $p_r(t)$ are the known functions and y is unknown function.The boundary value problems in terms of integro-differential equations have many practical applications. A physical event can be modelled by the differential equation, an integral equation or an integro-differential equation or a system of these equations.some authors have proposed numerical methods to approximate the solutions of linear and nonlinear Fredholm and Volterra integro-differential equations, such as the direct method via block-pulse functions [5], the sinc-collocation method [16,18,23], the variational iteration method [22],the Chebyshev wavelet approximating method [10],the formulation of the piecewise Tau method [11,12], the fast Galerkin scheme [4],the Modified Homotopy Perturbation Method[21].Using a global approximation to the solution of equations and functions is constructed by means of the spline quadrature in [1,2,3,8,13,19,20].
In this paper, we use cubic B-spline collocation for approximation unknown function and use of the Newton-Cotes rules for approximating integrand.

2. Cubic B-Spline

We introduce the cubic B-spline space and basis functions to construct an interpolant s to be used in the formulation of the cubic B-spline collocation method. Let $\pi=\{a=t_0<t_1<\cdots<t_N=b\}$, be a uniform partition of the interval $[a,b]$ with step size $h=\frac{b-a}{N}$. The cubic B-spline space is denoted by

$$S_3(\pi)=\{s\in C^2[a,b]: s\mid_{[t_i,t_{i+1}]}\in P_3, \quad i=0,1,\ldots,N\},$$

where P_3 is the class of cubic polynomials. The construction of the cubic B-spline interpolate s to the analytical solution y for (1) can be performed with the help of the four additional knots such that $t_{-2}<t_{-1}<t_0$ *and* $t_N<t_{N+1}<t_{N+2}$. We can define a cubic B-spline $s(t)$ of the form

$$s(t)=\sum_{i=-1}^{N+1}c_i\beta_i^3(t), \tag{2}$$

where

$$\beta_i^3(t)=\frac{1}{6h^3}$$

$$\begin{cases}(t-t_{i-2})^3 & t\in[t_{i-2},t_{i-1}]\\ h^3+3h^2(t-t_{i-1})+3h(t-t_{i-1})^2-3(t-t_{i-1})^3 & t\in[t_{i-1},t_i]\\ h^3+3h^2(t_{i+1}-t)+3h(t_{i+1}-t)^2-3(t_{i+1}-t)^3 & t\in[t_i,t_{i+1}]\\ (t_{i+2}-t)^3 & t\in[t_{i+1},t_{i+2}]\\ 0 & otherwise,\end{cases} \tag{3}$$

satisfying the following interpolatory conditions

$$s(t_i) = y(t_i), \quad 0 \leqslant i \leqslant N,$$

and the boundary conditions

$\mathcal{C}_1.s^{'}(t_0) = y^{'}(t_0), \quad s^{'}(t_N) = y^{'}(t_N),$

$\mathcal{C}_2.D^j s(t_0) = D^j s(t_N), \quad j = 1, 2,$

$$\mathcal{C}_3.s''(t_0) = 0, \quad s''(t_N) = 0. \tag{4}$$

3. The Collocation Method

3.1 *Nonlinear Fredholm integro-differential equation*

In the given nonlinear Fredholm integro-differential Eq. (1), we can approximate the unknown function by cubic B-spline (2), then we obtain:

$$\sum_{r=0}^{m} p_r(t)s^{(r)}(t) = g(t) + \int_a^b K(t, x, s(x))dx, \quad t \in [a, b], m = 1, 2, \tag{5}$$

with the boundary conditions,

$$\sum_{r=0}^{m-1} [\alpha_{i,r}s^{(r)}(a) + \beta_{i,r}s^{(r)}(b)] = \gamma_i, \qquad 0 \leqslant i \leqslant m-1.$$

We now collocate Eq. (5) at collocation points $t_j = a+jh,\ h = \frac{b-a}{N}, j = 0, 1, \ldots, N,$ and we obtain

$$\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j) + \int_a^b K(t_j, x, s(x))dx,\ j = 0, \ldots, N,\ m = 1, 2. \tag{6}$$

To approximate the integro-differential Eq. (6), we can use the Newton- Cotes formula[9], when n is even then the Simpson rule can be used and when n is multiple of 3 ,we have to use the three-eighth rule,then we get the following nonlinear system:

$$\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j) + h\sum_{i=0}^{N} w_{j,i}K(t_j, x_i, s(x_i)), j = 0, \ldots, N, m = 1, 2, \tag{7}$$

with the boundary conditions,

$$\sum_{r=0}^{m-1}[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)]=\gamma_i, \qquad 0\leqslant i\leqslant m-1,$$

where $x_i = a+ih, i = 0,1,\dots,N$,we need more equations to obtain the unique solution for Eq. (7). Hence by associating Eq. (7) with (4) , we have the following nonlinear system $(N+3)\times(N+3)$:

$$\begin{cases} \sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j)+h\sum_{i=0}^{N} w_{j,i}K(t_j,x_i,s(x_i)), j=0,\dots,N, m=1,2, \\ \\ \sum_{r=0}^{m-1}[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)]=\gamma_i, \qquad 0\leqslant i\leqslant m-1, \end{cases} \tag{8}$$

where $w_{j,i}$ represents the weights for a quadrature rule of Newton-Cotes type.

By solving the above nonlinear system , we can determine the coefficients $c_i, i = -1,\dots,N+1$, by setting c_i in (2), we obtain the approximate solution for Eq. (1).

3.2 *Nonlinear Volterra integro-differential equation*

Now we consider nonlinear Volterra integro-differential equation

$$\sum_{r=0}^{m} p_r(t)y^{(r)}(t) = g(t)+\int_a^t K(t,x,y(x))dx, \quad t\in[a,b], m=1,2, \tag{9}$$

with the boundary conditions,

$$\sum_{r=0}^{m-1}[\alpha_{i,r}y^{(r)}(a)+\beta_{i,r}y^{(r)}(b)]=\gamma_i, \qquad 0\leqslant i\leqslant m-1,$$

the solutions of Eq. (9)can be replaced with cubic B-spline and so we collocate Eq. (9) at collocation points $t_j = a+jh,\ h=\frac{t-a}{N},\ \ j=0,1,\dots,N$, then we obtain

$$\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j)+\int_a^{t_j} K(t_j,x,s(x))dx, j=0,\dots,N,\ m=1,2. \tag{10}$$

To approximate the integro-differential Eq. (10), we can use the Newton- Cotes formula, when n is even then the Simpson rule can be used and when n is multiple of 3, we have to use the three-eighth rule,then we get the following nonlinear system:

$$\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j)+h\sum_{i=0}^{j} w_{j,i}K(t_j,x_i,s(x_i)), j=1,\dots,N, m=1,2, \tag{11}$$

with the boundary conditions,

$$\sum_{r=0}^{m-1}[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)]=\gamma_i, \qquad 0\leqslant i\leqslant m-1.$$

We need more equations to obtain the unique solution for Eq. (11). Hence by associating Eq. (11) with (4) , we have the following nonlinear system $(N+3)\times(N+3)$:

$$\begin{cases}\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j) + h\sum_{i=0}^{j} w_{j,i}K(t_j,x_i,s(x_i)), j=1,\dots,N, m=1,2,\\ \sum_{r=0}^{m} p_r(t_0)s^{(r)}(t_0) = g(t_0)\\ \sum_{r=0}^{m-1}[\alpha_{i,r}s^{(r)}(a)+\beta_{i,r}s^{(r)}(b)] = \gamma_i, \qquad 0\leqslant i\leqslant m-1. \end{cases} \tag{12}$$

By solving the above nonlinear system , we can determine the coefficients $c_i, i = -1,\dots,N+1$, by setting c_i in (2), we obtain the approximate solution for Eq. (9).

4. Error analysis: convergence of the approximate solution

In this section, we consider the error analysis of the Fredholm and Volterra integro-differential equation of the second kind , first we recall the following definition in [17].

Definition : Let $s(t)$ be the cubic B-spline interpolate $f\in C^4[a,b]$, then for all admissible h,there is a number $M_j<\infty$,independent of h, such that

$$\|D^j(f-s)\|_2 \le M_j\|f^{(4)}\|_2 h^{4-j-1/2}, \quad j=0,\dots,3\ ,$$

where

$$M_j=\frac{2}{j!}, \qquad\qquad j=0,\dots,3,$$

and D^j is the j-th derivative and if $p=4-j-1/2$ is the largest number for which such an inequality holds, then p is called the order of convergence of the method.

Theorem : The approximate method

$$\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j) + h\sum_{i=0}^{j} w_{j,i}K(t_j,x_i,s(x_i)), j=1,\dots,N, m=1,2, \tag{13}$$

for solution of the nonlinear Volterra integro-differential Eq .(9) is converge and the error bounded is

$$|e_j^{(m)}| \leqslant \frac{1}{|p_{mj}|}\sum_{r=0}^{m-1}|p_{rj}||e_j^{(r)}| + \frac{hWL}{|p_{mj}|}\sum_{i=0}^{j}|e_i| + \frac{|E(h,t_j)|}{|p_{mj}|},$$

where $e_j^{(r)} = s_j^{(r)} - y_j^{(r)},\ \ r=0,\dots,m, j=1,\dots,N.$

Proof: We know that at $t_j = a+jh, h=\frac{t-a}{N},\ \ j=1,\dots,N$, the corresponding approximation method for nonlinear Volterra integro-differential equation (9) is

$$\sum_{r=0}^{m} p_r(t_j)s^{(r)}(t_j) = g(t_j) + h\sum_{i=0}^{j} w_{j,i}K(t_j,x_i,s(x_i)), j=1,\dots,N, m=1,2. \tag{14}$$

By discretizing (9) and approximating the integrand by the Newton- Cotes formula, we obtain

$$\sum_{r=0}^{m} p_r(t_j)y^{(r)}(t_j) = g(t_j) + h\sum_{i=0}^{j} w_{j,i}K(t_j, x_i, y(x_i)) + E(h, t_j), j = 1, \ldots, N,$$

$$m = 1, 2, \qquad (15)$$

where $E(h,t_j) = \int_a^{t_j} K(t_j, x, y(x))dx - h\sum_{i=0}^{j} w_{j,i}K(t_j, x_i, y(x_i))$.

By subtracting (15) from (14) and using interpolatory conditions of cubic B-spline, we get

$$\sum_{r=0}^{m} p_r(t_j)[s^{(r)}(t_j) - y^{(r)}(t_j)] = h\sum_{i=0}^{j} w_{j,i}[K(t_j, x_i, s(x_i)) - K(t_j, x_i, y(x_i))] - E(h, t_j).$$

We suppose that $W = max_{i,j}|w_{j,i}|$ and $s^{(m)}(t_j) = s_j^{(m)},\ \ y^{(m)}(t_j) = y_j^{(m)},\ \ j = 1, \ldots, N, m = 1, 2.$ *and kernel* K satisfies a Lipschitz condition in its third argument of the form

$$|K(t, x, s) - K(t, x, y)| \leq L|s - y|,$$

where L is independent of t, x, s and y. We get

$$|p_{mj}||s_j^{(m)} - y_j^{(m)}| \leq \sum_{r=0}^{m-1} |p_{rj}||s_j^{(r)} - y_j^{(r)}| + hWL\sum_{i=0}^{j} |s(x_i) - y(x_i)| + |E(h, t_j)|.$$

Since that $|p_{mj}| \neq 0$ then we have

$$|e_j^{(m)}| \leqslant \frac{1}{|p_{mj}|}\sum_{r=0}^{m-1} |p_{rj}||e_j^{(r)}| + \frac{hWL}{|p_{mj}|}\sum_{i=0}^{j} |e_i| + \frac{|E(h, t_j)|}{|p_{mj}|},$$

where $e_j^{(r)} = s_j^{(r)} - y_j^{(r)},\ \ r = 0, \ldots, m, j = 1, \ldots, N.$
Since by assumption both the quadrature error and the function approximate error are zero in the limit ,it follows when $h \to 0,\ \ \lim\max|E(h, t_j)| = 0$, and the above second term is zero and the first term in the above tend to zero because this term is due to interpolating of $y(t)$by cubic B-spline. We get for a fixed j,

$$|e_j^{(m)}| \to 0 \ \ as \ \ h \to 0, m = 0, 1, 2.$$

5. Numerical Examples

In order to test the applicability of the presented method, we consider four examples of linear and nonlinear Volterra and Fredholm integro-differential equations with the boundary conditions. The absolute errors in the solution for various values of

N are tabulated in Tables.The RMS error in the solutions:

$$E = \sqrt{\frac{1}{N}\sum_{i=0}^{N}[s(x_i) - y(x_i)]^2}$$

is computed by our purposed method where $y(t)$ is the exact solution and $s(t)$ is the approximated solution of integral equation. Programs preformed by Mathematica for all the four examples.

Example 1. Consider the following linear Fredholm integro-differential equation with exact solution $y(t) = e^t$,

$$y''(t) = e^t - t + \int_0^1 xty(x)dx,$$

with boundary conditions:$y(0) = 1, y'(0) = 1$.
This equation has been solved by our method with $N = 10, 30, 60$, the absolute errors at the particular grid points and the RMS errors are tabulated in table 1 , which shows that the error in the solutions for our method decreases by reducing the values of h.
Example 2. Consider the following linear Volterra integro-differential equation with exact solution $y(t) = e^{-t}\cosh t$,

$$y'(t) + y(t) = \int_0^t e^{x-t}y(x)dx,$$

with boundary conditions :$y(0) = 1$.
This equation has been solved by our method with $N = 10, 30, 60$, the absolute errors at the particular grid points and the RMS errors are tabulated in table 2 , which shows that the error in the solutions for our method decreases by reducing the values of h.
Example 3. Consider the following nonlinear Fredholm integro-differential equation with exact solution $y(t) = t$,

$$y'(t) = \frac{5}{4} - \frac{t^2}{3} + \int_0^1 (t^2 - x)y^2(x)dx,$$

with boundary conditions :$y(0) = 0$.
The absolute errors at the particular grid points are tabulated in table 3 , and compared with the absolute errors obtained by [14,15]. This table verified that our results are more accurate in comparison.

Example 4. Consider the following nonlinear Volterra integro-differential equation with exact solution$y(t) = \cos t$,

$$y'(t) = 2\sin t\cos t - \int_0^t 3\cos(t - x)y^2(x)dx,$$

with boundary conditions :$y(0) = 1$.

Table 1. *The error $\|E\|$ in solution of example 1 at particular points*

t	$N = 10$	$N = 30$	$N = 60$
0	0	0	0
0.1	4.39(−6)	4.82(−7)	1.20(−7)
0.2	1.81(−5)	2.0(−6)	5.0(−7)
0.3	4.24(−5)	4.69(−6)	1.17(−6)
0.4	7.83(−5)	8.67(−6)	2.16(−6)
0.5	1.27(−4)	1.41(−5)	3.52(−6)
0.6	1.91(−4)	2.11(−5)	5.28(−6)
0.7	2.70(−4)	2.99(−5)	7.49(−6)
0.8	3.67(−4)	4.07(−5)	1.01(−5)
0.9	4.85(−4)	5.37(−5)	1.34(−5)
1	6.25(−4)	6.91(−5)	1.72(−5)
The RMS error in our method	2.61(−4)	2.89(−5)	7.24(−6)

Table 2. *The error $\|E\|$ in solution of example 2 at particular points*

t	$N = 10$	$N = 30$	$N = 60$
0	6.66(−16)	4.44(−16)	0
0.1	3.62(−4)	4.03(−5)	8.01(−7)
0.2	5.10(−5)	5.51(−6)	1.37(−6)
0.3	4.21(−4)	4.67(−5)	1.75(−6)
0.4	7.32(−5)	7.90(−6)	1.97(−6)
0.5	4.56(−4)	5.06(−5)	2.04(−6)
0.6	7.48(−5)	8.04(−6)	2.0(−6)
0.7	4.76(−4)	5.28(−5)	1.86(−6)
0.8	6.18(−5)	6.58(−6)	1.63(−6)
0.9	4.86(−4)	5.38(−5)	1.34(−6)
1	3.87(−5)	4.02(−6)	9.98(−7)
The RMS error in our method	2.84(−4)	3.14(−5)	7.86(−6)

Table 3. *The error $\|E\|$ in solution of example 3 at particular points*

t	Our method $with N = 15$	method in [14]	method in [15]
0	1.73(−18)	5.06(−15)	1.55(−3)
0.1	1.38(−17)	3.63(−14)	4.01(−3)
0.2	2.77(−17)	7.36(−14)	3.98(−3)
0.3	5.55(−17)	1.11(−13)	1.47(−3)
0.4	2.22(−16)	1.41(−13)	6.40(−3)
0.5	5.55(−17)	1.75(−13)	8.87(−3)
0.6	0	2.29(−13)	3.82(−3)
0.7	2.22(−16)	2.51(−13)	3.75(−3)
0.8	4.44(−16)	2.67(−13)	7.70(−4)
0.9	3.33(−16)	2.85(−13)	2.14(−3)
1	0	2.86(−13)	1.37(−3)

The absolute errors at the particular grid points are tabulated in table 4 and compared with the absolute errors obtained by [6,7] .This table verified that our results are more accurate in comparison.

Table 4. *The error $\|E\|$ in solution of example 4 at particular points*

t	Our method with$N = 12$	Our method with$N = 34$	method in [7]	method in[6]
0	0	0	0	0
0.1	$4.93(-4)$	$4.21(-5)$	$1.16(-3)$	$1.41(-4)$
0.2	$8.63(-5)$	$5.31(-5)$	$1.63(-3)$	$4.21(-3)$
0.3	$2.01(-5)$	$1.67(-5)$	$1.23(-3)$	$5.05(-3)$
0.4	$2.95(-4)$	$5.51(-6)$	$2.20(-4)$	$2.62(-3)$
0.5	$3.27(-4)$	$3.71(-5)$	$1.33(-3)$	$1.54(-2)$
0.6	$1.71(-4)$	$2.08(-5)$	$9.93(-2)$	$3.63(-3)$
0.7	$2.26(-4)$	$4.72(-5)$	$7.66(-4)$	$1.19(-2)$
0.8	$2.79(-4)$	$7.84(-6)$	$1.30(-3)$	$1.37(-2)$
0.9	$1.07(-5)$	$1.75(-5)$	$2.26(-3)$	$4.38(-3)$
1	$5.94(-4)$	$7.35(-5)$	$3.18(-3)$	$2.65(-2)$

Conclusions

In the present work, a technique has been developed for solving linear and nonlinear Fredholm and Volterra integro-differential equations by using the Newton-Cotes formula and collocating by cubic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations in terms of the linear combination coefficients appearing in the representation of the solution in spline basic functions.This method tested on 4 examples .The absolute errors in the solutions of these examples show that our approach is more accurate in comparison with the methods given in [6,7,14,15] and our results verified the accurate nature of our method.

References

[1] Amirfakhrian. M, Shakibi. K, Solving Integro-Differential Equation by Using B- Spline Interpolation,International Journal of Mathematical Modelling & Computations, **3 (3)** (2013) 237-244.

[2] Amirfakhrian. M, Approximation of 3D-Parametric Functions by Bicubic B-spline Functions, International Journal of Mathematical Modelling & Computations, **2 (3)** (2012) 211- 220.

[3] Amirfakhrian. M, Nouriani. H, Interpolation by Hyperbolic B-spline Functions, International Journal of Mathematical Modelling & Computations, 1 (3) (2011) 175 -181.

[4] Babolian. E, Delves. L. M, A fast Galerkin scheme for linear integro-differential equations, J. Numer. Anal. **1** (1981) 193-213.

[5] Babolian. E, Masouri. Z, Numerical solution of Volterra integro-differential equation by direct method via block-pulse functions, J. Sci. Tarbiat Moallem Univ, 9 (1) (2010) 1-9.

[6] Babolian. E, Masouri. Z, Hatamzadeh-Varmazyar. S, New direct method to solve nonlinear Volterra-Fredholm integral and integro differential equation using operational matrix with Block-pulse functions, Progress In Electromagnetics Research B **8** (2008) 59-76.

[7] Babolian. E, Masouri. Z, Hatamzadeh-Varmazyar. S, Numerical solution of nonlinear Volterra-Fredholm integro-differential equations via direct method using triangular functions, Computers and Mathematics with Applications **58** (2009) 239-247.

[8] Dabounou. J, El Khayyari. O, Lamnii. A, Solving Linear Sixth-Order Boundary Value Problems by Using Hyperbolic Uniform Spline Method,International Journal of Mathematical Modelling & Computations **3 (3)** (2013) 169 -180.

[9] Delves. L. M, Mohamad. J. L, Computational methods for integral equations, Cambridge University Press, 1985.

[10] Fariborzi Araghi. M. A, Daliri. S, Bahmanpour. M, Numerical Solution of Integro-Differential Equation by using Chebyshev Wavelet Operational Matrix of Integration, International Journal of Mathematical Modelling & Computations, **2 (2)** (2012) 127- 136.

[11] Hosseini. S. M, Shahmorad. S, Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method, Appl. Math. Model. **29** (2005) 1005-1021.

[12] Hosseini. S. M, Shahmorad. S, Numerical solution of a class of integro-differential equations by the Tau Method with an error estimation, Appl. Math. Comput. **136** (2003) 559-570.

[13] Lamnii. A, Mraoui. H, Spline Collocation Method for Solving Boundary Value Problems, International Journal of Mathematical Modelling & Computations, **3 (1)** (2013) 11-23.

[14] Mahmoodi. Z, Rashidinia. J, Babolian. E, B-Spline collocation method for linear and nonlinear Fredholm and Volterra integro-differential equations, Applicable Analysis, (2012) 1-16.

[15] Mirzaee. F, The RHFs for solution of nonlinear Fredholm integro-differential equations, Appl. Math. Sci. **5 (70)** (2011) 3453-3464.

[16] Mohsen. A, El-Gamel. M, A sinc-collocation method for the linear Fredholm integro-differential equations, ZAMP. Z. Angew. Math. Phys. **58** (2007) 380-390.

[17] Prenter. P. M, Spline and Variational Methods, Wiley & Sons, New-York (1975).

[18] Rashidinia. J, Zarebnia. M, The numerical solution integro-differential equation by the sinc method, Appl. Math. Comput. **188** (2007) 1124-1130.

[19] Rashidinia. J, Esfahani. F, Jamalzadeh. S, B-spline Collocation Approach for Solution of Klein-Gordon Equation, International Journal of Mathematical Modelling & Computations,**3 (1)** (2013) 25- 33.

[20] Rashidinia. J, Mahmoodi. Z, Babolian. E, Spline Collocation for Nonlinear Fredholm Integral Equations, International Journal of Mathematical Modelling & Computations, **1 (1)** (2011) 69-75.

[21] Sotoodeh. M, Fariborzi Araghi. M. A, A New Modified Homotopy Perturbation Method for Solving Linear Second-Order Fredholm Integro-Differential Equations, International Journal of Mathematical Modelling & Computations, **2 (4)** (2012) 299-308.

[22] Sweilam. N. H, Fourth order integro-differential equations using variational iteration method, Comput. Math. Appl. **54** (2007) 1086-1091 .

[23] Zarebnia. M, Sinc numerical solution for the Volterra integro-differential equation, Commun. Nonlinear Sci. Numer. Simul. **15 (3)** (2010) 700-706 .