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Solving Nonlinear Two-Dimensional Volterra Integral Equations of the First-kind Using the Bivariate Shifted Legendre Functions

S. Nemati [a,*] and Y. Ordokhani[b]

[a] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[b] *Department of Mathematics, Alzahra University, Tehran, Iran.*

Abstract. In this paper, a method for finding an approximate solution of a class of two-dimensional nonlinear Volterra integral equations of the first-kind is proposed. This problem is transformed to a nonlinear two-dimensional Volterra integral equation of the second-kind. The properties of the bivariate shifted Legendre functions are presented. The operational matrices of integration together with the product operational matrix are utilized to reduce the solution of the second-kind equation to the solution of a system of linear algebraic equations. Finally, a system of nonlinear algebraic equations is obtained to give an approximate solution of the main problem. Also, numerical examples are included to demonstrate the validity and applicability of the method.

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1. Introduction

In this paper, we present a numerical method for the solution of nonlinear two-dimensional Volterra integral equation (2D-VIE) of the first-kind in the form

$$\int_0^t \int_0^x k(x,t,y,z)u^p(y,z)dydz = f(x,t), \quad (x,t) \in \Omega := [0,l] \times [0,T], \tag{1}$$

*Corresponding author. Email: s.nemati@umz.ac.ir

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where p is a positive integer number and f and K are smooth functions. Also, we require that

$$f(x,0)=0, \quad for\, all\, x\in[0,l], \tag{2}$$

$$f(0,t)=0, \quad for\, all\, t\in[0,T], \tag{3}$$

$$k(x,t,x,t)\neq 0, \quad for\, all\, (x,t)\in\Omega. \tag{4}$$

Integral equations of the first-kind are inherently ill-posed problems, meaning that the solution is generally unstable, and small changes to the problem can make very large changes to the answers obtained [8, 13]. This ill-posedness makes numerical solutions very difficult, a small error may leads to an unbounded error. To overcome the ill-posedness, we transform equation (1) with conditions (2)-(4) to a nonlinear 2D-VIE of the second-kind.

By differentiating (1) with respect to t and x, we get the following nonlinear 2D-VIE of the second-kind

$$\begin{aligned} u^p(x,t) = \textstyle\int_0^t\int_0^x k_1(x,t,y,z)u^p(y,z)dydz + \int_0^x k_2(x,t,y)u^p(y,t)dy \\ + \textstyle\int_0^t k_3(x,t,z)u^p(x,z)dz + F(x,t), \end{aligned} \tag{5}$$

where

$$k_1(x,t,y,z) = -\frac{\partial^2}{\partial x\partial t}k(x,t,y,z)/k(x,t,x,t),$$

$$k_2(x,t,y) = -\frac{\partial}{\partial x}k(x,t,y,t)/k(x,t,x,t),$$

$$k_3(x,t,z) = -\frac{\partial}{\partial t}k(x,t,x,z)/k(x,t,x,t),$$

$$F(x,t) = \frac{\partial^2 f}{\partial x\partial t}/k(x,t,x,t).$$

Since (2) and (3) hold, integrating (5) in x and t yields equation (1). Thus (1) and (5) are equivalent.

There are many works on developing and analyzing numerical methods for solving 2D-VIEs of the second-kind (see for example [2, 3, 6, 9–11, 17, 18]). But little work has been done to solve the first-kind cases. The numerical solution of equations of the type (1) have been considered in [4, 5, 15]. Maleknejad et. al [14] considered the numerical solution of equation (1) using 2D block-pulse functions.

In this work, we extend the method introduced in [16] to solve equation (1) with conditions (2)-(4). The bivariate shifted Legendre orthogonal functions are used to solve the considered problem. The main characteristic of this technique is that it reduces the main problem to those of solving two systems of algebraic equations thus greatly simplifies the problem. In [16] a similar method have been applied to solve a class of 2D nonlinear Volterra integral equations of the second-kind.

The paper is organized as follows: In Section 2, we discuss how to approximate functions in terms of the bivariate shifted Legendre orthogonal functions and also, the operational matrices of integration and the product operational matrix are introduced. In Section 3, we give an approximate solution for problem (1)–(4) using the bivariate shifted Legendre functions. Numerical examples are given in Section 4 to illustrate the accuracy of our method and conclusions are presented in Section 5.

2. Basic Concepts

2.1 *Definition and Function Approximation*

The bivariate shifted Legendre functions are defined on Ω as

$$\psi_{mn}(x,t) = L_m(\frac{2}{l}x-1)L_n(\frac{2}{T}t-1), \qquad m,n = 0,1,2,\ldots,$$

and are orthogonal with respect to the weight function $\omega(x,t) = 1$ such that

$$\int_0^T\int_0^l \omega(x,t)\psi_{mn}(x,t)\psi_{ij}(x,t)dxdt = \begin{cases} \frac{l}{(2m+1)}\frac{T}{(2n+1)}, & i = m \text{ and } j = n, \\ 0, & otherwise. \end{cases}$$

Here, L_m and L_n are the well-known Legendre polynomials of order m and n respectively, which are defined on the interval $[-1,1]$ and can be determined with the aid of the following recursive formula [1]

$$\begin{aligned} &L_0(x) = 1, \\ &L_1(x) = x, \\ &L_{m+1}(x) = \tfrac{2m+1}{m+1}xL_m(x) - \tfrac{m}{m+1}L_{m-1}(x), \;\; m = 1,2,3,\cdots. \end{aligned}$$

We note that $\{\psi_{mn}(x,t)\}_{m,n=0}^{\infty}$ are total orthogonal basis for the space $L^2(\Omega)$ [12]. The inner product in this space is defined by

$$\langle f(x,t), g(x,t)\rangle = \int_0^T\int_0^l f(x,t)g(x,t)dxdt, \tag{6}$$

and the norm is as follows:

$$\|f(x,t)\|_2 = \langle f(x,t), f(x,t)\rangle^{\frac{1}{2}} = (\int_0^T\int_0^l |f(x,t)|^2dxdt)^{\frac{1}{2}}.$$

For every $f(x,t) \in L^2(\Omega)$ we have

$$f(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\psi_{mn}(x,t), \tag{7}$$

where

$$c_{mn} = \frac{\langle f(x,t), \psi_{mn}(x,t)\rangle}{\|\psi_{mn}(x,t)\|_2^2}.$$

Let $\Pi_{M,N}(\Omega)$ be the space of all polynomials of degree less than or equal to M in variable x and degree less than or equal to N in variable t. Then the functions $\{\psi_{mn}(x,t)\}$, $m = 0, 1, \ldots, M$, $n = 0, 1, \ldots, N$, form an orthogonal basis for the space $\Pi_{M,N}(\Omega)$.
If the infinite series in equation (7) is truncated, then it can be written as

$$f(x,t) \simeq f_{M,N}(x,t) = \sum_{m=0}^{M} \sum_{n=0}^{N} c_{mn}\psi_{mn}(x,t) = C^T \psi(x,t),$$

where C and $\psi(x,t)$ are $(M+1)(N+1) \times 1$ vectors, respectively given by

$$C = [c_{00}, c_{01}, \ldots, c_{0N}, c_{10}, \ldots, c_{1N}, \ldots, c_{M0}, \ldots, c_{MN}]^T, \tag{8}$$

$$\psi(x,t) = [\psi_{00}(x,t), \psi_{01}(x,t), \ldots, \psi_{0N}(x,t), \psi_{10}(x,t), \ldots, \psi_{1N}(x,t), \ldots , \psi_{M0}(x,t), \ldots, \psi_{MN}(x,t)]^T. \tag{9}$$

The function $f_{M,N}(x,t)$ is the orthogonal projection of $f(x,t)$ onto the polynomial space $\Pi_{M,N}(\Omega)$ with respect to the inner product (6) and is the best approximation to $f(x,t)$ (see [12]).

Similarly, any functions k_1 in $L^2(\Omega \times \Omega)$, k_2 in $L^2(\Omega \times [0,l])$ and k_3 in $L^2(\Omega \times [0,T])$ can be expanded in terms of the bivariate shifted Legendre functions respectively as

$$k_1(x,t,y,z) \simeq \psi^T(x,t) K_1 \psi(y,z),$$

$$k_2(x,t,y) \simeq \psi^T(x,t) K_2 \psi(y,t),$$

$$k_3(x,t,z) \simeq \psi^T(x,t) K_3 \psi(x,z),$$

where K_1, K_2 and K_3 are block matrices of the form

$$K_q = [K_q^{(i,m)}]_{i,m=0}^{M}, \quad q = 1, 2, 3,$$

in which

$$K_q^{(i,m)} = [k_{ijmn}^q]_{j,n=0}^{N}, \quad i, m = 0, 1, \ldots, M, \quad q = 1, 2, 3,$$

and Legendre coefficients k_{ijmn}^q, $q = 1, 2, 3$ are given by

$$k_{ijmn}^1 = \frac{\langle\langle k_1(x,t,y,z), \psi_{mn}(y,z)\rangle, \psi_{ij}(x,t)\rangle}{\|\psi_{ij}(x,t)\|_2^2 \|\psi_{mn}(y,z)\|_2^2}, \quad i, m = 0, 1, \ldots, M, \ j, n = 0, 1, \ldots, N,$$

$$k_{ijmn}^2 = \frac{\langle\langle k_2(x,t,y), \psi_{mn}(y,t)\rangle, \psi_{ij}(x,t)\rangle}{\|\psi_{ij}(x,t)\|_2^2 \|\psi_{mn}(y,t)\|_2^2}, \quad i, m = 0, 1, \ldots, M, \ j, n = 0, 1, \ldots, N,$$

$$k_{ijmn}^3 = \frac{\langle\langle k_3(x,t,z), \psi_{mn}(x,z)\rangle, \psi_{ij}(x,t)\rangle}{\|\psi_{ij}(x,t)\|_2^2 \|\psi_{mn}(x,z)\|_2^2}, \quad i, m = 0, 1, \ldots, M, \ j, n = 0, 1, \ldots, N.$$

2.2 *Operational Matrices*

The integration of the vector $\psi(x,t)$ defined by (9) can be approximately obtained as:

$$\int_0^t \int_0^x \psi(x',t')dx'dt' \simeq Q_1\psi(x,t), \tag{10}$$

$$\int_0^x \psi(x',t)dx' \simeq Q_2\psi(x,t), \tag{11}$$

$$\int_0^t \psi(x,t')dt' \simeq Q_3\psi(x,t), \tag{12}$$

where $x \in [0,l]$, $t \in [0,T]$ and Q_1, Q_2 and Q_3 are the $(M+1)(N+1)\times(M+1)(N+1)$ operational matrices of integration which have been introduced in [16], respectively as

$$Q_1 = P_1 \otimes P_2,$$

$$Q_2 = \frac{l}{2}\begin{bmatrix} I & I & O & \cdots & O & O & O \\ \frac{-I}{3} & O & \frac{I}{3} & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & \frac{-I}{2M-1} & O & \frac{I}{2M-1} \\ O & O & O & \cdots & O & \frac{-I}{2M+1} & O \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} P_2 & O & O & \cdots & O \\ O & P_2 & O & \cdots & O \\ O & O & P_2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & P_2 \end{bmatrix},$$

such that, I and O are the identity and zero matrix of order $N+1$, respectively and P_1 and P_2 are the operational matrices of 1D shifted Legendre polynomials, respectively defined on $[0,l]$ and $[0,T]$ as follows [7]:

$$P_1 = \frac{l}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2M-1} & 0 & \frac{1}{2M-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M+1} & 0 \end{bmatrix},$$

$$P_2 = \frac{T}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{2N-1} & 0 & \frac{1}{2N-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2N+1} & 0 \end{bmatrix}.$$

The following property of the product of two vectors $\psi(x,t)$ and $\psi^T(x,t)$ will also be used:

$$\psi(x,t)\psi^T(x,t)C \simeq \tilde{C}\psi(x,t), \tag{13}$$

where C is defined by (8) and $\tilde{C}$ is an $(M+1)(N+1) \times (M+1)(N+1)$ matrix as follows [16]

$$\tilde{C} = [C^{(i,j)}]_{i,j=0,1,\dots,M}, \tag{14}$$

such that in equation (14), $C^{(i,j)}$, $i,j = 0,1,\dots,M$ are given by

$$C^{(i,j)} = \frac{2j+1}{l}\sum_{m=0}^{M}\omega_{i,j,m}A_m,$$

in which $\omega_{i,k,m}$ is defined as

$$\omega_{i,k,m} = \int_0^l L_i(\frac{2}{l}x-1)L_k(\frac{2}{l}x-1)L_m(\frac{2}{l}x-1)dx,$$

and A_m, $m = 0,1,\dots,M$ are $(N+1)\times(N+1)$ matrices as

$$[A_m]_{kh} = \frac{2h+1}{T}\sum_{n=0}^{N}\omega'_{k,h,n}f_{mn}, \quad k,h = 0,1,\dots,N,$$

where

$$\omega'_{j,h,n} = \int_0^T L_j(\frac{2}{T}t-1)L_h(\frac{2}{T}t-1)L_n(\frac{2}{T}t-1)dt.$$

Finally, for an $(M+1)(N+1) \times (M+1)(N+1)$ matrix $K = [K^{(i,j)}]$, $i,j = 0,1,\dots,M$, in which

$$K^{(i,j)} = [k_{imjn}]_{m,n=0}^{N}, \quad i,j = 0,1,\dots,M,$$

we have

$$\psi^T(x,t)K\psi(x,t) \simeq \hat{K}\psi(x,t), \tag{15}$$

where $\hat{K}$ is a $1 \times (M+1)(N+1)$ vector as

$$\hat{K} = [K_{oo},\dots,K_{0N},K_{10},\dots,K_{1N},\dots,K_{M0},\dots,K_{MN}],$$

in which K_{mn}, $m = 0, 1, \dots, M$, $n = 0, 1, \dots, N$ have been introduced in [16] as

$$K_{mn} = \frac{(2m+1)(2n+1)}{lT} \sum_{i=0}^{M} \sum_{j=0}^{N} \sum_{r=0}^{M} \sum_{s=0}^{N} \omega_{i,r,m} \omega'_{j,s,n} k_{ijrs}, \ m = 0, 1, \dots, M, \ n = 0, 1, \dots, N.$$

3. Method of Solution

In this section, we present a numerical method to find an approximate solution to problem (1)–(4), which corresponds with equation (5). We assume that the known functions in equation (1) satisfy the conditions that this equation has a unique solution. Using the way mentioned in the previous section, the functions $u^p(x,t)$, $F(x,t)$, $k_1(x,t,y,z)$, $k_2(x,t,y)$ and $k_3(x,t,z)$ can be approximated by the bivariate shifted Legendre functions as:

$$u^p(x,t) \simeq C^T \psi(x,t) = \psi^T(x,t) C, \tag{16}$$

$$F(x,t) \simeq F^T \psi(x,t), \tag{17}$$

$$k_1(x,t,y,z) \simeq \psi^T(x,t) K_1 \psi(y,z), \tag{18}$$

$$k_2(x,t,y) \simeq \psi^T(x,t) K_2 \psi(y,t), \tag{19}$$

$$k_3(x,t,z) \simeq \psi^T(x,t) K_3 \psi(x,z). \tag{20}$$

Substituting equations (16)-(20) into equation (5) yields:

$$C^T \psi(x,t) = \psi^T(x,t) K_1 \int_0^t \int_0^x \psi(y,z) \psi^T(y,z) C dy dz + \psi^T(x,t) K_2 \int_0^x \psi(y,t) \psi^T(y,t) C dy$$

$$+ \psi^T(x,t) K_3 \int_0^t \psi(x,z) \psi^T(x,z) C dz + F^T \psi(x,t).$$

Using equations (10)–(13), we obtain

$$C^T \psi(x,t) = \psi^T(x,t) K_1 \tilde{C} Q_1 \psi(x,t) + \psi^T(x,t) K_2 \tilde{C} Q_2 \psi(x,t) + \psi^T(x,t) K_3 \tilde{C} Q_3 \psi(x,t) + F^T \psi(x,t).$$

We assume that

$$\Lambda_1 = K_1 \tilde{C} Q_1,$$

$$\Lambda_2 = K_2 \tilde{C} Q_2,$$

$$\Lambda_3 = K_3 \tilde{C} Q_3.$$

Then, by applying (15) we have

$$C^T = \hat{\Lambda}_1 + \hat{\Lambda}_2 + \hat{\Lambda}_3 + F^T$$

which corresponds with a system of linear algebraic equations in terms of the unknown elements of the vector C and can be solved easily using direct methods.

The unknown function $u(x,t)$ can be approximated in terms of the bivariate shifted Legendre functions as

$$u(x,t) \simeq A^T \psi(x,t), \tag{21}$$

such that, the entries of the vector A are unknown. Using equation (13) and (21) it is easily obtained that

$$u^p(x,t) \simeq A^T \tilde{A}^{p-1} \psi(x,t). \tag{22}$$

Finally, using equations (16) and (22), we get

$$A^T \tilde{A}^{p-1} = C^T. \tag{23}$$

Equation (23) forms a system of $(M+1)(N+1)$ nonlinear equations which can be solved for the elements of A using numerical methods such as Newton's iterative method.

4. Numerical Examples

In this section, we give some computational results of numerical experiments using the method presented in Section 4. In order to demonstrate the error of the method, we introduce the notation:

$$e_{M,N}(x,t) = |u(x,t) - u_{M,N}(x,t)|, \qquad (x,t) \in \Omega,$$

where $u(x,t)$ is the exact solution and $u_{M,N}(x,t)$ is the computed result with M and N.

To solve the examples, we consider $M = N$ and the Newton's iterative method is used to solve the nonlinear system. The initial guess in Newton's method for these examples is considered to be $A^{(0)} = C$, but the number of iterations can be reduced by choosing a more closed $A^{(0)}$ to the exact solution.

Example 1. As the first example, consider the following nonlinear 2D-VIE

$$\int_0^t \int_0^x \frac{6}{1+y+z} u^3(y,z)dydz = xt(2t^2+3t(2+x)+2(3+3x+x^2)), \qquad (x,t) \in [0,1]\times[0,1].$$

The exact solution is $u(x,t) = x+t+1$. We apply the numerical method presented in this paper with $M = 1$ and obtain the linear system in terms of the unknown coefficients of the function $u^3(x,t)$ as:

$$\begin{cases} c_{00} - 9 = 0 \\ c_{01} - \frac{31}{5} = 0 \\ c_{10} - \frac{31}{5} = 0 \\ c_{11} - 3 = 0, \end{cases}$$

and get

$$c_{00} = 9, \;\; c_{01} = \frac{31}{5}, \;\; c_{10} = \frac{31}{5}, \;\; c_{11} = 3.$$

Substituting obtained values for c_{ij}, $i, j = 0, 1$ into the equation

$$A^T \tilde{A}^2 = C^T,$$

the following nonlinear system in terms of the unknown coefficients of the function $u(x,t)$ yields:

$$\begin{cases} {a_{00}}^3 + \frac{2}{3}a_{01}a_{10}a_{11} + a_{00}{a_{01}}^2 + a_{00}{a_{10}}^2 + \frac{1}{3}a_{00}{a_{11}}^2 - 9 = 0 \\ 3{a_{00}}^2a_{01} + \frac{1}{3}{a_{01}}^3 + 2a_{00}a_{10}a_{11} + a_{01}{a_{10}}^2 + \frac{1}{3}a_{01}{a_{11}}^2 - \frac{31}{5} = 0 \\ 3{a_{00}}^2a_{10} + {a_{01}}^2a_{10} + 2a_{00}a_{01}a_{11} + \frac{1}{3}{a_{10}}^3 + \frac{1}{3}a_{10}{a_{11}}^2 - \frac{31}{5} = 0 \\ 6a_{00}a_{01}a_{10} + 3{a_{00}}^2a_{11} + {a_{01}}^2a_{11} + {a_{10}}^2a_{11} + \frac{1}{9}{a_{11}}^3 - 3 = 0. \end{cases}$$

Solving this system of nonlinear equations using Newton's iterative method with initial guess $A^{(0)} = C$, eight iterations and precision 10^{-6}, we obtain

$$a_{00} = 1.998890, \;\; a_{01} = 0.503593, \;\; a_{10} = 0.503593, \;\; a_{11} = -0.003328,$$

therefore, we have

$$u_{1,1}(x,t) = 0.988376 + 1.013842(x+t) - 0.013312xt,$$

and

$$e_{1,1}(x,t) = |0.011624 - 0.013842(x+t) + 0.013312xt| \leqslant 0.052620.$$

Also, with $M = 2$ we get

$$a_{00} = 2, \;\; a_{01} = 0.5, \;\; a_{10} = 0.5, \;\; a_{02} = a_{11} = a_{12} = a_{20} = a_{21} = a_{22} = 0,$$

so

$$u_{2,2}(x,t) = x + t + 1,$$

which is the exact solution.

Example 2. Consider the following nonlinear 2D integral equation [14]

$$\int_0^t \int_0^x 2e^{x+t}u^3(y,z)dydz = \frac{1}{9}(e^{x+t} - e^{x+7t} - e^{4x+t} + e^{4x+7t}), \quad (x,t) \in [0,1] \times [0,1],$$

which has the exact solution $u(x,t) = e^{x+2t}$. Numerical results are given in Table 1 and Figure 1. Table 1 shows the error $e_{M,M}(x,t)$ with $M = 2, 4, 8$ at some selected grid points using the presented method together with the results obtained by the method of [14] using 2D block-pulse functions (2D-BPFs) with $m = 64$.

Table 1. Numerical results for Example 2.

$(x,t) = (2^{-i}, 2^{-i})$	Method of [14] with $m = 64$	Present Method with $M = 2$	Present Method with $M = 4$	Present Method with $M = 8$
$i = 1$	1.0×10^{-1}	1.9×10^{-1}	2.9×10^{-3}	2.6×10^{-6}
$i = 2$	4.6×10^{-2}	1.4×10^{-1}	6.2×10^{-4}	4.6×10^{-6}
$i = 3$	2.9×10^{-2}	6.6×10^{-2}	3.5×10^{-3}	6.3×10^{-7}
$i = 4$	2.3×10^{-2}	2.1×10^{-1}	2.1×10^{-3}	1.2×10^{-5}
$i = 5$	2.0×10^{-2}	2.8×10^{-1}	3.1×10^{-5}	3.8×10^{-6}
$i = 6$	3.1×10^{-2}	3.3×10^{-1}	1.4×10^{-3}	9.0×10^{-6}

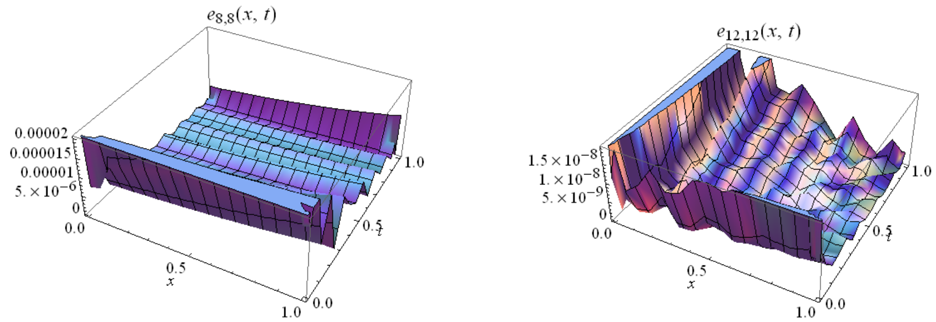

Figure 1. Plot of the $e_{M,M}(x,t)$ for Example 2; left: $M = 8$, right: $M = 12$.

Example 3. Consider a 2D integral equation of the form [14]

$$\int_0^t \int_0^x u^2(y,z)dydz = \frac{1}{45}xt(9x^4 + 10x^2t^2 + 9t^4), \qquad (x,t) \in [0,1] \times [0,1].$$

The exact solution is $u(x,t) = x^2 + t^2$. Figure 2 shows the numerical result for this example with $M = 1$. With $M = 2$, we find the exact solution of this equation.

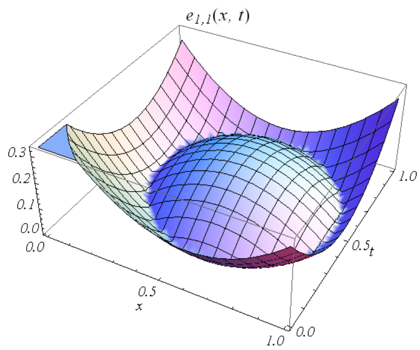

Figure 2. Plot of the $e_{1,1}(x,t)$ for Example 3.

Example 4. Consider the following linear 2D Volterra integral equation of the first-kind [14, 15]

$$\int_0^t \int_0^x (sin(t+y) + sin(x+z) + 3)u(y,z)dydz = f(x,t), \qquad (x,t) \in [0,3] \times [0,3],$$

where $f(x,t)$ is selected so that the exact solution is $u(x,t) = cos(x+t)$. Table 2 presents the numerical results with $M = 2, 4$ using the presented method together with the results obtained in [15] and [14], respectively using Euler's method and 2D-BPFs method.

Table 2. Numerical results for Example 4.

(x,t)	Euler's Method [15] with $h = 0.05$	2D-BPFs Method[14] with $m = 32$	Present Method with $M = 2$	Present Method with $M = 4$
$(1,1)$	4.06×10^{-2}	6.08×10^{-2}	3.70×10^{-2}	4.96×10^{-6}
$(1,2)$	1.23×10^{-2}	4.00×10^{-3}	6.70×10^{-2}	5.98×10^{-6}
$(2,1)$	1.23×10^{-2}	4.00×10^{-3}	6.70×10^{-2}	5.98×10^{-6}
$(2,2)$	4.06×10^{-2}	4.74×10^{-2}	4.86×10^{-2}	1.22×10^{-5}

5. Conclusions

In the present method, problem (1)–(4) has been transformed to a nonlinear 2D-VIE of the second-kind. The bivariate shifted Legendre functions operational matrices Q_1, Q_2 and Q_3, together with the product operational matrix $\tilde{C}$ and product vector $\hat{K}$ have been used to solve this problem. This approach transformed the nonlinear 2D-VIE of the second-kind to a linear system of algebraic equations with unknown coefficients which can be easily solved by direct methods. Finally, a system of nonlinear algebraic equations with unknown coefficients of the solution of the main problem has been obtained which can be solved using the Newton's iterative method. Applicability and accuracy of the method have been checked on some examples. Examples 2 and 4 show that the present method gives more accurate results than the methods presented in [14, 15] even when we use a small number of basis functions.

The method can be applied to the first-kind integral equations of the form

$$\int_0^t \int_0^x k(x,t,y,z)G(u(y,z))dydz = f(x,t),$$

where G is a polynomial function of the solution. Also, we believe that it will not difficult to extend this approach to nonlinear integral equations of different forms.

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