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Unboundedness in MOILP and its Efficient Solutions

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Abstract. In this paper we investigate Multi-Objective Integer Linear Programming (MOILP) problems with unbounded feasible region and introduce recession direction for MOILP problems. Then we present necessary and sufficient conditions to have unbounded feasible region and infinite optimal values for objective functions of MOILP problems. Finally we present some examples with unbounded feasible region and finite and infinite efficient solution.

Keywords: L_1-norm, Multi-Objective Integer Linear Programming, Recession Direction, Efficient Solutions.

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1. Introduction

Since most real-life problems include conflicting objectives, multiple objective optimization provides a means for obtaining more realistic models. Multi-Objective Integer Linear Programming (MOILP) problem is an important research area as many practical situations require discrete representations by integer variables and many decision makers have to deal with several objectives [6]. Some note-worthy practical environments where the MOILP problems find their applications are supply chain design, logistics planning, scheduling and financial planning. The MOILP problems are theoretically challenging as well, as most of them, even their single objective versions, fall into the class of computationally intractable problems.

Numerous algorithms have been designed to solve MOILP [3–6] and multiple objective mixed integer linear programs [2, 5]. Using a straightforward theoretical approach Sylva and Crema's [4] algorithm enumerates all efficient solutions

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of MOILP models with bounded feasible regions. Sylva and Crema's [4] approach, solves the problem using a sequence of progressively more constrained integer linear programs and generates a new solution at each step.

But, in some cases feasible region of a MOILP problem is unbounded. Therefore, a MOILP problem can have infinite objective values. These cases haven't been considered in [4] and [5]. This paper introduces recession direction to the MOILP problem and provides necessary and sufficient conditions to have unbounded feasible region, and infinite values of MOILP problem.

The paper is organized as follows. Section 2 presents a brief background about MOILP problem. Section 3 introduces a necessary and sufficient conditions to have unbounded feasible region and infinite optimal values. Illustration with some numerical examples are given in Section 4. Finally, the concluding results are presented.

2. Background

A MOILP problem is a special case of multi objective program and with s-objectives is defined as:

$$\begin{array}{ll} \max & \{C_1W, C_2W, \cdots, C_sW\} \\ \text{s.t.} & A_iW \leqslant b_i, \quad i = 1, 2, \cdots, m \\ & W \in Z_n^+ \end{array} \tag{1}$$

where, $C_r = (c_{1r}, c_{2r}, \cdots, c_{nr})$ $(r = 1, 2, \cdots, s)$, $A_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ $(i = 1, 2, \cdots, m)$, $Z_n^+ = \{(e_1, \cdots, e_n) | e_j \in Z^+ = \{0, 1, 2, \cdots\}, j = 1, \cdots, n\}$ and $W = (w_1, w_2, \cdots, w_n)^T$. The set X, which is defined as follows:

$$X = \Big\{W \ \mid \ A_iW \leqslant b_i, i = 1, 2, \cdots, m, W \in Z_n^+\Big\} \tag{2}$$

is called the set of feasible solutions of problem (1). Corresponding to each $W \in X$ the vector Y is defined as follows:

$$Y = (y_1, \cdots, y_s)^T = (C_1W, C_2W, \cdots, C_sW)^T. \tag{3}$$

Definition 2.1 *The vector $Y = (y_1, y_2, \cdots, y_s)^T$ dominates the vector $Y^o = (y_1^o, y_2^o, \cdots, y_s^o)^T$ if for each $r (r = 1, 2, \cdots, s)$, $y_r \geqslant y_r^o$ and there is at least one l such that $y_l > y_l^o$.*

Definition 2.2 *Let $F = \big\{Y \mid Y = (C_1W, C_2W, \cdots, C_sW)^T,\ A_iW \leqslant b_i,\ i = 1, \cdots, m,\ W \in Z_n^+\big\}$. F is called the values space of objective functions in problem (1).*

3. MOILP Problem with Unbounded Feasible Region and its Efficient Solutions

In some cases feasible region of a MOILP problem is unbounded. For instance consider the following MOILP problem

$$\begin{array}{ll} \max & w_1 + w_2 \\ \max & +2\, w_1 + w_2 \\ \text{s.t.} & -5\, w_1 + 4w_2 \leqslant 20 \\ & -6\, w_1 + 7w_2 \leqslant 42 \\ & w_1\,,\ \ w_2 \ \in Z^+. \end{array}$$

To explain the unbounded case we define recession direction for MOILP problems similar to recession direction for linear programming problem, [1].

DEFINITION 3.1 *Definition: Let $d \neq 0$ and $d \in Z_n^+$, then d is a recession direction of the MOILP problem if and only if for all $W \in X$, and for all $\lambda \in Z^+$ we have $W + \lambda d \in X$.*

THEOREM 3.2 *Let $d \neq 0$, then d is a recession direction of the problem (1) if and only if $A_i d \leqslant 0$, $i = 1, \cdots, m$, and $d \in Z_n^+$.*

Proof If W, $d \in Z_n^+$ and $\lambda \in Z^+$, then $W + \lambda d \in Z_n^+$. Let d be a recession direction. So, for each $\lambda \in Z^+$ we have $A_i(W + \lambda d) \leqslant b_i$. But, $\lim\limits_{\lambda \to +\infty} (b_i - A_i W)/\lambda = 0$, for $i = 1, \ldots, m$. So, for each $i (i = 1, \ldots, m)$, $A_i d \leqslant 0$. The converse of the theorem is evident. ■

THEOREM 3.3 *If there is $d \neq 0$ such that $A_i d \leqslant 0$, $i = 1, \cdots, m, C_r d \geqslant 0, r = 1, \cdots, s$ with at least one $p (p \in \{1, \cdots, s\})$ such that $C_r d > 0$ and $d \in Z_n^+$ then optimal values of the objective functions is infinite, i.e. the problem (1) has no efficient solution.*

Proof By contradiction let $\widehat{W}$ be an efficient solution of problem (1), so there is no $W \in X$ such that $C_r W \geqslant C_r \widehat{W}$, $r = 1, \cdots, s$ with at least one strictly inequality. According to $A_i d \leqslant 0$, $i = 1, \cdots, m$ and $d \in Z_n^+$, d is a recession direction of the model (1). Therefore, for each $\lambda \in Z^+$, we have $\overline{W} = \widehat{W} + \lambda d \in X$. But, using $C_r d \geqslant 0, r = 1, \cdots, s$, with at least one $p (p \in \{1, \cdots, s\})$ such that $C_r d > 0$ we have $\overline{W} = C_p(\widehat{W} + \lambda d) = C_p \widehat{W} + \lambda C_p d > C_p \widehat{W}$. This is a contradiction. That is, problem (1) has no efficient solution. ■

THEOREM 3.4 *Suppose that for each recession direction of the model (1), say $\widetilde{d}$, there is $p \in \{1, \ldots, s\}$ such that $C_p \widetilde{d} < 0$, then model (1) has efficient solution.*

Proof Because, for each $d \neq 0$ as a recession direction of the problem (1) there is $p \in \{1, \ldots, s\}$ such that $C_p d < 0$. Therefore, all objective functions values con not be infinite together. So, problem (1) has efficient solution. ■

4. Examples

The following examples illustrate the above definitions and theorems.

Example 4.1 Consider the following MOILP problem with two objective functions.

$$\begin{array}{ll} \max & w_1 + w_2 \\ \max & +4\, w_1 + 3w_2 \\ \text{s.t.} & -3\, w_1 + 2w_2 \leqslant 6 \\ & -6\, w_1 + 10w_2 \leqslant 60 \\ & w_1\,,\ \ w_2 \ \in Z^+. \end{array}$$

As can be seen there is $d = \binom{d_1}{d_2}$ such that $A_i d \leqslant 0,\ i = 1,2, C_r d > 0, r = 1,2$ and $d \in Z_2^+$, where $A_1 = (-3,2), A_2 = (-6,10), C_1 = (1,1)$ and $C_2 = (4,3)$. That is, feasible region is unbounded and objective functions can become infinite together. Therefore, there isn't any efficient solution for this problem.

Example 4.2 Consider the following MOILP problem

$$\begin{array}{ll} \max & -2\, w_1 + w_2 \\ \max & \quad w_1 - 3w_2 \\ \text{s.t.} & -4\, w_1 + w_2 \ \leqslant 4 \\ & -9\, w_1 + 5w_2 \leqslant 45 \\ & \quad w_1\ ,\ \ w_2 \ \in Z^+. \end{array} \tag{4}$$

It is evident that there is d, say $d = \binom{d_1}{d_2} = \binom{1}{1}$, such that $A_i d \leqslant 0,\ i = 1,2, d \in Z_2^+$ where $A_1 = (-4,1), A_2 = (-9,5)$. That is, feasible region of this problem is unbounded. But, there is no recession direction such that, $C_r d \geqslant 0, r = 1,2, \exists p \in \{1,2\}, C_p d > 0, d \in Z_2^+$, where $A_1 = (-1,1), A_2 = (-4,6), C_1 = (-2,1)$ and $C_2 = (1,-3)$. Therefore, this problem has efficient solution.

5. Conclusion

This paper considered MOILP problems with bounded and unbounded feasible regions and introduced recession direction. As another research in future, we use the provided necessary and sufficient conditions to find all efficient solutions of any kind of MOILP problems, i.e. MOILP problems with bounded and unbounded feasible region,and finite and infinite optimal values, too.

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