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Unsteady Convective Diffusion in a Herschel-Bulkley Fluid in a Conduit with Interphase Mass Transfer

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Abstract. The combined effect of non-Newtonian rheology and irreversible boundary reaction on dispersion in a Herschel-Bulkley fluid through a conduit (pipe/channel) is studied by using generalized dispersion model. The study explains the development of dispersive transport following the injection of a tracer in terms of three effective transport coefficients namely exchange, convective and dispersion coefficients. It is found that the exchange coefficient does not depend on yield stress and power law index but the convective and dispersion coefficients depend on yield stress and power law index.

Keywords: Generalized Dispersion model, Non Newtonian fluid, Power index.

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1. Introduction

The study of longitudinal dispersion of a tracer in a tubular flow when it is either irreversibly absorbed or undergoes an exchange process at the boundary has many applications in the fields of chemical engineering, environmental dynamics, biomedical engineering and physiological fluid dynamics.

The dispersion of a bolus of a tracer in a straight circular tube was first studied by Taylor [20]. Using the method of moments, Aris [2] extended Taylor theory by considering axial diffusion. Bailey and Gogarty [3] made an experimental study on the dispersion of a solute in a fluid flow through a tube. Ananthakrishnan et al. [1] obtained the numerical solution for the complete convective diffusion equation,

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which includes the radial and molecular diffusion. Gill [7] generalized Taylor's work by proposing a series expansion about the mean concentration to describe the local concentration distribution. Later Gill and Sankarasubramanian [8] studied the unsteady convective diffusion problem for laminar flow in a circular tube by the method of series solution. This model is referred as generalized dispersion model. By using Taylor's [20] model Gupta and Gupta [9] analyzed the phenomenon of dispersion of reactive contaminants in a liquid flowing through a channel in the presence of first order heterogeneous chemical reaction. Sankarasubramanian and Gill [16] extended this analysis with the interphase mass transfer. He showed that the three effective transport coefficients, namely the exchange, the convective and dispersion coefficients are influenced by the interphase transport. The effect of boundary absorption on the longitudinal dispersion in shear flow using delay diffusion equation was studied by smith [19]. Purnama [14] analyzed the case of wall reaction when the contaminant is chemically reactive. Mazumdar and Das [12] explained the concept of wall absorption on dispersion in an oscillatory flow through a pipe.

The study of dispersion through non - Newtonian fluids has abundant applications in polymer and biochemical processing and in cardio vascular system. Jayaraman et al. [10] studied the dispersion of a solute in a curved tube with absorbing walls using two phase models and showed that the influence of secondary flow on dispersion is reduced if the tracer is very soluble in the wall. In subsequent papers Balasubramanian et al. [4, 5] studied the boundary retention effects upon dispersion in secondary flows. In most of the investigations, blood is treated as a Newtonian fluid. But it is well known that blood being suspension of cells behaves like a non-Newtonian fluid at lower shear rates or during its flow through narrow blood vessels. Nagarani et al. [13] studied the dispersion of a solute in a Casson fluid flowing in a conduit by including the effect of wall absorption using the generalized dispersion model.

It is reported [18] that blood obeys Casson equation only in a limited range, except at very high and very low shear rate and that there is no difference between the Casson plots and the Herschel-Bulkley plots of experimental data over the range where the Casson plot is valid. It is observed that the Casson fluid model can be used for moderate shear rates $\gamma < 10/s$ in smaller diameter tubes whereas the Herschel- Bulkley fluid model can be used at still lower shear rate of flow in very narrow arteries where the yield stress is high [18, 21]. Further, the mathematical model of Herschel-Bulkley fluid also describes the behaviour of Newtonian fluid, Bingham fluid and power law fluid by taking appropriate values of the parameters viz. yield stress and power law index.

In this paper the combined effect of non-Newtonian rheology and irreversible boundary reaction on dispersion in a Herschel-Bulkley fluid through a conduit (pipe/channel) is studied by using the generalized dispersion model proposed by Sankarasubramanian and Gill [16]. In the absence of yield stress and for power law index is unity the results reduce to those of Sankarasubramanian and Gill [16]. The mathematical analysis of the problem in pipe flow and channel flow is presented in sections 2 and 3 respectively. The results showing the effect of yield stress, power law index of the fluid and the interphase mass transport are presented in section 4. The application of the results of this mathematical model to blood flows is discussed in section 5. Conclusions are given in section 6.

2. Pipe Flow Analysis

2.1 *Mathematical Formulation*

Let consider the dispersion of a bolus of solute that is initially distributed in a circular tube of radius 'a'. The flow in the tube is considered to be axi symmetric fully developed, steady laminar and the fluid is Herschel-Bulkley fluid. The non dimensional unsteady convective diffusion equation which describes the local concentration C of the solute as a function of axial coordinate z, radial coordinate r and time t can be written in the form

$$\frac{\partial C}{\partial t} + w\frac{\partial C}{\partial z} = \left(L^2 + \frac{1}{Pe^2}\frac{\partial^2}{\partial z^2}\right)C \tag{1}$$

with the non-dimensional variables

$$C = \frac{\bar{C}}{C_0},\ w = \frac{\bar{w}}{w_0},\ r = \frac{\bar{r}}{a},\ t = \frac{D_m \bar{t}}{a^2} \tag{2}$$

where $L^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})$, t is the non-dimensional time, C_0 is the reference concentration, w is the non dimensional axial velocity of the fluid in a axial direction z, $w_0 = -\frac{a^2}{4\mu}\frac{d\bar{p}}{d\bar{z}}$ is the characteristic velocity, μ is the Newtonian viscosity of the fluid, $\frac{d\bar{p}}{d\bar{z}}$ is the applied pressure gradient along the axis of the pipe, D_m is coefficient of molecular diffusion (molecular diffusivity) which is assumed to be constant and $Pe = \frac{aw_0}{D_m}$, Peclet number. The variables with bars represent the corresponding dimensional quantities.

Initial and Boundary Conditions

At an instant of time, the amount of tracer left in the system, its convective velocity, and the extent of shear distribution depend upon the initial discharge. Following Sankarasubramanian and Gill [16], we consider the initial distribution at $t = 0$ as the case when the solute of mass m is introduced instantaneously at the plane $z = 0$ uniformly over the cross section of a circle of radius d (where $0 < d \leqslant 1$) concentric with the tube. Hence, in terms of nondimensional quantities, the initial distribution assumed to be in a variable separable form is given by

$$C(O, z, r) = \Psi(z)Y(r) \tag{3a}$$

$$\text{with } \Psi(z) = \frac{\delta(z)}{d^2 Pe} \tag{3b}$$

$$\text{and } Y(r) = 1,\ 0 < r < d \tag{3c}$$

$$= 0,\ d < r < 1$$

where $\delta(z)$ is the Dirac delta function.

In the present model, we consider the reaction mechanism occurring at the wall of the tube, such that

$$\frac{\partial(C)}{\partial(r)}(t, z, 1) = -\beta C(t, z, 1) \tag{3d}$$

where β is the non-dimensional wall absorption parameter.

As the amount of solute in the system is finite

$$C(t,\infty,r) = \frac{\partial C}{\partial z}(t,\infty,r) = 0 \tag{3e}$$

$$\text{and } C(t,z,0) = \text{finite} \tag{3f}$$

The constitutive equation for a Herschel-Bulkley fluid relating the stress (τ) and rate of strain ($\frac{dw}{dr}$) in the non dimensional form is given by

$$\tau = \tau_y + (-\frac{dw}{dr})^{\frac{1}{n}} \quad \text{if} \quad \tau \geqslant \tau_y \tag{4a}$$

$$\frac{dw}{dr} = 0, \qquad \text{if} \quad \tau \leqslant \tau_y$$

$$\text{where } \tau = \frac{\bar{\tau}}{2\mu(w_0/a)} \quad \tau_y = \frac{\bar{\tau}_y}{2\mu(w_0/a)} \tag{5}$$

where 'n' is the power law index. $\bar{\tau}$ and $\bar{\tau}_y$ are the dimensional shear stress and yield stress respectively. τ_y is the dimensionless yield stress and is called yield stress parameter. The above relations correspond to vanishing of velocity gradient in the region where the shear stress τ is less than the yield stress τ_y which implies a plug flow whenever $\tau \leqslant \tau_y$. This model describes the Newtonian fluid if $\tau_y = 0$ and $n = 1$, a power law fluid model by taking $\tau_y = 0$ and $n \neq 1$ and a Bingham plastic fluid by taking $\tau_y \neq 0$ and $n = 1$.

The velocity distribution for axi-symmetric, fully developed, steady, laminar flow of a Herschel-Bulkley fluid in a pipe, in non- dimensional form is obtained as

$$w = w_+ = \frac{2}{n+1}\left\{(1-r_p)^{n+1} - (r-r_p)^{n+1}\right\} \text{ if } r_p \leqslant r \leqslant 1 \tag{6a}$$

$$w = w_- = w_p = \frac{2}{n+1}(1-r_p)^{n+1} \qquad \text{if } 0 \leqslant r \leqslant r_p \tag{6b}$$

$$\text{where } r_p = \frac{a\bar{\tau}_H}{2\mu w_0} = \tau_y \tag{7}$$

is the dimensionless radius of plug flow region. The subscripts '$+'$ and '$-'$ correspond to the values for the shear flow ($r_p \leqslant\leqslant 1$) and plug flow ($0 \leqslant r \leqslant r_p$) regions respectively. The fluid particles in the plug flow region, do not move by themselves, but are merely carried along by the fluid particles in the adjacent shear flow region as a solid body with a constant velocity w_p which is the plug flow velocity.

2.2 *Method of Solution*

In order to solve the convective diffusion equation (1) along with the associated sets of initial and boundary conditions $(3a, d, e, f)$, we introduce the derivative expansion method developed by Sankarasubramanian and Gill [16]. Following their solution procedure, we assume the concentration $C(r,t,z)$ as a series expansion in $\frac{\partial^j C_m}{\partial z^j}$ and express $C(r,t,z)$ as

$$C = \sum_{j=0}^{\infty} f_j(t,r)\frac{\partial^j C_m}{\partial z^j} \tag{8}$$

where the average concentration C_m is defined as

$$C_m = 2\int_0^1 Cr\ dr \tag{9}$$

Multiplying equation (1) by $2r$ and integrating with respect to r from 0 to 1, we get

$$\frac{\partial C_m}{\partial t} = \frac{1}{Pe^2}\frac{\partial^2 C_m}{\partial z^2} + 2\frac{\partial C}{\partial r}(t,z,1) - 2\frac{\partial}{\partial z}\int_0^1 w(t,r)C(t,z,r)r\ dr \tag{10}$$

Introducing (8) into (10), we get the dispersion model for C_m as

$$\frac{\partial C_m}{\partial t} = \sum_{j=0}^{\infty} K_j(t)\frac{\partial^j C_m}{\partial z^j} \tag{11}$$

where K_j's are given by

$$K_j(t) = \frac{\delta_{j,2}}{Pe^2} + 2\frac{\partial f_j}{\partial r}(t,1) - 2\int_0^1 f_{j-1}(t,r)w(t,r)r\ dr \quad j = 0,1,\ldots, f_{-1} = 0 \tag{12}$$

$\delta_{j,2}$ denotes the Kronecker delta.

It was shown by Gill and Sankarasubramanian [16] that equation (11) can be truncated after the term involving K_2. The resulting model for the mean concentration C_m, can be described by the generalized dispersion model as

$$\frac{\partial C_m}{\partial t} = K_0(t)C_m + K_1(t)\frac{\partial C_m}{\partial z} + K_2(t)\frac{\partial^2 C_m}{\partial z^2} \tag{13}$$

The term $K_0(t)$ corresponds to the absorption parameter. This term arises due to the nonzero solute flux at the flow boundary and will be zero if there is no wall absorption. This will be negative in this problem to account for the depletion of solute in the system with time caused by the irreversible reaction occurring at the tube wall. If the solute were to be generated at the wall according to first-order process, β in equation (3d) would assume negative sign and then the exchange coefficient $K_0(t)$ would be positive. $K_1(t)$ and $K_2(t)$ correspond to the convective and dispersion coefficients, respectively. The convection coefficient $K_1(t)$ accounts for the velocity of the reactive tracer and the dispersion coefficient $K_2(t)$ provides the modifications in the convective dispersion, occurring because of absorption.

Substituting (8) in equation (1), using (11) and equating the coefficients of $\frac{\partial' C_m}{\partial z^j}$, $j = 0,1,2,\ldots$, gives the partial differential equations for f_j as

$$\frac{\partial f_j}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f_j}{\partial r}) - w(t,r)f_{j-1} + \frac{1}{Pe^2}f_{j-2} - \sum_{i=0}^{j} K_i f_{j-i},\ j = 0,1,2,\ldots \tag{14}$$

where $f_{-1} = f_{-2} = 0$.

From Equation (3) we obtain the initial and boundary conditions on C_m and f_j
From (3a) and (9) we get

$$C_m(0,z) = 2\Psi(z)\int_0^1 Y(r)r\ dr \tag{15a}$$

$$\text{which gives } f_0(0,r) = \frac{Y(r)}{2\int_0^1 Y(r)r\ dr} \tag{15b}$$

$$f_j(0,r) = 0,\ \ j = 1,2,\ldots \tag{15c}$$

From Equation (5), (6a) and (6b), the boundary conditions are

$$\frac{\partial f_j}{\partial r}(t,l) = -\beta f_j(t,l), \qquad j = 0,1,2,\ldots \tag{15d}$$

$$f_j(t,0) = \text{finite} \qquad j = 0,1,2,\ldots \tag{15e}$$

$$C_m(t,\infty) = \frac{\partial C_m}{\partial z}(t,\infty) = 0 \tag{15f}$$

Substituting (9) in (8), we get

$$\int_0^1 f_j(t,r)r\ dr = \frac{1}{2}\delta_{j,0} \text{ for } j = 0,1,2,\ldots \tag{15g}$$

Since the Equations (12) and (14) are coupled, to determine the dispersion coefficient $K_2(t)$ we need to obtain the pair of functions (f_j, K_j), $j = 0,1,\ldots,$ one after the other. The function f_0 and exchange coefficient K_0 are independent of velocity field and can be solved directly. From (12) we have

$$K_0(t) = 2\frac{\partial f_0}{\partial t}(t,1) \tag{16}$$

Thus, $f_0(t,r)$ has to be evaluated first to determine $K_0(t)$. The equation for f_0 may be written from Equation (14) as

$$\frac{\partial f_0}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_0}{\partial r}\right) - f_0K_0 \tag{17}$$

Equations (15b, d, e) are the initial and boundary conditions on f_0. From (15g) we have

$$\int_0^1 f_0(t,r)r\ dr = \frac{1}{2} \tag{18}$$

Using (17), the solution of $f_0(t,r)$ satisfying the initial and boundary conditions (15 b, d, e) and (18) is given by

$$f_0(t,r) = \frac{\sum_0^\infty A_jJ_0(\mu_jr)e^{-\mu_j^2}}{2\sum_0^\infty(\frac{A_j}{\mu_j})J_1(\mu_j)e^{-\mu_j^2t}} \tag{19}$$

where

$$A_j = \frac{\mu_j^2 \int_0^1 rY(r) J_0(\mu_j r) dr}{(\mu_j^2 + \beta^2) J_0^2(\mu_j r) \int_0^1 rY(r)\ dr} \quad (j = 0, 1, 2, \dots) \tag{20a}$$

μ_j's are the eigne values satisfying the equation

$$\mu_j j_1(\mu_j) = \beta J_0(\mu_j) \quad (j = 0, 1, 2, \dots) \tag{20b}$$

where J_0, J_1 are Bessel functions of orders zero and one, respectively. From Equation (16) and (19) the exchange coefficient can be written as

$$K_0(t) = -\frac{\sum\limits_0^\infty A_j \mu_j J_1(\mu_j) \exp(-\mu_j^2 t}{\sum\limits_0^\infty \left(\frac{A_j}{\mu_j}\right) J_1(\mu_j) \exp(-\mu_j^2 t)} \tag{21}$$

which is exactly the same as derived by Sankarasubramanian and Gill [16].

Asymptotic Expansions for f_j's and K_j's for $j = 0, 1, 2, \dots$ for steady flow

To generate the dispersion coefficient, it is necessary to determine the remaining functions f_j introduced in Equation (14). In view of the tedious calculations of higher order dispersion coefficients and time dependent parts of the dispersion coefficients, we confine to the asymptotic steady-state representations of $f_j(t, r)$ and $K_j(t)$ for the case of steady flow, as these asymptotic values provide useful physical insight into the behavior of the system. Hence, we will obtain solutions $(f_j, K_j), j = 0, 1, 2, \dots$ for large times, so that the dispersion model defined in (13) is a representation of the asymptotic results under steady state conditions.

As $t \to \infty$, Equations (19) and (21) give the following asymptotic representation for f_0 and K_0:

$$f_0(\infty, r) = \frac{\mu_0}{2J_1(\mu_0)} J_0(\mu_0 r) \tag{22}$$

$$K_0(\infty) = -\mu_0^2 \tag{23}$$

where is the first root of Equation (20b) with least magnitude. From Equation (14) the function $f_j(r)$ for large values of time, satisfies the equation

$$\frac{1}{d}\frac{d}{dr}\left(r\frac{df_j}{dr}\right) + \mu_0^2 f_j = w(r) f_{j-1} - \frac{1}{(pe)^2} f_{j-2} + \sum_{i=1}^n k_i f_{j-i},\ j = 1, 2, 3, \dots \tag{24}$$

The boundary conditions on $f_j(r)$ are

$$f_j(0) = \text{finite} \qquad f_j'(I) = -\beta f_j(1)\ j = 1, 2, 3, \dots \tag{25}$$

$$\int\limits_0^1 f_j r dr = 0 \qquad (j = 1, 2, 3, \dots) \tag{26}$$

For large times the Equation (12) for K_j's reduces to

$$K_j = \frac{\delta_{j,2}}{(Pe)^2} + 2f_j'(1) - 2\int_0^1 w(r) f_{j-1}(r)dr \qquad (j = 1, 2, 3, \dots) \tag{27}$$

The use of the solvability condition in Equation (24) gives the expression for K_j as

$$K_j = \frac{\int_0^1 r J_0(\mu_0 r)\left\{\frac{1}{Pe^2} f_{j-2}(r) - w(r) f_{j-1}(r) - \sum_{i=1}^{j-1} K_i f_{j-i}(r)\right\} dr}{\int_0^1 r f_0(r) J_0(\mu_0 r) dr} \quad (n = 1, 2, 3, \dots) \tag{28}$$

For $j = 1$, the expression for the asymptotic convective coefficient K_1 can be obtained as

$$K_1 = \frac{-2}{n+1}\left\{(1-r_p)^{n+1} - \frac{2\mu_0^2}{(\mu_0^2+\beta^2)J_0^2(\mu_0)}\int_{r_0}^1 \left[(r-r_p)^{n+1} r J_0^2(\mu_0 r) dr\right]\right\} \tag{29}$$

For large times, the differential equation for f1 from Equation (24) can be written as

$$\frac{1}{r}\frac{d}{dr}(r\frac{df_1}{dr}) + \mu_0^2 f_1 = w(r) f_0 + K_1 f_0 \tag{30}$$

and the boundary conditions for f_1 are

$$f_1(0) = \text{finite}, \quad f_1'(1) = -\beta f_1(1) \tag{31}$$

$$\text{and} \quad \int_0^1 f_1 r dr = 0 \tag{32}$$

Using equations (29) and (30), we get the solution for f1 satisfying the corresponding boundary conditions (31) and (32) as

$$f_1(r) = \sum_{j=0}^{\infty} B_j J_0(\mu_j r) \tag{33}$$

With the help of the condition (32) and from equation (33) we get

$$B_0 = \frac{-\mu_0}{J_1(\mu_0)}\sum_{j=1}^{\infty} B_j \frac{J_1(\mu_j)}{\mu_j} \tag{34}$$

Using equations (33) and (34), the expression for can be written as

$$f_1 = \sum_{j=1}^{\infty} B_j \left[J_0(\mu_j r) - \frac{\mu_0}{J_1(\mu_0)} \frac{J_1(\mu_j)}{\mu_j} J_0(\mu_0 r) \right] \tag{35}$$

where

$$B_j = \frac{2\mu_j^2 \mu_0}{(\mu_0^2 - \mu_j^2)(\mu_j^2 + \beta^2)} \frac{1}{(j+1)J_0^2(\mu_j)J_1(\mu_)} \int_{r0}^{1} \left[r(r - r_p)^{n+1} J_0(\mu_n r) J_0(\mu_0 r) \right] dr \tag{36}$$

We obtain the dispersion coefficient by using (28), (35) and (36) as

$$K_2 = \frac{1}{Pe^2} + \frac{4\mu_0 J_1(\mu_0)}{(\mu_0^2 + \beta^2)J_0^2(\mu_0)} \sum_{j=1}^{\infty} B_j \left[\frac{2}{n+1}(P1 + P2 - P3) + P4 \right] \tag{37}$$

where

$$P_1 = \int_{r_0}^{1} \left[r(r - r_p)^{n+1} J_0(\mu_j r) J_0(\mu_j r) \right] dr \tag{38a}$$

$$P_2 = \frac{\mu_0}{J_1(\mu_0)} \frac{J_1(\mu_j)}{\mu_j} \frac{(\mu_0^2 + \beta^2)J_0^2(\mu_0)}{2\mu_0^2} (1 - r_p)^{n+1} \tag{38b}$$

$$P_3 = \frac{\mu_0}{J_1(\mu_0)} \frac{J_1(\mu_j)}{\mu_j} \int_{r_0}^{1} \left[r(r - r_p)^{n+1} J_0^2(\mu_0 r) \right] dr \tag{38c}$$

$$P_4 = K_1 \left(\frac{\mu_0}{J_1(\mu_0)} \frac{J_1(\mu_j)}{\mu_n} \frac{(\mu_0^2 + \beta^2)J_0^2(\mu_0)}{2\mu_0^2} \right) \tag{38d}$$

The integrals involving in equation (29), (36), (38 a, c) can be evaluated numerically.

Solution for mean concentration

The mean concentration Cm is obtained from equation (10) with initial and boundary conditions given by (15a) and (15f) and is given by

$$C_m(t, z) = \frac{1}{2(Pe)\sqrt{\pi\zeta}} \exp(\zeta - \frac{z_1^2}{4\zeta}) \tag{39}$$

where

$$\zeta(t) = \int_0^t K_0(\eta)d\eta \tag{40a}$$

$$z_1(t,z) = z + \int_0^t K_1(\eta)d\eta \tag{40b}$$

$$\zeta(t) = \int_0^t K_2(\eta)d\eta \tag{40c}$$

3. Channel Flow Analysis

3.1 *Mathematical Formulation*

We use the Cartesian co-ordinate system (x, y) to describe the dispersion of a solute in a Herschel-Bulkley fluid flowing in a channel. Assuming that the flow is steady, fully developed, laminar and axi-symmetric, the unsteady convective diffusion equation for the dispersion of the solute in the channel flow, in non-dimensional form is

$$\frac{\partial C}{\partial t} + w\frac{\partial c}{\partial}z = \left(\delta^2 + \frac{1}{Pe^2}\frac{\partial^2}{\partial z^2}\right)C \tag{41}$$

where $\delta^2 = \frac{\partial^2}{\partial x^2}$ and

$$Pe = \frac{aw_0}{D_m} \quad \text{Peclet number} \tag{42}$$

x is the transverse coordinate, z is the axial co-ordinate and, 'a' is half of channel width, and w_0 is the characteristic velocity given by

$$w_0 = -\frac{a^2}{2\mu}\frac{d\bar{p}}{d\bar{z}} \tag{43}$$

The non-dimensional initial and boundary conditions are

$$C(0,z,x) = \psi(z)X(x) \tag{44a}$$

$$\frac{\partial C}{\partial r}(t,z,1) = -\beta C(t,z,1) \tag{44b}$$

$$C(t,\infty,x) = \frac{\partial C}{\partial z}(t,\infty,x) = 0 \tag{44c}$$

$$\frac{\partial C}{\partial x}(t,z,0) = 0 \tag{44d}$$

The expression for the velocity distribution, in non dimensional form, for axi-symmetric, fully developed, steady, laminar flow of a Herschel-Bulkley fluid in a channel, is obtained as

$$w(x) = w_+ = \frac{2}{n+1}\left\{(1-x_p)^{n+1} - (x-x_p)^{n+1}\right\} \qquad \text{if} \quad x_p \leqslant x \leqslant 1 \qquad (45a)$$

$$w(x) = w_- = w_p = \frac{2}{n+1}(1-x_p)^{n+1} \qquad \text{if} \quad 0 \leqslant x \leqslant x_p \qquad (45b)$$

where

$$x_p = \frac{a\bar{\tau}_y}{2\mu w_o} = \tau_y \qquad (46)$$

denotes half of dimensionless thickness of the plug flow region in a channel which is equal to the dimensionless yield stress τ_y of the fluid in the channel.

3.2 *Method of Solution*

To analyze the dispersion of a solute in a Herschel-Bulkley fluid flowing in a channel, the unsteady convective diffusion equation (41) has to be solved for the local concentration C subject to conditions (44) with axial velocity w given by equation (45). In the present analysis the mean concentration is defined as

$$C_m^{=}\int_0^1 C\,dx \qquad (47)$$

Following the procedure adopted in the pipe flow analysis to find the solution for the problem, the expressions for K_j's and f_j's are given by

$$\frac{\partial f_j}{\partial t} = \frac{\partial^2 f_j}{\partial x^2} - w(r)f_{j-1} + \frac{1}{Pe^2}f_{j-2} - \sum_{i=0}^{n} K_i f_{j-i} \qquad (48)$$

where $j = 0, 1, 2, \ldots$

$$K_j = \frac{\int_0^1 \cos(\mu_0 x)\left\{\frac{1}{Pe^2}f_{j-2}(x) - w(x)f_{j-1}(x) - \sum_{i=1}^{j-1} K_i f_{j-i}(r)\right\}dx}{\int_0^1 f_0(x)\cos(\mu_0 x)dx}, \quad j = 1, 2, \ldots\ldots \qquad (49)$$

and

$$K_0 = \frac{\partial f_0(t,1)}{\partial x} \qquad (50)$$

The initial and boundary conditions are

$$f_0(0,x) = \frac{X(x)}{2\int\limits_0^1 X(x)\,dx} \tag{51a}$$

$$f_j(0,x) = 0, \quad j = 0,1,2\ldots \tag{51b}$$

$$\frac{\partial f_j}{\partial x}(t,1) = -\beta f_j(t,1), \quad j = 0,1,2,\ldots \tag{51c}$$

$$\frac{\partial f_j}{\partial x}(t,0) = 0, \quad j = 0,1,2,\ldots \tag{51d}$$

$$C_m(t,\infty) = \frac{\partial C_m}{\partial z}(t,\infty) = 0 \tag{51e}$$

$$\int\limits_0^1 f_j(t,x)\,dx = \delta_{j0} \tag{51f}$$

The change in the definition of C_m and the operator L^2 will necessitate some minor modifications in the solvability condition and the expression for K_j, $j = 0,1,2\ldots.$
With out going into details again, the solutions for K_0, K_1, K_2 and f_0, f_1 are written as given below:

$$f_0(x,t) = \frac{\sum_0^\infty A_j\cos(\mu_j x)e^{-\mu_j^2 t}}{\sum_0^\infty (A_j/\mu_j)\sin(\mu_j)e^{-\mu_j^2 t}} \tag{52}$$

$$K_0(t) = \frac{-\sum_0^\infty A_j\mu_j\sin(\mu_j)e^{-\mu_j^2 t}}{\sum_0^\infty (A_j/\mu_j)\sin(\mu_j)e^{-\mu_j^2 t}} \tag{53}$$

$$\text{where} \quad A_j = \frac{2\mu_j^2}{\mu_j^2+\beta\cos^2(\mu_j)}\frac{\int_0^1 X(x)\cos(\mu_j x)dx}{\int_0^1 X(x)dx} \tag{54}$$

and μ_j's are the roots of the equation

$$\mu_j\sin(\mu_j) = \beta\cos(\mu_j) \tag{55}$$

Same as in the case of pipe flow analysis, the absorption, convection and dispersion coefficients are obtained for large times. Hence, the asymptotic expansions for f_j's and K_j's for $j = 0,1,2$ in channel analysis are

$$f_0(\infty,x) = \frac{\mu_0\cos(\mu_0 x)}{\sin\mu_0} \tag{56}$$

$$K_0(\infty) = -\mu_0^2(\mu_1 > \mu_0) \tag{57}$$

$$K_1 = \frac{-2}{n+1}\left\{(1-x_p)^{n+1} - \frac{2\mu_0^2}{\mu_0^2+\beta\cos^2\mu_0}\int_{x_p}^1 (x-x_p)^{n+1}\cos^2(\mu_0 x)dx\right\} \tag{58}$$

$$f_1 = \sum_1^\infty B_j[\cos(\mu_j x) - \frac{\mu_0}{\sin\mu_0}\frac{\sin\mu_j}{\mu_j}\cos(\mu_0 x)] \tag{59}$$

where B_j's are given by

$$B_j = \frac{-2\mu_j^2}{(\mu_0^2 - \mu_j^2)(\mu_j^2 + \beta\cos^2\mu_j)} \frac{\mu_0}{\sin\mu_0} \frac{2}{n+1} \int_{x_p}^{1} (x - x_p)^{n+1} \cos(\mu_0 x)\cos(\mu_j x)dx \tag{60}$$

$$K_2 = \frac{1}{pe^2} + \frac{2\beta\cos\mu_0}{(\mu_j^2 + \beta\cos^2\mu_j)} \sum_1^\infty B_j \left[\frac{2}{n+1}(C_1 - C_2 + C_3) + C_4\right] \tag{61}$$

where

$$C_1 = \int_{x_p}^{1} (x - x_p)^{n+1} \cos(\mu_0 x)dx\cos(\mu_j x)dx \tag{62a}$$

$$C_2 = \frac{\mu_0}{\sin\mu_0} \frac{\sin\mu_j}{\mu_j} \int_{x_p}^{1} (x - x_p)^{n+1} \cos^2(\mu_0 x)dx \tag{62b}$$

$$C_3 = \frac{\mu_0 \sin\mu_j}{\sin\mu_0\mu_j} \frac{\mu_0^2 + \beta\cos^2\mu_0}{2\mu_0^2} (1 - x_p)^{n+1} \tag{62c}$$

$$C_4 = K_1 \left(\frac{\mu_0 \sin\mu_j}{\sin\mu_0\mu_j} \frac{\mu_0^2 + \beta\cos^2\mu_0}{2\mu_0^2}\right) \tag{62d}$$

The integrals involving in equation (58), (60), (62a,b) can be evaluated numerically.

4. Results and Discussions

In the present investigation, the development of the dispersive transport following the injection of a chemically active tracer in a solvent flowing through a conduit (pipe/channel) with a reactive boundary wall has been analyzed. The study uses the generalized dispersion model proposed by Sankarasubramanian and Gill [16]. The effect of moderate boundary absorption through Herschel-Bulkley fluid on the three effective transport coefficients, viz. the exchange (absorption), the convective and dispersion coefficients is studied. Integrals involved in obtaining the transport coefficients are evaluated numerically using Simpson's rule.

The value of wall absorption parameter β is taken to range from 0.01 to 100 to account for small to large absorption, r_p (plug flow radius) and x_p (half thickness of the plug flow region) varying from 0.02 to 0.2 in pipe and channel for the power law index $n = 1, 2$. The results correspond to the case of (i) Newtonian fluid for $n = 1$ and $r_p(x_p) = 0$, (ii) Power law fluid for $n = 2$ and $r_p(x_p) = 0$, (iii) Bingham plastics for $n = 1$ and $r_p(x_p) = 0.1$ are compared.

Asymptotic Absorption Coefficient K_0

Due to irreversible reaction occurring at the boundary wall, the total amount of solute is no longer a conserved quantity. The coefficient $K_0(t)$ accounting for this non-zero solute flux at the tube wall is negative indicating the depletion of solute in the system. For large times the absorption coefficient is $K_0 \cong -\mu_0^2$. Figure 1 presents the variation of negative asymptotic coefficient $-K_0$ with β for large times in a pipe and channel. The magnitude of the absorption coefficient K_0steadily

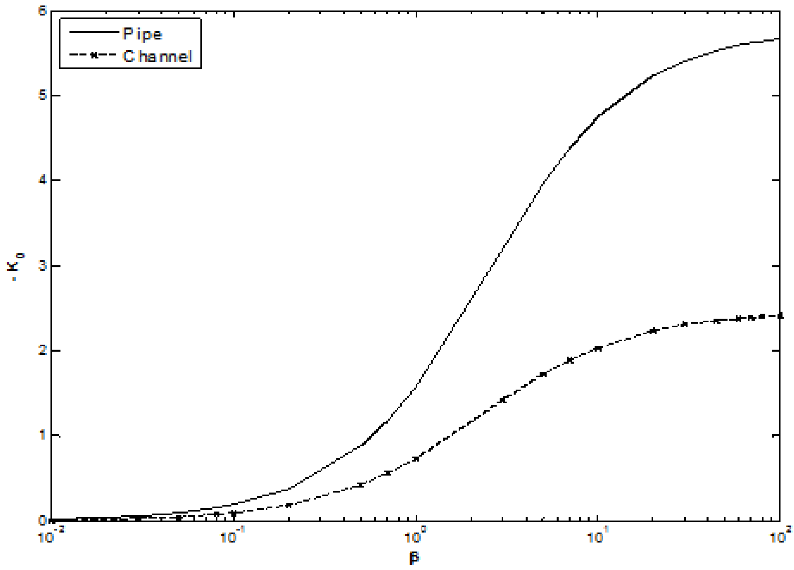

Figure 1. Variation of negative asymptotic absorption coefficient-K_0 with absorption parameter β in pipe and channel.

increases with the absorption parameter β and attains a value 5.7(2.4) as β assumes very large values (say 100) in pipe (channel). As β increases, the reaction at the wall of the conduit consumes the material more rapidly than it can be supplied by molecular diffusion. Thus, the process of mass transport in the system becomes diffusion controlled. The behaviour of $-K_0$ in a channel is similar to that of pipe and the values are less than half of the corresponding values in the pipe flow. Thus, there is more absorption of solutes at the wall in pipe when compared to channel. It is evident from equation (21) the asymptotic absorption coefficient K_0 is independent of the yield stress and nature of fluid.

Asymptotic convective coefficient K_1

Figure 2(a,b) shows the variation of negative asymptotic convective coefficient $-K_1$ versus wall absorption parameter β for various central core radii $r_p(x_p)$. Figure 3(a) and 3(b) presents the comparative behaviour of $-K_1$ for Newtonian, Power law, Bingham plastics and Herschel-Bulkley fluid in pipe and channel. For different vessels in the cardiovascular system the possible values of the plug core radii are in the range 0.0112-0.0737 [17]. When the plug radius $r_p(x_p)$ is one tenth of the tube (channel) radius and $\beta = 100$ in the tube (channel) $-K_1$ is observed to be increased by 1.4(1.21) times of the values corresponding to $\beta = 0.01$. For a Newtonian fluid the corresponding enhancement in pipe (channel) is observed to be 1.56(1.29). This enhancement is due to the depletion of solute as a consequence of wall reaction in the slower moving shear region and hence the solute distribution is weighted in favor of the faster moving central region. Thus the solute is convected along at a velocity higher than the average velocity of flow. The more rapid the wall reaction, the greater this effect will be. With increase in $r_p(x_p)$ it is found that $-K_1$ reduces significantly due to the reduction in velocities. These values are found to be lesser in the case of Casson fluid [13]. The values of $-K_1$ are $(1.41 - 1.52)$ times higher than those corresponding to the case of Bingham plastic, (1.30-1.33) times higher than in the case of Power law fluid when compared to Herschel-Bulkley fluid case when β ranges from 0.01 to 100. In all cases the values of $-K_1$ in channel are higher than those of the pipe.

Asymptotic Dispersion coefficient K_2 The asymptotic dispersion coefficient

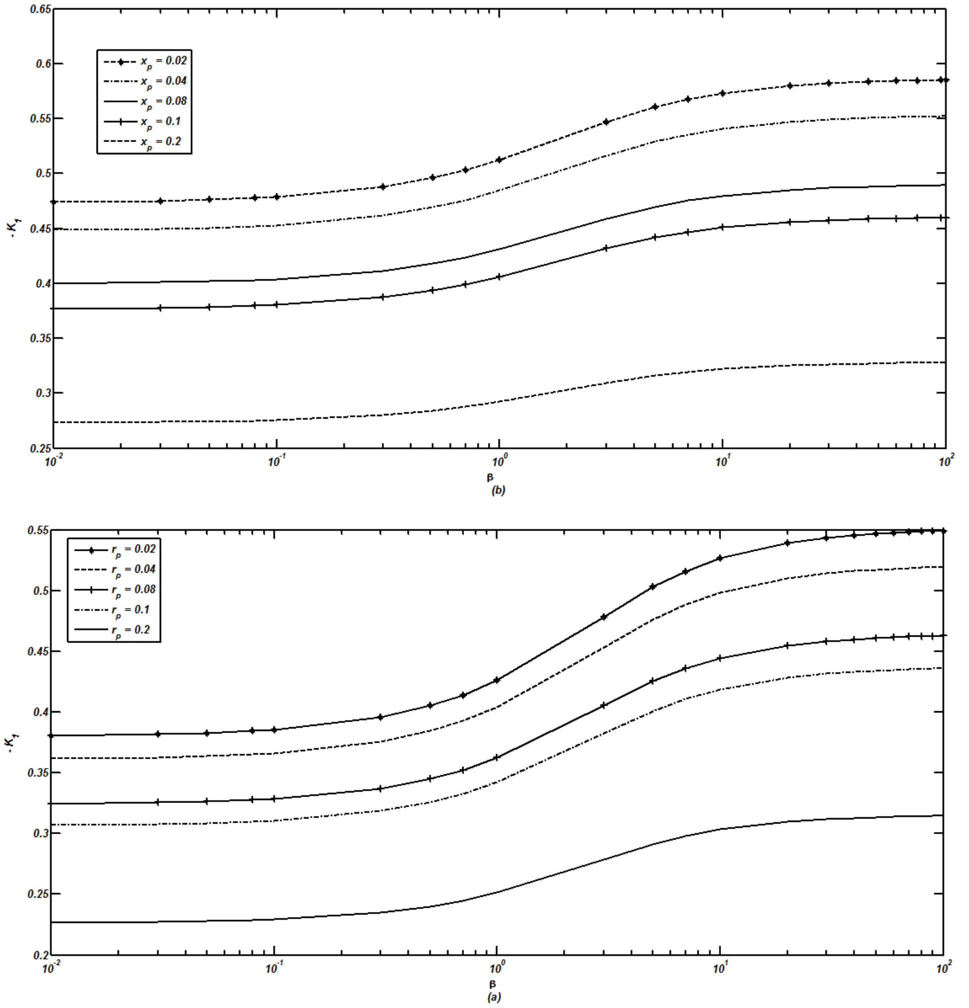

Figure 2. Variation of negative asymptotic convection coefficient-K_1 with absorption parameter β for different values of (a) r_p in pipe when $n = 2$ (b) x_p in channel when $n = 2$.

K_2 (from which the additive contribution of the axial diffusion $1/Pe^2$ has been deducted) against the wall absorption parameter β is plotted in figures 4(a, b) and 5 (a, b) for different values $r_p(x_p)$ and other fluids respectively. It is seen that the axial dispersion is significantly decreased with an increase in the absorption parameter.

When the plug radius $r_p(x_p)$ is one tenth of the tube (channel) radius and $\beta = 100$ the dispersion coefficient is reduced by 7.9(14.32) times of the corresponding value for $\beta = 0.01$. In a Casson fluid for the same value of yield stress, the dispersion coefficient when $\beta = 100$ is reduced by 14.62 (28.93) times of the corresponding value for $\beta = 0.01$ [13]. For a Newtonian fluid the corresponding reduction in pipe (channel) is found to be 3.95 (6.71). The values of dispersion coefficient are (3.29 – 5.36) times higher than in the case of Bingham plastic, (1.91-2.36) times higher than in the case of Power law fluid when compared to Herschel-Bulkley fluid case when β ranges from 0.01 to 100.

Mean Concentration

The variation of mean concentration ($C_m \times Pe$) versus time for $\beta = 0.01$ is plotted in Figure 6(a, b) for different fluids at $z = 0.5$ in pipe (channel).These profiles are obtained by solving equation (39). The transport coefficients involved are approximated by the corresponding asymptotic equivalents.

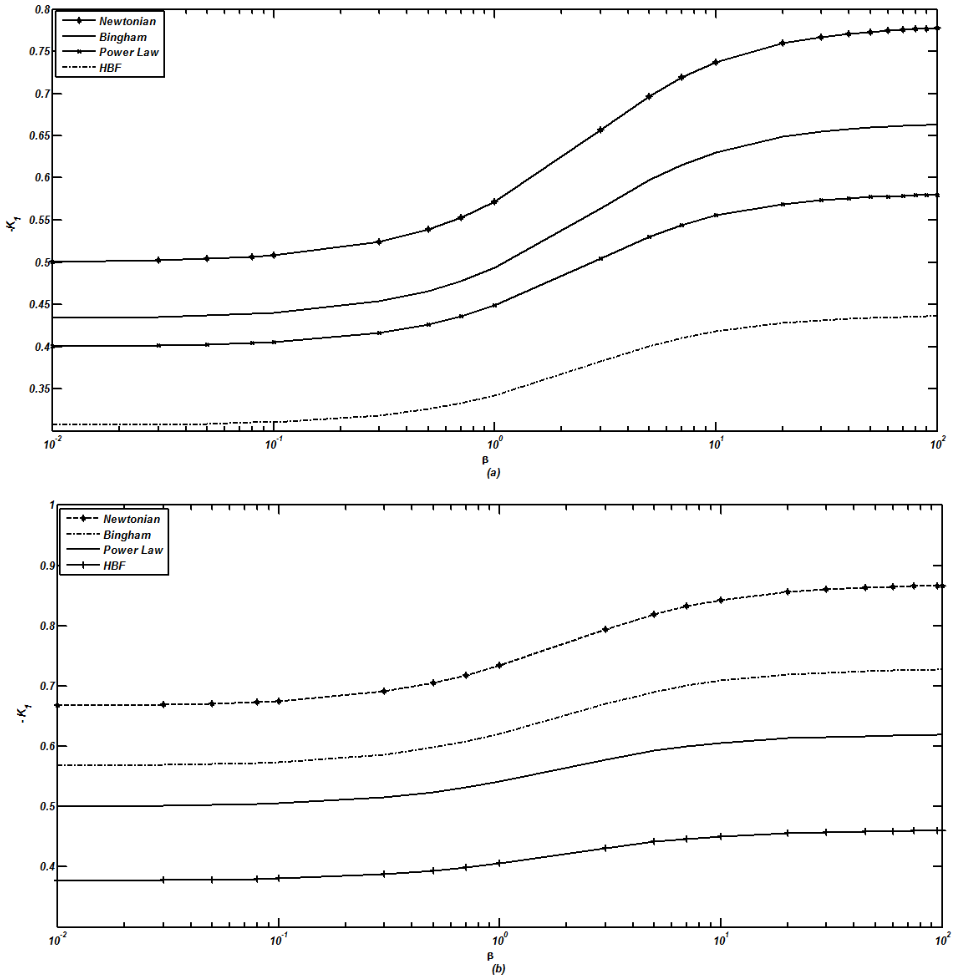

Figure 3. Variation of negative asymptotic convection coefficient-K_1 with absorption parameter β for different fluids (a) in pipe (b) in channel.

The mean concentration C_m reduces with time due to the constant depletion taking place at the boundary. It is seen that the peak mean concentration decreases as plug width increases. It is observed that the peak concentration in the case of Herschel-Bulkley fluid is almost double to that in the case of Newtonian fluid and it is 2.36 (1.15) times in a Bingham fluid (Power law fluid).

However, in channel case it is noticed that the absorption in the channel is more when compared to pipe. In the case of Newtonian and Bingham fluids this enhancement is very marginal, while in Power law fluid the peak concentration in pipe (channel) it is 8.649 (6.52) and in Herschel-Bulkley fluid it is 10.02 (8.124). From Figure 7(a, b), as β increases we notice that the concentration decreases and the absorption is found to be more in non-Newtonian fluids.

5. Applications to Blood Flows

The present mathematical model may be relevant to understand several physiological processes such as dispersion of drugs and nutrients in the human circulatory system. This also has applications in artificial blood such as blood oxygenators.

It is known that the blood flow in the human circulatory system is affected by several complexities that arise due to the elastic properties of the arterial wall,

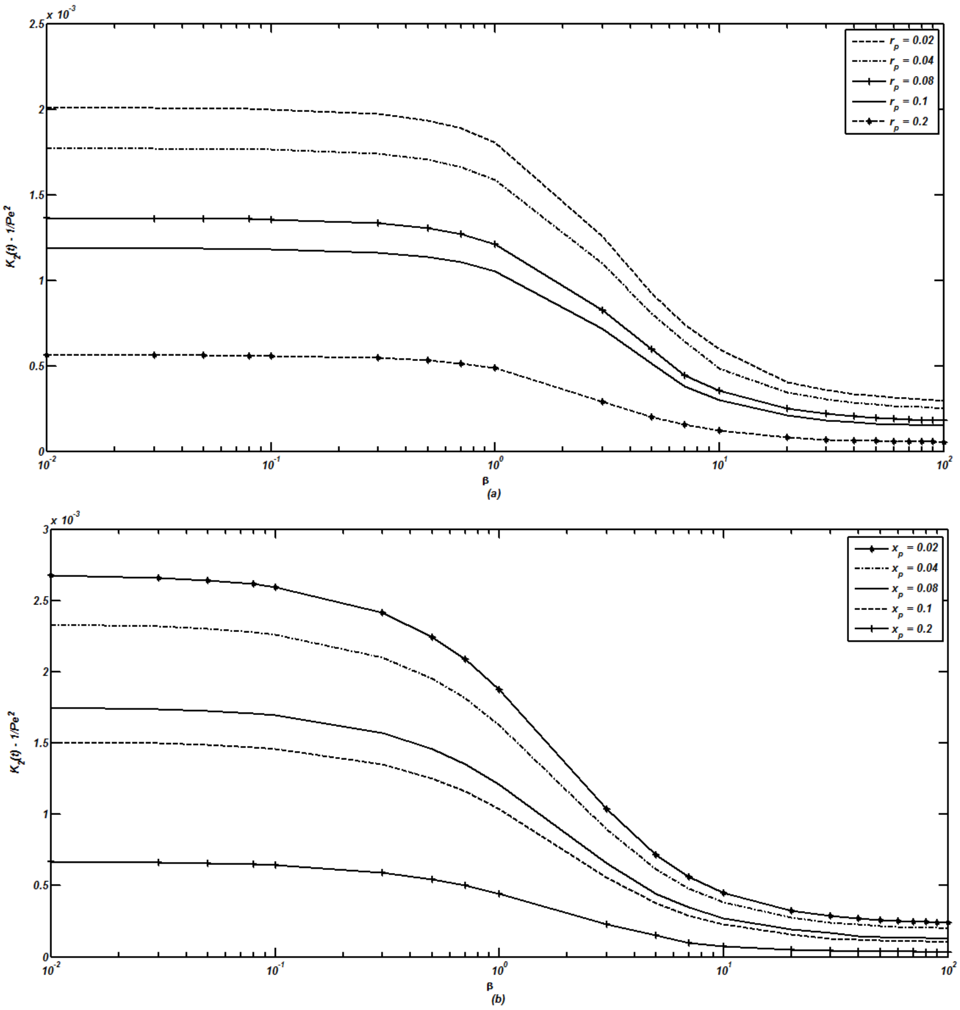

Figure 4. Variation of asymptotic dispersion coefficient $(K_2 - 1/Pe^2)$ with absorption parameter β for different values of (a) r_p in pipe when $n = 2$ (b) x_p in channel when $n = 2$.

branching and curvature pulsatile flow etc. Besides the influence of these complexities on the transfer of any passive species in blood stream, the non Newtonian character of the blood also plays a vital role on the transport.

Lighthill [11] discussed the applicability of the Taylor-Aris's theory dispersion in the study of the dispersion phenomenon in cardiovascular system. The theoretical work of Ananthakrishnan et al. [1] experimental computational study of Reejhsinghani *et al.* [15] revealed that the analysis of Taylor-Aris's gives a good description of the dispersion process provided the time after injection of the solute is greater than $0.5(a^2/D_m)$ that corresponds to a very large time of dispersion in actual blood flow conditions. However, the typical dimensionless time for oxygen transport in aorta is of the order of 10^{-5} [6] which are very small compared to the value 0.5.

The applicability condition of the Taylor-Aris's study to the dispersion process in the cardiovascular system makes the analysis valid for very small flow rates. In view of these limitations, a more general analysis than the Taylor-Aris's analysis to study the dispersion is required. By using the dispersion model of Gill and Sankarasubramanian, the entire dispersion process describes appropriately in terms of a simple diffusion process with effective dispersion coefficient K_2 as a function of time t. This facilitates the validity of the results for times smaller than $0.5(a^2/D_m)$. This may correspond to relatively higher rates of flow, i.e. blood flow in small

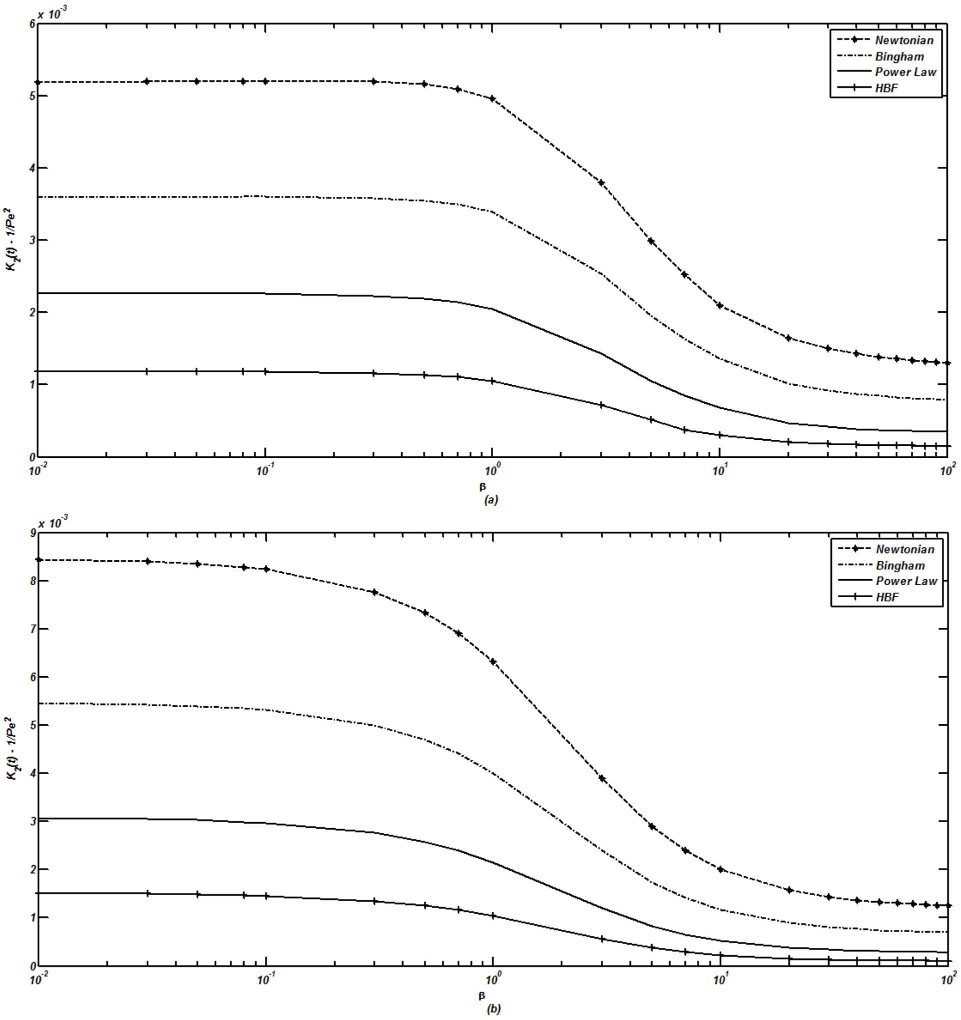

Figure 5. Variation of asymptotic dispersion coefficient $(K_2 - 1/Pe^2)$ with absorption parameter β for different fluids (a) in pipe (b) in channel.

arteries where blood is modeled as Herschel Bulkley fluid. The dispersion coefficient is influenced by non-Newtonian rheological parameters and power law index.

As the dispersion coefficients K_i, $i \geqslant 2$ are evaluated as functions of time, and they become negligible at higher values of time, this model can be considered to describe the entire process of dispersion to a first approximation for small time. i.e. $t < 0.5$, but exact for large time i.e. $t > 0.5$.

Although the present study explains the effects of non-Newtonian rheology on the dispersion of a passive species in a Herschel Bulkley fluid flowing through a pipe with boundary reaction, this model can be further redefined by including the transport of some solutes like oxygen combining with hemoglobin resulting in oxyhemoglobin, may increase the complexities in the analysis of dispersion.

6. Conclusions

Dispersion of a solute in a Herschel-Bulkley fluid with boundary retention effects in a conduit is discussed using the generalized dispersion model of Sankarasubramanian and Gill [16]. Thus the dispersion process is described through the three transport coefficients viz., exchange (absorption) coefficient, convection coefficient

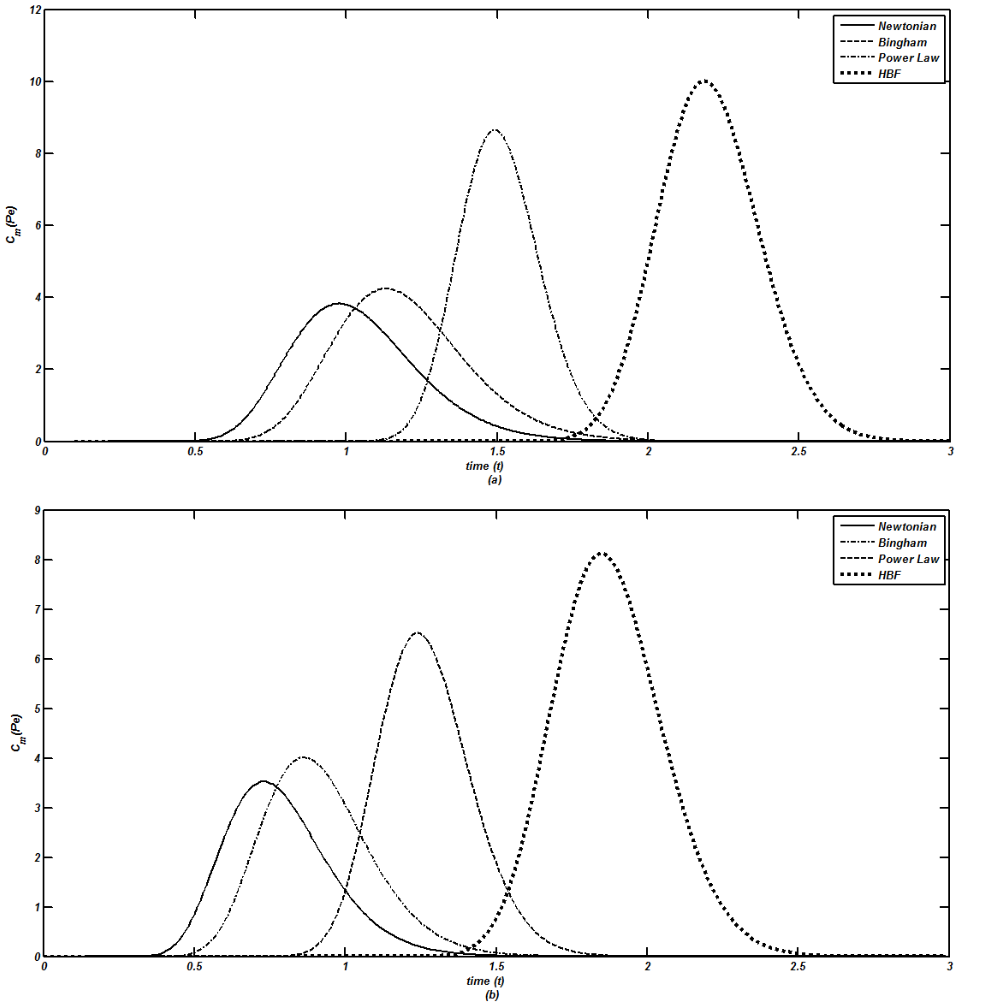

Figure 6. Variation of mean concentration C_m with time t for different fluids when $\beta = 0.01$, $Pe = 1000$ (a) in pipe (b)in channel.

and dispersion coefficient. The absorption coefficient is seen to be independent of non-Newtonian rheology. The convection coefficient is influenced by the yield stress and Power law index. The negative asymptotic convection coefficient decreases with increase in yield stress and increases with wall absorption parameter. The power law index also shows similar impact. It is observed that the convection coefficient reduces in magnitude in Bingham, Power law and Herschel-Bulkley fluid consecutively. When the plug radius $r_p(x_p)$ is one tenth of the tube (channel) radius and $\beta = 100$ in the tube (channel) $-K_1$ is observed to be increased by 1.4(1.21) times of the values corresponding to $\beta = 0.01$. For a Newtonian fluid the corresponding enhancement in pipe (channel) is observed to be 1.56(1.29). It is seen that the axial dispersion is significantly decreased with an increase in the absorption parameter.

When the plug radius $r_p(x_p)$ is one tenth of the tube (channel) radius and $\beta = 100$ the dispersion coefficient is reduced by 7.9(14.32) times of the corresponding value for $\beta = 0.01$. For a Newtonian fluid the corresponding reduction in pipe (channel) is found to be 3.95 (6.71). The mean concentration C_m reduces with time due to the constant depletion taking place at the boundary. As β increases we notice that the concentration decreases and the absorption is found to be more in non-Newtonian Fluids.

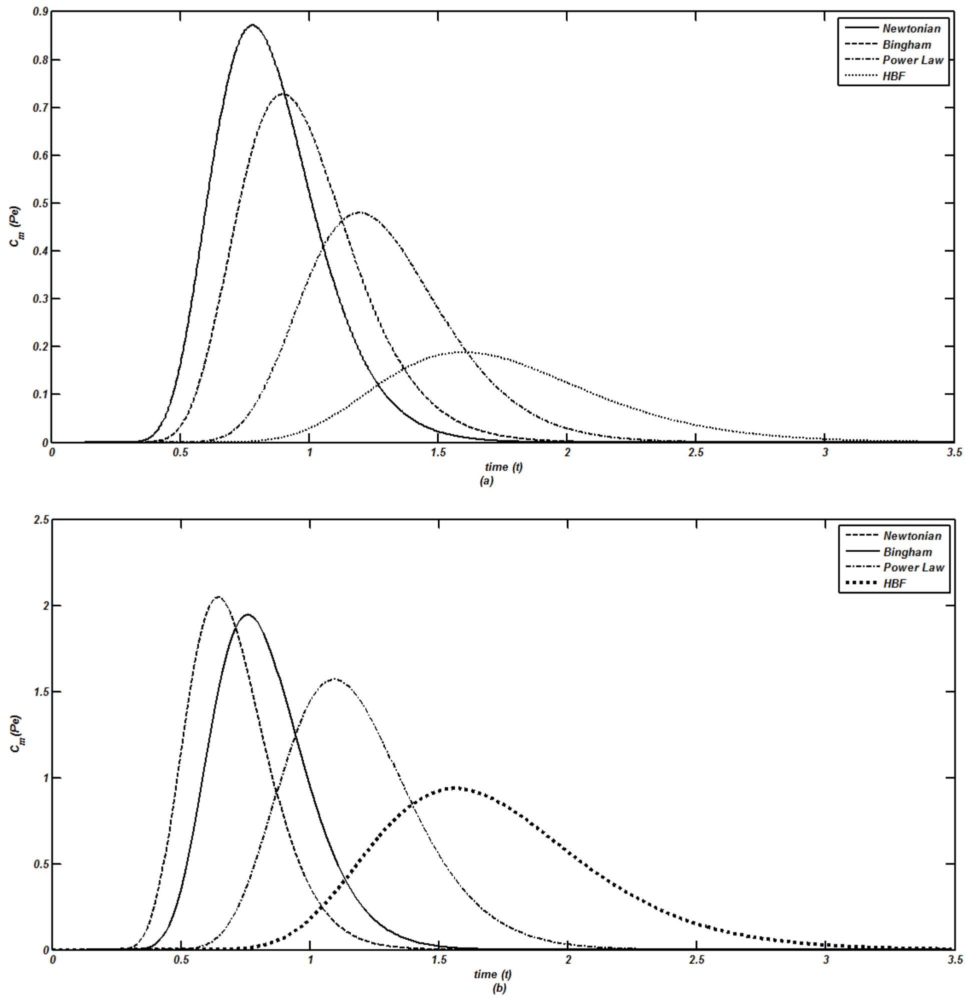

Figure 7. Variation of mean concentration C_m with time t for different fluids when $\beta = 1$, $Pe = 1000$ (a) in pipe (b)in channel.

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