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Numerical Solution of Delay Integral Equations by Using Block Pulse Functions Arises in Biological Sciences

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Abstract. This article proposes a direct method for solving Volterra and Fredholm integral equations with time delay without using any projection method. By using operational matrix of integration, these equations can be reduced to a linear lower triangular system of algebraic equations which can be directly solved by forward substitution. Numerical examples show that the approximate solutions have a good degree of accuracy.

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Index to information contained in this paper

1. Introduction

Integral equations with time delay utilized in physical and biological modeling processes. Delays occurs in biological, chemical, transportation, electronic, communication, manufacturing and power systems. In [2, 10, 11], delay integral equations (DIEs) and delay integro-differential equations (DIDEs) are solved by different methods. In [6, 7], the approximate solutions of optimal control of time delay systems are derived by Block pulse functions. Nowadays, basis functions were used to

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derive solutions of integral equations and differential equations that can be seen in [1, 4, 5, 9].
In this paper we use Block pulse functions for numerical solving of two types of integral equations with constant time delay $\tau > 0$,

1) Volterra integral equations with time delay :

$$g(t) = f(t) + \int_0^t k(t,s)g(s-\tau)ds \quad t \in [0,T], \quad \tau \in (0,t)$$

This Volterra integral equation is the modelling of human population that the $g(t)$ is the number of population in time t and all children born during the time interval $0 < \tau < t$ who survive to time t. Also $f(t)$ is the survival function, which is the function of the number of people that survive to age t.

2) Fredholm integral equations with time delay :

$$g(t) = f(t) + \int_a^b k(t,s)g(s-\tau)ds \quad t \in [a,b], \quad \tau \in (a,b)$$

This integral equation is similar to Lotka integral equation for periodicity in the surge of birthrates. The study of population dynamics includes determination of the surge in the birthrate $g(t)$ at any time t to allow for future necwssary planning. The dependence of the birthrate $g(t)$ on previous birthrate $g(t-\tau)$, for woman in the childbearing age range $a < \tau < b$ is given by Lotka integral equation. $k(s,t)$ is the probility that a female lives to age τ and she will give birth to a female in the interval $\Delta\tau$. $f(t)$ is a term added to allow for girls already born before the oldest childbearing woman (of age $\tau = b$) was born.

This article is organized as follows. In section 2, we explain block pulse functions and integration operational matrix and functions containing time delay $f(t-\tau)$. Section 3 is devoted to solving Volterra integral equations with time delay. In Section 4, we solve Fredholm integral equations with time delay. Section 5 is devoted to error estimation and rate of convergence and in section 6, we achieve numerical examples to show the accuracy of the method and the culmination of paper in section 7 is the conclusion.

2. Preliminaries

The aim of this section is to interprate notations and definition of the block pulse functions that have expressed entirely in [8].

2.1 *Definition*

We define the m-set of BPFs as,

$$\phi_i^{(m)}(t) = \begin{cases} 1 & (i-1)h \leqslant t < ih, \\ 0 & otherwise. \end{cases} \tag{1}$$

with $t \in [0,T)$, $i = 1,2,...,m$ and $h = \frac{T}{m}$.

The primary properties of BPFs are disjointness and orthogonality that can be expressed as follows

$$\phi_i^{(m)}(t)\phi_j^{(m)}(t) = \delta_{ij}\phi_i^{(m)}(t), \tag{2}$$

$$\int_0^T \phi_i^{(m)}(t)\phi_j^{(m)}(t)dt = h\delta_{ij}, \qquad i,j = 1,2,...,m. \tag{3}$$

where $i,j = 1,2,...,m$ and δ_{ij} is Kronecker delta.

also If $m \to \infty$, then the BPFs set is complete; i.e. for every $f \in L^2([0,T))$, Parseval's identity holds,

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\phi_i^{(m)}(t)\|^2, \tag{4}$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\phi_i^{(m)}(t)dt. \tag{5}$$

By considering first m terms of BPFs, we can write them brevity as m-vector form

$$\Phi(t) = \left(\phi_1(t), \phi_2(t), \ldots, \phi_m(t)\right)^T, \quad t \in [0,T).$$

2.2 *Functions Approximation*

A real bounded function $f(t)$, which $f(t) \in L^2[0,T)$, can be expanded into a block pulse series as

$$f(t) \simeq \hat{f}_m(t) = \sum_{i=1}^{m} f_i\phi_i^{(m)}(t), \tag{6}$$

where f_i is the block pulse coefficient with respect to the ith BPF $\phi_i^{(m)}(t)$. In the vector form we have,

$$f(t) \simeq \hat{f}_m(t) = F^T\Phi(t) = \Phi^T(t)F, \tag{7}$$

where

$$F = \left(f_1, f_2, \ldots, f_m\right)^T.$$

Let $k(s,t) \in L^2\big([0,T_1) \times [0,T_2)\big)$. It can be expanded as

$$k(s,t) = \Psi^T(s)K\Phi(t) = \Phi^T(t)K^T\Psi(s), \tag{8}$$

where $\Psi(s)$ and $\Phi(t)$ are m_1 and m_2 dimensional BPFs vectors respectively, and $K = \big(k_{ij}\big), i = 1,2,...,m_1, j = 1,2,...,m_2$ is the $m_1 \times m_2$ block pulse coefficient matrix with

$$k_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} k(s,t) \Psi_i^{(m_1)}(s) \Phi_j^{(m_2)}(t) dt ds,$$

where $h_1 = \frac{T_1}{m_1}$, $h_2 = \frac{T_2}{m_2}$. For convenience, we put $m_1 = m_2 = m$.

2.3 *Integration operational matrix*

Computing $\int_0^t \phi_i^{(m)}(s)ds$ follows

$$\int_0^t \phi_i^{(m)}(s)ds = \begin{cases} 0 & 0 \leqslant t < (i-1)h, \\ t-(i-1)h & (i-1)h \leqslant t < ih, \\ h & ih \leqslant t < T. \end{cases} \tag{9}$$

From [3], We will have:

$$\int_0^t \Phi(s) ds \simeq P\Phi(t), \tag{10}$$

where operational matrix of integration is given by

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{m \times m}. \tag{11}$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s) ds \simeq \int_0^t F^T \Phi(s) ds \simeq F^T P \Phi(t). \tag{12}$$

2.4 *Functions Containing Time Delay $f(t-\tau)$*

In order to approximate a function containing time delay, we consider a block pulse function containing time delay $\tau = (q+\lambda)h$ with a nonnegative integer q and $0 \leqslant \lambda < 1$ that can be expressed as

$$\phi_i^{(m)}(t-\tau) = \begin{cases} \phi_{i+q}^{(m)}(t) + \phi_\lambda^{(m)}(t-(i+q)h) - \phi_\lambda^{(m)}(t-(i+q-1)h) & for \quad i < m-q \\ \phi_{i+q}^{(m)}(t) - \phi_\lambda^{(m)}(t-(i+q-1)h) & for \quad i = m-q \\ 0 & for \quad i > m-q. \end{cases} \tag{13}$$

or in a vector form :

$$\phi_i^{(m)}(t-\tau) = \Delta_i^T H^q \Phi(t) - \Delta_i^T H^q \Phi_\lambda(t) + \Delta_i^T H^{q+1} \Phi_\lambda(t), \tag{14}$$

To avoid the expression $\Phi_\lambda(t)$ in the above equation, we expand the function $\phi_i^{(m)}(t-\tau)$ into its block pulse series :

$$\phi_i^{(m)}(t-\tau) = \big(c_{i1}, c_{i2}, \ldots, c_{im}\big)\,\Phi(t),$$

where the block pulse coefficients c_{ij} $(i, j = 1, 2, \cdots, m)$ are :

$$\begin{aligned}
c_{ij} &= \frac{1}{h}\int_0^T \phi_i^{(m)}(t-\tau)\phi_j^{(m)}(t)dt,\\
&= \frac{1}{h}\int_{(j-1)h}^{jh} \phi_i^{(m)}(t-\tau)dt,\\
&= \frac{1}{h}\Delta_i^T H^q\Big(\int_{(j-1)h}^{jh}\Phi(t)dt - \int_{(j-1)h}^{jh}\Phi_\lambda(t)dt + H\int_{(j-1)h}^{jh}\Phi_\lambda(t)dt\Big),\\
&= \Delta_i^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Delta_j.
\end{aligned} \tag{15}$$

Noticing that the expression $\Delta_i^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Delta_j$ is just the single entry positioned in the ith row and jth column of the matrix $(1-\lambda)H^q+\lambda H^{q+1}$, we can expand the whole block pulse function vector containing time delay $\tau = (q+\lambda)h$ into its block pulse series in a vector form :

$$\Phi(t-\tau) = \big((1-\lambda)H^q + \lambda H^{q+1}\big)\Phi(t). \tag{16}$$

In the above equation, the matrix $(1-\lambda)H^q + \lambda H^{q+1}$ is usually called the block pulse operational matrix for time delay, or simply the delay operational matrix. Expressing concretely, it is :

$$(1-\lambda)H^q + \lambda H^{q+1} = \begin{array}{c} \begin{array}{c}(q+1)\text{th-column}\\ \downarrow\end{array} \\ \begin{pmatrix}
0 & \cdots & 0 & 1-\lambda & \lambda & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 1-\lambda & \lambda & \cdots & 0\\
\vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda\\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1-\lambda\\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
\vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{m\times m}\end{array}. \tag{17}$$

Therefore, the block pulse series of a function containing time delay $\tau = (q+\lambda)h$ can easily be obtained as :

$$f(t-\tau) \simeq F^T\Phi(t-\tau) = F^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Phi(t). \tag{18}$$

3. Solving Volterra Integral Equations with Time Delay

We consider following Volterra integral equation with constant time delay $\tau > 0$,

$$g(t) = f(t) + \int_0^t k(t,s)g(s-\tau)ds, \quad t \in [0,T], \quad \tau \in (0,T), \tag{19}$$

where the function $g \in L^2[0,T]$ is the unknown function, while the functions $f \in L^2[0,T]$ and $k(t,s) \in L^2\big([0,T]\times[0,T]\big)$ are the known functions.
We approximate $g(t)$, $f(t)$, $k(t,s)$ by relations (7), (8) as follows :

$$g(t) \simeq G^T\Phi(t) = \Phi^T(t)G,$$

$$f(t) \simeq F^T\Phi(t) = \Phi^T(t)F,$$

$$k(t,s) \simeq \Phi^T(t)K\Psi(s) = \Psi^T(s)K^T\Phi(t),$$

We approximate $g(s-\tau)$ by ralation (18) as follows,

$$g(s-\tau) \simeq G^T\Psi(s-\tau) \simeq G^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Psi(s),$$

and by letting $A = (1-\lambda)H^q + \lambda H^{q+1}$,we can write,

$$g(s-\tau) \simeq G^T A\Psi(s).$$

With substituting above approximation in equation (19), we have

$$\begin{aligned} G^T\Phi(t) &\simeq F^T\Phi(t) + \int_0^t G^T A\Psi(s)\Psi^T(s)K^T\Phi(t)ds, \\ &\simeq F^T\Phi(t) + G^T A\big(\int_0^t \Psi(s)\Psi^T(s)ds\big)K^T\Phi(t). \end{aligned} \tag{20}$$

Let K_i be the ith row of the constant matrix K^T, R_i be the ith row of the integration operational matrix P, and D_{K_i} be a diagonal matrix with K_i as its diagonal entries. By the previous relations and assuming $m_1 = m_2$, we will have,

$$\begin{aligned} \big(\int_0^t \Psi(s)\Psi^T(s)ds\big)K^T\Phi(t) &= \big(\int_0^t \Phi(s)\Phi^T(s)ds\big)K^T\Phi(t) \\ &= \begin{pmatrix} R_1\Phi(t) & 0 & \cdots & 0 \\ 0 & R_2\Phi(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_m\Phi(t) \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{pmatrix} \Phi(t) \\ &= \begin{pmatrix} R_1\Phi(t)K_1\Phi(t) \\ R_2\Phi(t)K_2\Phi(t) \\ \vdots \\ R_m\Phi(t)K_m\Phi(t) \end{pmatrix} = \begin{pmatrix} R_1\Phi(t)\Phi^T(t)K_1^T \\ R_2\Phi(t)\Phi^T(t)K_2^T \\ \vdots \\ R_m\Phi(t)\Phi^T(t)K_m^T \end{pmatrix} \\ &= \begin{pmatrix} R_1 D_{K_1} \\ R_2 D_{K_2} \\ \vdots \\ R_m D_{K_m} \end{pmatrix} \Phi(t) = B\Phi(t), \end{aligned} \tag{21}$$

where

$$B = \frac{h}{2}\begin{pmatrix} k_{11} & 2k_{21} & \cdots & 2k_{m1} \\ 0 & k_{22} & \cdots & 2k_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{mm} \end{pmatrix}_{m\times m}. \qquad (22)$$

With substituting relation (21) in (20), we have

$$G^T\Phi(t) \simeq F^T\Phi(t) + G^T AB\Phi(t),$$

Then,

$$G^T(I - AB) \simeq F^T, \qquad (23)$$

So, by setting $M = I - AB$ and replacing $\simeq$ by $=$, we will have,

$$M^T G = F. \qquad (24)$$

Which is a linear system of equations with lower triangular coefficients matrix that gives the approximate block pulse coefficient of the unknown function $g(t)$.

4. Solving Fredholm Integral Equations with Time Delay

We consider following Fredholm integral equation with constant time delay $\tau > 0$,

$$g(t) = f(t) + \int_a^b k(t,s)g(s-\tau)ds, \quad t \in [a,b], \quad \tau \in (0, b-a), \qquad (25)$$

Our problem is to determine block pulse coefficients of $g(t)$ in the interval $t \in [a,b]$ from the known functions $f(t)$ and $k(t,s)$. usually we set $a = 0$ to facilities the use of block pulse functions. In case $a \neq 0$ we set $s = \frac{t-a}{b-a}T$ where $T = mh$.
Approximating functions $g(t)$, $f(t)$ and $k(t,s)$ by BPFs by relations (7), (8), (18) gives,

$$g(t) \simeq G^T\Phi(t) = \Phi^T(t)G,$$

$$f(t) \simeq F^T\Phi(t) = \Phi^T(t)F,$$

$$k(t,s) \simeq \Phi^T(t)K\Psi(s) = \Psi^T(s)K^T\Phi(t),$$

$$g(s-\tau) \simeq G^T\Psi(s-\tau) \simeq G^T\big((1-\lambda)H^q + \lambda H^{q+1}\big)\Psi(s) = G^T A\Psi(s),$$

where vectors F, G and matrix K are BPFs coefficients of $f(t)$, $g(t)$ and $k(t,s)$,respectively.

With substituting above approximation in equation (25), we have

$$\begin{aligned} G^T\Phi(t) &\simeq F^T\Phi(t) + \int_0^{mh} G^T A\Psi(s)\Psi^T(s)K^T\Phi(t)ds \\ &\simeq F^T\Phi(t) + G^T A\big(\int_0^{mh} \Psi(s)\Psi^T(s)ds\big)K^T\Phi(t), \end{aligned} \tag{26}$$

by the relation $\int_0^{mh} \Psi(s)\Psi^T(s)ds = hI$, we have

$$G^T\Phi(t) \simeq F^T\Phi(t) + G^T AhIK^T\Phi(t),$$

or

$$G^T(I - hAK^T) \simeq F^T, \tag{27}$$

So, by assuming $M = I - hAK^T$ and replacing $\simeq$ by $=$, we can write,

$$M^T G = F. \tag{28}$$

Therefore we will have a liner system of equations that gives the approximate Block pulse coefficients of the unknown function $g(t)$.

5. Error Estimation and Rate of Convergence

In this section, we will show that the rate of convergence presented method for solving integral equations with time delay is $O(h)$ and because of it we can obtain good degree of accuracy.

THEOREM 5.1 *Suppose that $f(t)$ is an arbitrary real bounded function, which is square integrable in the interval $[0,1)$, and $e(t) = f(t) - \hat{f}_m(t)$, $t \in I = [0,1)$, which $\hat{f}_m(t) = \sum_{i=1}^m f_i\phi_i^{(m)}(t)$ is the block pulse series of $f(t)$. Then,*

$$\|e(t)\| \leqslant \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|. \tag{29}$$

Proof Let,

$$e_i(t) = \begin{cases} f(t) - f_i & t \in D_i, \\ 0 & t \in I - D_i. \end{cases} \tag{30}$$

where $D_i = \{t : (i-1)h \leqslant t < ih, h = \frac{1}{m}\}$ and $i = 1, 2, ..., m$.
We have,

$$e_i(t) = f(t) - \frac{1}{h}\int_{(i-1)h}^{ih} f(s)ds = \frac{1}{h}\int_{(i-1)h}^{ih} \Big(f(t) - f(s)\Big)ds,$$

now by mean value theorem, we get,

$$e_i(t) = \frac{f'(\eta_i)}{h} \int_{(i-1)h}^{ih} (t-s)ds = f'(\eta_i)\Big(t+(-i+\frac{1}{2})h\Big), \qquad t, \eta_i \in D_i, \quad i = 1, 2, ..., m.$$

then,

$$\begin{aligned} \|e_i(t)\|^2 &= \int_{(i-1)h}^{ih} |e_i(t)|^2 dt = \big(f'(\eta_i)\big)^2 \int_{(i-1)h}^{ih} \Big(t + (-i + \frac{1}{2})h\Big)^2 dt \\ &= \frac{h^3}{12}\big(f'(\eta_i)\big)^2, \qquad \eta_i \in D_i, \qquad i = 1, 2, ..., m. \end{aligned} \tag{31}$$

Consequently

$$\begin{aligned} \|e(t)\|^2 &= \int_0^1 |e(t)|^2 dt = \int_0^1 \Big(\sum_{i=1}^m e_i(t)\Big)^2 dt \\ &= \int_0^1 \Big[\sum_{i=1}^m e_i^2(t) + 2\sum_{i<j} e_i(t)e_j(t)\Big]dt = \sum_{i=1}^m \int_0^1 e_i^2(t)dt = \sum_{i=1}^m \|e_i(t)\|^2 \\ &= \frac{h^3}{12}\sum_{i=1}^m \big(f'(\eta_i)\big)^2 \leqslant \frac{h^2}{12}\sup_{t\in I} |f'(t)|^2, \end{aligned} \tag{32}$$

or,

$$\|e(t)\| \leqslant \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|.$$

hence, $\|e(t)\| = O(h)$. ∎

THEOREM 5.2 *Suppose that* $f(s,t) \in L^2\big([0,1) \times [0,1)\big)$ *and* $e(s,t) = f(s,t) - \hat{f}_m(s,t)$, $(s,t) \in D = [0,1) \times [0,1)$, *which* $\hat{f}_m(s,t) = \sum_{i=1}^m \sum_{j=1}^m f_{ij}\psi_i^{(m)}(s)\phi_j^{(m)}(t)$ *is the block pulse series of* $f(s,t)$. *Then,*

$$\|e(s,t)\| \leqslant \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D} |f'_s(x,y)|^2 + \sup_{(x,y)\in D} |f'_t(x,y)|^2\Big)^{\frac{1}{2}}. \tag{33}$$

Proof Let,

$$e_{ij}(s,t) = \begin{cases} f(s,t) - f_{ij} & (s,t) \in D_{ij}, \\ 0 & (s,t) \in D - D_{ij}. \end{cases} \tag{34}$$

where $D_{ij} = \{(s,t) : (i-1)h \leqslant s < ih,\ (j-1)h \leqslant t < jh,\ h = \frac{1}{m}\}$ and $i, j = 1, 2, ..., m$.
For $i, j = 1, 2, ..., m$, we have,

$$e_{ij}(s,t) = f(s,t) - \frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh} f(x,y)dydx = \frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh} \Big(f(s,t) - f(x,y)\Big)dydx,$$

now by mean value theorem, we get,

$$e_{ij}(s,t)=\frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}\Big((s-x)f'_s(\eta_i,\eta_j)+(t-y)f'_t(\eta_i,\eta_j)\Big)dydx$$
$$=f'_s(\eta_i,\eta_j)\Big(s+(-i+\frac{1}{2})h\Big)+f'_t(\eta_i,\eta_j)\Big(t+(-j+\frac{1}{2})h\Big),\qquad (s,t),(\eta_i,\eta_j)\in D_{ij}.$$

then,

$$\|e_{ij}(s,t)\|^2=\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}|e_{ij}(s,t)|^2dtds$$
$$=\frac{h^4}{12}\Big(f'^2_s(\eta_i,\eta_j)+f'^2_t(\eta_i,\eta_j)\Big),\quad (\eta_i,\eta_j)\in D_{ij},\quad i,j=1,2,...,m. \tag{35}$$

Consequently

$$\|e(s,t)\|^2=\int_0^1\int_0^1|e(s,t)|^2dtds=\int_0^1\int_0^1\Big(\sum_{i=1}^m\sum_{j=1}^m e_{ij}(s,t)\Big)^2dtds$$
$$=\sum_{i=1}^m\sum_{j=1}^m\int_0^1\int_0^1 e^2_{ij}(s,t)dtds=\sum_{i=1}^m\sum_{j=1}^m\|e_{ij}(s,t)\|^2$$
$$=\frac{h^4}{12}\sum_{i=1}^m\sum_{j=1}^m\Big(f'^2_s(\eta_i,\eta_j)+f'^2_t(\eta_i,\eta_j)\Big)\leqslant\frac{h^2}{12}\Big(\sup_{(x,y)\in D}|f'_s(x,y)|^2+\sup_{(x,y)\in D}|f'_t(x,y)|^2\Big), \tag{36}$$

or,

$$\|e(s,t)\|\leqslant\frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D}|f'_s(x,y)|^2+\sup_{(x,y)\in D}|f'_t(x,y)|^2\Big)^{\frac{1}{2}}.$$

hence, $\|e(s,t)\|=O(h)$. ∎

6. Numerical Examples

To illustrate the theoretical results stated in Sections 3, 4 we consider below examples. The computations associated with the examples were performed using Matlab 7. Let G_i denote the Block pulse coefficient of exact solution of the given examples, and let g_i be the Block pulse coefficient of computed solutions by the presented method. The error is defined as

$$\|E\|_\infty=max_{1\leqslant i\leqslant m}|G_i-g_i|$$

Example 6.1 Consider the following Volterra integral equation with (constant) time delay $\tau>0$,

$$g(t)=-\frac{t^4}{12}+\tau\frac{t^3}{3}+\big(1-\frac{\tau^2}{2}\big)t^2+\int_0^t(t-s)g(s-\tau)ds\quad s,t\in[0,T],\tau\in(0,T) \tag{37}$$

With the exact solution $g(t) = t^2$, for $0 \leqslant t \leqslant T$. The numerical results are shown in Table 1.

Table 1: Results for Example 6.1 with $m = 32$

T=0.1		T=0.5		T=1	
τ	$\|E\|_\infty$	τ	$\|E\|_\infty$	τ	$\|E\|_\infty$
0.001	$2.25253702323E-10$	0.005	$1.43236475159E-7$	0.01	$2.41781324660E-7$
0.004	$7.89018951737E-10$	0.020	$5.05392178898E-7$	0.04	$8.89548092253E-7$
0.007	$1.05137312606E-8$	0.035	$6.77549461664E-6$	0.07	$1.18904409627E-5$
0.010	$3.23552150853E-8$	0.050	$2.07573266779E-5$	0.10	$3.59412759183E-5$
0.013	$7.10063631885E-8$	0.065	$4.54085551807E-5$	0.13	$7.78747910139E-5$
0.016	$1.31142679048E-7$	0.080	$8.36313265580E-5$	0.16	$1.42228093536E-6$
0.019	$2.17278255504E-7$	0.095	$1.38209253211E-8$	0.19	$2.33265255085E-6$
0.022	$3.33766241622E-7$	0.110	$2.11814666396E-8$	0.22	$3.55022878247E-6$
0.025	$4.84799248938E-7$	0.125	$3.07014865478E-8$	0.25	$5.11348138626E-6$

Example 6.2 Consider the following Fredholm integral equation with (constant) time delay $\tau > 0$,

$$g(t) = t\big(Tcos(T-\tau)-sin(T-\tau)-sin(\tau)\big)+sin(t)+\int_0^T (ts)g(s-\tau)ds \quad s,t \in [0,T], \tau \in (0,T) \tag{38}$$

With the exact solution $g(t) = sin(t)$, for $0 \leqslant t \leqslant T$. The numerical results are shown in Table 2.

Table 2: Results for Example 6.2 with $m = 32$

T=0.1		T=0.5		T=1	
τ	$\|E\|_\infty$	τ	$\|E\|_\infty$	τ	$\|E\|_\infty$
0.001	$2.58568237865E-8$	0.005	$1.79098655889E-7$	0.01	$4.75634508642E-6$
0.004	$2.06631349057E-8$	0.020	$1.43564755172E-7$	0.04	$3.78276743845E-6$
0.007	$3.90494418458E-9$	0.035	$3.29688101492E-6$	0.07	$1.18920567226E-6$
0.010	$3.31892173962E-8$	0.050	$2.09017956946E-5$	0.10	$4.14221930711E-6$
0.013	$9.93901366404E-8$	0.065	$6.38269549703E-5$	0.13	$1.32377076222E-5$
0.016	$2.03467811041E-7$	0.080	$1.31020801974E-6$	0.16	$2.70420910648E-5$
0.019	$3.54191473459E-7$	0.095	$2.27981003931E-6$	0.19	$4.64284955478E-5$
0.022	$5.60329447516E-7$	0.110	$3.60161712474E-6$	0.22	$7.22073765667E-5$
0.025	$8.30649143506E-7$	0.125	$5.32974941630E-6$	0.25	$1.05134960953E-4$

7. Conclusion

Using Block pulse functions as basis functions to solve the Volterra and Fredholm integral equations with constant time delay is very simple and effective in comparison with other methods. Its applicability and accuracy is checked on some examples. In these examples the norm infinity of error is given only for 10 specific values of τ . The benefits of this method are low cost of setting up the equations without applying any projection method such as Galerkin, collocation, etc.

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