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ABS Method for Solving Fuzzy Sylvester Matrix Equation

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Abstract. The main aim of this paper intends to discuss the solution of fuzzy Sylvester matrix equation $AX + XB = C$ where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $B = (b_{ij}) \in \mathbb{R}^{m\times m}$ are crisp M-matrices, $C = (c_{ij}) \in \mathbb{R}^{n\times m}$ is fuzzy matrix and $X \in \mathbb{R}^{n\times m}$ is unknown,by applying a special algorithm based on a class of ABS algorithms called Huang algorithm. At first, we transform this system to an $(mn) \times (mn)$ fuzzy system of linear equations. Then, we convert this system to three $(mn) \times (mn)$ crisp linear systems of equations, and we solve them simultaneously by ABS algorithm.

Keywords: Fuzzy Sylvester matrix equation, Fuzzy number, ABS algorithm, Huang algorithm.

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1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [26] and Dubois and Prade [11]. Some information on fuzzy numbers and fuzzy arithmetic can be found in Kanfmann [20]. One of the important topics in fuzzy set theory is to solve fuzzy linear system of equations (FLSE). Fuzzy Linear systems of equations play a major role in several applications in various areas such as economics, physics, statistics, engineering, finance and social sciences.
Friedman et al. in [15] proposed a general model for solving fuzzy linear systems by using the embedding approach. Allahviranloo in [2] proposed Jacobiand Gauss-Seidel methods to solve FLSE. Also, he in [3] used the successive over-relaxation iterative method for solving FLSE. Abbasbandy et al. [1], considered the existence of a minimal solution of general dual fuzzy linear systems of the form where, are real matrix, and are fuzzy vectors, and the unknown vector is vector consisting

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of fuzzy numbers.The Sylvester matrix equation is important in control theory, signal processing, model reduction, decoupling techniques for ordinary and partial differential equations, implementation of implicit numerical methods for ordinary differential equations, and block-diagonalization of matrices [8, 9][10, 19, 23].
Standard solution methods for Sylvester equations of the form $AX + XB = C$ where, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $B = (b_{ij}) \in \mathbb{R}^{m\times m}$, $C = (c_{ij}) \in \mathbb{R}^{n\times m}$ and $X \in \mathbb{R}^{n\times m}$ is unknown are the Bartels-Stewart method [7] and the Hessenberg- Schurmethod method [12, 17]. Also, KhojastehSalkuyeh in [21] computed approximate solution of fuzzy Sylvester matrix equations by AOR method. In this work, we use ABS method which was introduced by Abaffy et al. [4, 5] in order to solve a fuzzy Sylvester matrix equation.
In section 2, we introduce some main definitions and theorems in fuzzy sets theory. In section 3, we recall the Huang algorithm as a class of ABS algorithms. In section 4, we propose an algorithm based on the Huang algorithm for finding the exact solution of the fuzzy Sylvester matrix.

2. Preliminaries

Definition 2.1 *[27]. A fuzzy number a is of LR-type if there exist shape functions L (for left), R (for right) and scalars* $\alpha > 0, \beta > 0$ *with*

$$\mu_0(x) = \begin{cases} L(\frac{m-x}{\alpha}),\ x \leqslant m \\ R(\frac{x-m}{\beta}),\ x \geqslant m \end{cases}$$

$a = (m, \alpha, \beta)_{LR}$ *is a triangular fuzzy number if* $L = R = max(0, 1 - x)$*. A popular fuzzy number is a triangular fuzzy number* $a = (m, \alpha, \beta)$ *where* m *, is a real number, and* α, β *are called the left and right spreads, respectively. If* $\alpha' = m - \alpha$ *and* $\beta' = m + \beta$ *then we can also use the notation* $a = (m, \alpha', \beta')$ *in order to show the fuzzy number* a*.*

Definition 2.2 *A fuzzy number a is called positive (negative), denoted by* $m > 0 (m < 0)$*, if its membership function* $\mu_a(x)$ *satisfies* $\mu_a(x) = 0, \quad \forall x \leqslant 0 (\forall x \geqslant 0)$.

Definition 2.3 *[27]. Let* $x = (m, \alpha, \beta)$, $y = (n, \gamma, \delta)$ *,and* $\lambda \in \mathbb{R}$ *. Then*

(a) $\lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta) & \lambda > 0 \\ (\lambda m, -\lambda\beta, -\lambda\alpha) & \lambda < 0 \end{cases}$

(b) $x + y = (m + n, \alpha + \gamma, \beta + \delta)$

(c) $x - y = (m - n, \alpha + \delta, \beta + \gamma)$

Definition 2.4 *[1]. We represent an arbitrary fuzzy number by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r)), 0 \leqslant r \leqslant 1$ *which satisfy the following requirements:*

1) $\underline{u}(r)$*is a bounded left continuous non-decreasing function over* $[0, 1]$*,*

2) $\overline{u}(r)$ *is a bounded left continuous non-increasing function over* $[0, 1]$*,*

3) $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$

A crisp number is simply represented by $\underline{u}(r) = \overline{u}(r) = a$, $0 \leqslant r \leqslant 1$.

DEFINITION 2.5 *[15] The $m \times n$ linear system*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m, \end{cases} \tag{1}$$

where the given matrix of coefficients $A = (a_{ij}), 1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ is a real $m \times n$ matrix, the given $y_1 \in E, 1 \leqslant i \leqslant m$, with the unknowns $x_j \in E, 1 \leqslant j \leqslant n$, is called a fuzzy linear system (FLSE).
For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r)), y = (\underline{y}(r) = \overline{y}(r))$, and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as

(a) $x = y$ *if and only* $\underline{x}(r) = \underline{y}(r)$ *and* $\overline{x}(r) = \overline{y}(r)$

(b) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$

(c) $x - y = (\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r)$

(d) $kx = \begin{cases} (k\underline{x}, k\overline{x}) \ , k \geqslant 0 \\ (k\overline{x}, k\underline{x}) \ , k < 0 \end{cases}$

DEFINITION 2.6 *[15]. A fuzzy number vector $(x_1, x_2, \cdots, x_n)^t$ given by $x_j = (\underline{x_j}(r), \overline{x_j}(r))$; $1 \leqslant j \leqslant n, 0 \leqslant r \leqslant 1$ is called a solution of the FLSE (1) if*

$$\underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y_i}, \quad 1 \leqslant i \leqslant m$$

$$\overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y_i}.$$

DEFINITION 2.7 *[22] The fuzzy Sylvester matrix equation is defined as:*

$$AX + XB = C, \tag{2}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $B = (b_{ij}) \in \mathbb{R}^{m\times m}$ are crisp M-matrices, $C = (c_{ij}) \in \mathbb{R}^{n\times m}$ is fuzzy matrix and $X \in \mathbb{R}^{n\times m}$ is unknown.

DEFINITION 2.8 *[22] Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Then, the Kronecker product of A and B is defined as the matrix $A \otimes B = (a_{ij}B) \in \mathbb{R}^{mp\times nq}$.*

THEOREM 2.9 *[22] Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times s}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$. Then $(A \otimes B)(x \otimes y) = Ax \otimes By$.*

DEFINITION 2.10 *[22] Let $Z_i \in \mathbb{R}^n, i = 1, \cdots, m$ denotes the i-th column of $Z \in \mathbb{R}^{n\times m}$ so that $Z = (z_1, z_2, \cdots, z_m)$. Then $vec(Z)$ is defined to be the (mn)-vector defined as as$vec(Z) = (z_1, z_2, \cdots, z_m)^T$.*

From definition 2.8, we can write fuzzy Sylvester matrix equation (2) in the form

$$P\chi = Q \tag{3}$$

$$\begin{cases} P = (I_m \otimes A) + (B^T \otimes I_n) \in \mathbb{R}^{(mn)\times(mn)}, \\ \chi = vec(X), \\ Q = vec(C), \end{cases} \tag{4}$$

where I_m and I_n are the identity matrices of orders m and n , respectively. There exists a unique solution to (3) if and only if P is nonsingular.

Definition 2.11 *[24] A matrix $A = (a_{ij})$ is said to be an M-matrix if $a_{ii} > 0, i = 1, \cdots, n,\quad a_{ij} \leqslant 0$, for $i \neq j$,A is nonsingular and $A^{-1} \geqslant 0$.*

Theorem 2.12 *[6] A matrix $A = (a_{ij})$ with $a_{ij} \leqslant 0, i \neq j$,is an M-matrix if and only if there exists a positive vector x , such that Ax is positive.*

Theorem 2.13 *[22] Let A and B be two M-matrices. Then, the matrix P is also an M-matrix. This theorem shows that Eq. (3) is alinear system of equations with a crisp coefficient M-matrix and fuzzy righthand side vector. Therefore, the existence of a fuzzy vector χ for (3) should be investigated.*

3. ABS algorithm for solving linear system of equations

ABS methods were introduced by Abaffy et al. [4, 5]. The ABS algorithm contains direct iterative methods for computing the general solution of linear systems, linear least squares, nonlinear equations, diophantine equations and optimization problems. ABS algorithm is used, for solving a m linear equations and n unknowns with $m \leqslant n$ in a finite iteration (maximum m iterations) [4, 5, 13, 25]. Also, Huang algorithm is a special kind of ABS method that obtained with choosingthe specific parameters on ABS algorithm [18]. ABS method for solving underdetermined linear systems produces a specific answer with a basis for the null space.The authors in [14] used ABS algorithm for solving general dual fuzzy linear systems. Also, Ghanbari and MahdaviAmiri in [16] solved fuzzy linear systems by ABS algorithm.

3.1 ABS algorithm(Huang algorithm)

Step 1. Let $x_1 \in \mathbb{R}^n$ be arbitrary, and $H_1 = I$, set $i = 1, r_1 = 0$.

Step 2. Compute $\tau_i = a_i^T x_i - b_i$ and $s_i = H_i a_i$.

Step 3. If $s_i = 0$ and $\tau_i = 0$ set $x_{i+1} = x_i, H_{i+1} = H_i$,$r_{i+1} = r_i$ and go to step 7 (the$i$-th equation isredundant). If $s_i = 0$ and $\tau_i \neq 0$ then stop (the i-th equation and hence the system is incompatible).

Step 4. Compute the search direction $p_i = H_i^T z_i$ with $z_i = a_i$ satisfying $z_i^T H_i a_i \neq 0$. Compute $\alpha_i = \frac{\tau_i}{a_i^T p_i}$ and set $x_{i+1} = x_i - \alpha_i p_i$.

Step 5. Update H_i to H_{i+1} by $H_{i+1} = H_i - H_i a_i W_i^T H_i$, where $w_i = \frac{a_i}{a_i^T H_i a_i}$ such that $w_i^T H_i a_i = 1$.

Step 6. Set $r_{i+1} = r_i + 1$.

Step 7. If $i = m$ then stop (x_{m+1} is a solution) else set $i = i + 1$ and go step 2.

We note that after the completion of the algorithm, the general solution of system, if compatible, is written as $x = x_{m+1} + H_{m+1}^T q$, where $q \in \mathbb{R}^n$ is arbitrary

[18].

4. The proposed algorithm and numerical examples

Now, we introduce the following algorithm in order to solve fuzzy Sylvester matrix equation(2). In the algorithm, d is a positive given number like $d = 0.5$ and $\chi_j = (\chi_{m_j}, \chi_{\alpha_j}, \chi_{\beta_j}), 1 \leqslant j \leqslant mn$ are the unknowns of the (3). Also, $q_j = (q_{m_j}, q_{\alpha_j}, q_{\beta_j}), 1 \leqslant j \leqslant mn$ are the element of q .

4.1 Algorithm

Step 1. Convert the given fuzzy Sylvester matrix equation $AX + XB = C$ to $P\chi = Q$ where $Q = (q_{ij}) \in \mathbb{R}^{n\times m}$ according to (3) and (4) .

Step 2. For $r = 0(d)1$ solvethe three crisp $(mn) \times (mn)$ linear systems $P\chi_m = (q_m)^T, P\chi_\alpha = (q_\alpha)^T, P\chi_\beta = (q_\beta)^T$ where $\chi_m = (\chi_{m_j}), \chi_\alpha = (\chi_{\alpha_j}), \chi_\beta = (\chi_{\beta_j}), q_m = (q_{m_j}), q_\alpha = (q_{\alpha_j}), q_\beta = (q_{\beta_j}), \quad 1 \leqslant j \leqslant mn$ by using ABS algorithm 3.1to obtain $\chi = (\chi_m, \chi_\alpha, \chi_\beta)$.

Step 3. Write $\chi_j^{ABC} = (\chi_{m_j}, \chi_{\alpha_j}, \chi_{\beta_j}), \quad 1 \leqslant j \leqslant mn$.

We computethe exact solution of following examples by algorithm 4.1.The results are demonstrated in tables 1 and 2. The results in the tables show that the ABS method is faster method (with less number of iterations).Also, by algorithm 4.1, we can find the exact solution of the system with finite number of iteration (at most $3mn$).
In these examples, $d = 0.5$ and we consider the notation $A = (m, \alpha', \beta')$, where $\alpha' = m - \alpha$ and $\beta' = m + \beta$ for fuzzy numbers. The results have been obtained by $MATLAB$.

Example 4.1 [21] Consider the fuzzy Sylvester equation (2) with

$$\mathbf{A} = \begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & -2 \\ -3 & 4 \end{pmatrix}$$

and fuzzy right hand side

$$\mathbf{C} = \begin{pmatrix} (-21 + 11r, 4 - 14r) & (19r, 31 - 12r) \\ (-1 + 8r, 15 - 8r) & (-16 + 9r, 3 - 10r) \end{pmatrix}.$$

The exact solution of this system is

$$\mathbf{X}^* = \begin{pmatrix} x_1 = (r, 2 - r) & x_3 = (1 + 2r, 4 - r) \\ x_2 = (1 + r, 3 - r) & x_4 = (-1 + r, 1 - r) \end{pmatrix}.$$

Results of example 4.1 are demonstrated in table 1

Table 1. Results of example 4.1.

	ABS method with maximum 12 iterations
r= 0	$X^{ABS} = \begin{pmatrix} x_1 = (1,0,2) & x_3 = (3,1,4) \\ x_2 = (2,1,3) & x_4 = (0,-1,1) \end{pmatrix}$
r= 0.5	$X^{ABS} = \begin{pmatrix} x_1 = (1,0.5,1.5) & x_3 = (3,2,3.5) \\ x_2 = (2,1.5,2.5) & x_4 = (0,-0.5,0.5) \end{pmatrix}$
r=1	$X^{ABS} = \begin{pmatrix} x_1 = (1,1,1) & x_3 = (3,3,3) \\ x_2 = (2,2,2) & x_4 = (0,0,0) \end{pmatrix}$

Example 4.2 [21] Consider the fuzzy Sylvester equation (2) with

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 5 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & -5 \\ -3 & 5 \end{pmatrix}$$

and fuzzy right hand side

$$\mathbf{C} = \begin{pmatrix} (-25+16r, 10-19r) & (-28+26r, 14-16r) \\ (-15+22r, 26-19r) & (-18+20r, 31-29r) \\ (-10+17r, 24-17r) & (-4+20r, 35-19r) \end{pmatrix}.$$

The exact solution of the this Sylvester equation is

$$\mathbf{X}^* = \begin{pmatrix} x_1 = (1+r, 3-r) & x_4 = (1+2r, 4-r) \\ x_2 = (1+2r, 5-2r) & x_5 = (2+r, 5-2r) \\ x_3 = (2+r, 4-r) & x_6 = (3+r, 5-r) \end{pmatrix}.$$

Results of example 4.2 are demonstrated in table 2

Table 2. Results of example 4.2.

	ABS method with maximum 18 iterations
r= 0	$X^{ABS} = \begin{pmatrix} x_1 = (2,1,3) & x_4 = (3,1,4) \\ x_2 = (3,1,5) & x_5 = (3,2,5) \\ x_3 = (3,2,4) & x_6 = (4,3,5) \end{pmatrix}$
r= 0.5	$X^{ABS} = \begin{pmatrix} x_1 = (2,1.5,2.5) & x_4 = (3,2,3.5) \\ x_2 = (3,2,4) & x_5 = (3,2.5,4) \\ x_3 = (3,2.5,3.5) & x_6 = (4,3.5,4.5) \end{pmatrix}$
r=1	$X^{ABS} = \begin{pmatrix} x_1 = (2,2,2) & x_4 = (3,3,3) \\ x_2 = (3,3,3) & x_5 = (3,3,3) \\ x_3 = (3,3,3) & x_6 = (4,4,4) \end{pmatrix}$

In these examples we observe $X^{ABC} = X^{exact}$, therefore, ABS method is a exact method with less iterations for solving fuzzy Sylvester equation (2) .

5. Conclusion

In this paper, we suggested a fuzzy solution for a fuzzy Sylvester equation by introducing a new algorithm based on the ABS algorithm. This new idea is able to compute the exact solution.

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