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Optimal Resource Allocation in DEA with Integer Variables

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Abstract. Resource allocation and optimal leveling are among the top challenges in project management. This paper presents a DEA-based procedure for determining an optimal level of inputs to produce a fixed level of outputs. To achieve this goal, we assume that the levels of outputs can be forecasted in the next season and the procedure will determine optimal level of inputs for all DMUs. Such as some of them can only take integer values. So, after this design for the value of inputs and outputs all DMUs are placed on the optimal level. An illustrative example is used to show the applicability of the proposed method.

Keywords: Data envelopment analysis, Efficiency, Input, Output, Integer variable.

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1. Introduction

Data envelopment analysis (DEA) is procedure originally developed by Charnes et al. (1978) [3] to measure the relative efficiency of a set of decision making units (DMUs); each of them uses multiple inputs to produce multiple outputs. Whit this forecasted production for new operational unit, the paper develops a DEA-based production planning approach to determine the most favorable resources. Nevertheless, some of these inputs only take integer values and some of them can change arbitrary. A principal idea behind the procedure is to use the empirical production function defined by observed inputs and outputs to make the new operational units as most productive scale size (MPSS). We assume that we are given a finite set $j = 1, \dots, n$ of observed DMUs, Each of which uses m inputs to produce s outputs. Especially DMU_j consumes input $x_{ij} = (x_{1j}, \dots, x_{mj}) \geqslant 0$ to produce $y_{rj} = (y_{1j}, \dots, y_{sj}) \geqslant 0$. The production possibility set T_c is defined as:

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$$T_c = \left\{ (x,y) | \sum_{j=1}^{n} \lambda_j x_j \leqslant \mathbf{x},\ \sum_{j=1}^{n} \lambda_j x_j \geqslant \mathbf{y},\ j = 1,\dots,n \right\}$$

The original DEA radial efficiency of DMU$_o$ is calculated as

$$\begin{array}{ll} \min \theta & \\ \text{s.t. } y_{ro} \leqslant \sum_{j=1}^{n} \lambda_j y_{rj} & r = 1,\dots,s \\ \quad \theta x_{io} \geqslant \sum_{j=1}^{n} \lambda_j x_{ij} & i = 1,\dots,m \\ \quad \lambda_j \geqslant 0 & j = 1,\dots,n \end{array} \tag{1}$$

Suppose some of the inputs and outputs are deemed to be integer, while the others are not. Following Lozano and Villa (2006) [8], we partition the set of input variables as $I = I^I \cup I^{NI}$ and the set of output variables as $O = O^I \cup O^{NI}$, where subsets I^I and O^I are subject to the integrality condition and subsets I^{NI} and O^{NI} are real-valued. Subsets I^I and I^{NI}, as well as O^I and O^{NI}, are assumed to be mutually disjoint, $|I^I| = p \leqslant m$ and $|O^I| = q \leqslant s$. Based on the preceding notations, every feasible activity which is characterized by a pair of non-negative input and output vectors (X,Y) can be written as $X = \binom{X^I}{X^{NI}}$, $Y = \binom{Y^I}{Y^{NI}}$ and a DEA PPS that can be stated as

$$T_{HIDEA} = \left\{ \begin{pmatrix} X^I & Y^I \\ X^{NI} & Y^{NI} \end{pmatrix} | (X^I, Y^I) \in Z_+^{p+q}; \begin{pmatrix} X^I \\ X^{NI} \end{pmatrix} \geqslant \sum_{j=1}^{n} \begin{pmatrix} X_j^I \\ X_j^{NI} \end{pmatrix} \lambda_j; \right. \\ \left. \begin{pmatrix} Y^I \\ Y^{NI} \end{pmatrix} \leqslant \sum_{j=1}^{n} \begin{pmatrix} Y_j^I \\ Y_j^{NI} \end{pmatrix} \lambda_j; \lambda_j \geqslant 0\ \forall j \right\}$$

Based on this production possibility set T_{HIDEA} Kuosmanen and Kazemi Matin (2009) [7] proposed an input efficiency scores follows:

$$\begin{array}{ll} \min \theta & \\ \text{s.t. } y_{ro} \leqslant \sum_{j=1}^{n} \lambda_j y_{rj} & r \in O \\ \quad \theta x_{io} \geqslant \sum_{j=1}^{n} \lambda_j x_{ij} & i \in I^{NI} \\ \quad \tilde{x}_i \geqslant \sum_{j=1}^{n} \lambda_j x_{ij} & i \in I^I \\ \quad \theta x_{io} \geqslant \tilde{x}_i & i \in I^I \\ \quad \tilde{x}_i \in Z_+ & i \in I^I \\ \quad \lambda_j \geqslant 0 & j = 1,\dots,n \end{array} \tag{2}$$

This model is a MILP problem which is computable by standard MILP algorithms and solver software. The multiplier from of model (2) is as follows:

$$\begin{aligned}
\max \; & \sum_{r\in O} u_r y_{ro} \\
\text{s.t.} \; & \sum_{r\in O} u_r y_{rj} - \sum_{i\in I^{NI}} v_i x_{ij} - \sum_{I\in I^{I}} v'_i x_{ij} \leqslant 0 \; j=1,\dots,n \\
& \sum_{i\in I^{NI}} v_i x_{io} + \sum_{i\in I^{I}} v''_i x_{io} = 1 \\
& \sum_{i\in I^{I}} v'_i + \sum_{i\in I^{I}} v''_i \leqslant 0 \\
& u_r \geqslant 0 \, r\in O \quad v_i \geqslant 0 \, i \in I^{NI} \\
& v'_i \geqslant 0 \, i\in I^{I} \quad v''_i \geqslant 0 \, i\in I^{I}
\end{aligned} \tag{3}$$

Banker (1984) [1] takes the nature of returns to scale into account and introduced the concept of MPSS as follows:

DEFINITION 1.1 *A production possibility* $(x, y) \in T_c$ *is banker's MPSS if for all* $(\beta x, \alpha y) \in T_c$, *be* $\frac{\alpha}{\beta} \leqslant 1$.

This definition shows that banker's MPSS must be associated with boundary points and maximizes, average productivity, $\frac{\alpha}{\beta}$, for the DMU (x, y). Cooper et. al. (1996) [4] proposed the following linear fractional programming problem to determine the MPSS.

$$\begin{aligned}
\max \; & \frac{\alpha}{\beta} \\
\text{s.t.} \; & \alpha Y_o \leqslant \sum_{j=1}^{n} \lambda_j y_{rj} \;\; r=1,\dots,s \\
& \beta X_o \geqslant \sum_{j=1}^{n} \lambda_j x_{ij} \;\; i=1,\dots,m \\
& \alpha, \beta, \lambda_j \geqslant 0 \qquad j=1,\dots,n
\end{aligned} \tag{4}$$

They stated that DMU$_o$ is MPSS if (I) $\frac{\alpha^*}{\beta^*} = 1$ and (II) All slack variables must be zero in any optimal solution.

2. Production Planning Model

We assume that there are n DMUs which the i-th input and r-th output of DMU$_j$ are denoted by $x_{ij} : (i = 1, \dots, m)$ and $y_{rj} : (r = 1, \dots, s)$, respectively. Suppose that the production output $r = 1, \dots, s$ in the next season can be forecasted as $y_r : (r = 1, \dots, s)$.

For these forecasted outputs, we will determine the most favorable input consumption for operational unit. As some of them integer and some of them are real valued that are shown $i \in I^I$ and $i \in I^{NI}$ respectively as $I = I^I \cup I^{NI}$.We first solve the following linear programming problem in order to find the efficient surface of the production possibility set T_{HIDEA} that the new operational unit DMU is projected to it:

$$
\begin{array}{l}
\min \sum_{j=1}^{n} s_j \\
\sum_{r\in O} u_r y_{ro} = 1 \\
\sum_{r\in O} u_r y_{ro} - \sum_{i\in I^{NI}} v_i x_{ij} - \sum_{i\in I^{I}} v'_i x_{ij} + s_j = 0 \; j = 1,\dots,n \\
\sum_{i\in I^I} (v'_i - v''_i) \leqslant 0 \\
u_r \geqslant 0 \quad r\in O \quad v_i \geqslant 0 \quad i \in I^{NI} \\
v'_i, v''_i \geqslant 0 \quad i\in I^I
\end{array}
\tag{5}
$$

Let u^*, v^*, v'^*, v''^*, and s^* be an optimal solution to model (5). It is easy to show that $s^* = 0$ for some $j = 1\dots,n$.
Moreover, the optimal solution to model (5) gives a supporting surface of T_{HIDEA} as follows:

$$
F = \left\{ \begin{pmatrix} x_{ij} & i\in I^I \\ x_{ij} & i\in I^{NI} \\ y_{rj} & r\in O \end{pmatrix} : \sum_{r\in O} u^*_r y_{ro} - \sum_{I\in I^{NI}} v^*_i x_{ij} - \sum_{i\in I^I} v'^*_i x_{ij} = 0 \right\} \cap T_{HIDEA}
$$

DMUs that do belong to F are clearly MPSS in T_{HIDEA}.

Now, each production plan that lies on F makes the new operational unit as MPSS. The set all optimal production plans that make DMU$_o$ as MPSS is defined as follows:

$$
\bar{P} = \left\{ \begin{pmatrix} \bar{x}_i & i\in I^I \\ \bar{x}_i & i\in I^{NI} \end{pmatrix} : \sum_{i\in I^{NI}} v^*_i x_{io} + \sum_{I\in I^I} v''^*_i x_{io} = 1, \quad l_i \leqslant \bar{x}_i \leqslant u_i, \quad i\in I^I \cup I^{NI} \right\} \tag{6}
$$

$u_r \geqslant 0$ and $l_i \geqslant 0$ are user-defined constant to reflect upper and lower bounds on the i-th input.

A multi-objective linear programming (MOLP) model is developed as follows to simultaneously minimize all input consumptions:

$$
\begin{array}{l}
\min \{\bar{x}_i : i\in I^I, \bar{x}_i : i \in I^I \cup i^{NI}\} \\
\sum_{i\in I^{NI}} v^*_i x_{io} + \sum_{i\in I^I} v''^*_i x_{io} = 1 \\
l_i \leqslant \bar{x}_i \leqslant u_i \quad i\in I^I \cup I^{NI}
\end{array}
\tag{7}
$$

Taking the priority of the inputs into account, we can develop the following linear programming model as equivalence to model (5):

$$
\begin{array}{lll}
\min \phi & & \\
\text{s.t.} & \sum_{i\in I^{NI}} v^*_i x_{io} + \sum_{i\in I^I} v''^* x_{io} = 1 & \\
 & \phi \geqslant \mu_i \bar{x}_i & i\in I^{NI} \\
 & \phi \geqslant \mu'_i & i \in I^I \\
 & l_i \leqslant \bar{x}_i \leqslant u_i & i\in I^I \cup I^{NI}
\end{array}
\tag{8}
$$

where μ_i and μ'_i represent positive values that reflect the importance of $x_i \quad i\in I^{NI}$ and $x_i \quad i\in I^I$ respectively with $\sum_{i\in I^{NI}} + \sum_{i\in I^I} \mu'_i = 1$.

THEOREM 2.1 *Planned by the foregoing planning procedure, the new operational unit* $\mathrm{D\bar{M}U}$ *can become a MPSS when evaluated under the original production possibility Set.*

Proof since DMU$\in F$, the proof is clear. ∎

3. Numerical Example

Suppose we have data on 10 DMUs with two outputs and two inputs as given in Table 1 and suppose we are given an output production level $(\bar{y}_1, \bar{y}_2) = (3.8, 4.2)$.

Table 1. data set

DMU	X_1	X_2	Y_1	Y_2
1	4	6	2	1
2	12	8	3	1
3	8	2	2	2
4	6	6	4	2
5	2	8	2	3
6	3	9	1	2
7	3	7	2	4
8	8	12	3	1
9	4	10	1	3
10	6	5	2	2

We want to find an optimal level of inputs in such a way that $\mathrm{D\bar{M}U}$ become a MPSS. Solving LP model (3) numerically yields to:

$$u_1^* = 0.1342$$
$$u_2^* = 0.1168$$
$$v_1^* = 0.0408$$
$$v_2^* = 0.0875$$

This solution gives the following efficient surface of T_c:

$$F = \{(x_1, x_2, y_1, y_2) : 0.1342y_1 + 0.1168y_2 - 0.0408x_1 - 0.0875x_2 = 0\}$$

Now we want to find (x_1, x_2) in such a way that $\mathrm{D\bar{M}U}$ has been placed on this efficient surface. Therefore (x_1, x_2) should belong to the following set:

$$P = \{(x_1, x_2) : 0.0408x_1 + 0.0875x_2 = 1 \quad x_1, x_2 \geqslant 0\}$$

If we incorporate the priority of input as $\mu_1 = 0.6$ and $\mu_2 = 0.4$, the LP model (5), gives:

$$\bar{x}_1^* = 5.8123, \qquad \bar{x}_2^* = 8.7184$$

4. Conclusions

In this paper, a DEA-based approach has been developed for finding optimal input consumption for a new operational unit. To produce a fixed level of output a principal idea behind the procedure is to use the empirical production function defined by observed input and outputs to make the new operational unit as MPSS.

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