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Studying the Behavior of Solutions of a Second-Order Rational Difference Equation and a Rational System

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Abstract. In this paper we investigate the behavior of solutions, stable and unstable of the solutions a second-order rational difference equation. Also we will discuss about the behavior of solutions a the rational system, we show these solutions may be stable or unstable.

Keywords: Dierence equation, Rational system, Boundedness character, Nonnegative parameters.

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1. Introduction

Kulenovic and Ladas in their book [4] initiated a systematic study of the general second-order rational difference equation,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 1, 2, 3, ... \tag{1}$$

with nonnegative parameters so that $A+B+C > 0$ and nonnegative initial conditions chosen to avoid division by zero. The study of these special cases has attracted a great deal of attention in this literature. A large amount of work has been directed toward developing a complete picture of the qualitative behavior of the difference Equation (1). According to Ref. [1, 5], there remain only one special case of (1) for which the qualitative behavior has not been established yet. In [5] has been established that in all ranges of positive parameters the unique equilibrium of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}}, \qquad n = 1, 2, 3, \ldots . \tag{2}$$

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with positive parameters and nonnegative initial conditions is globally asymptotically stable. More recently, in [2], Camouzis, Kulenovic, Ladas, and Merino have initiated a systematic study of the general rational system of difference equations in the plane,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma y_n}{A + Bx_n + Cy_n}, \qquad y_{n+1} = \frac{p + \delta x_n + \varepsilon y_n}{q + Dx_n + Ey_n}, \qquad n = 1, 2, 3, \ldots. \tag{3}$$

with nonnegative parameters and nonnegative inital conditions chosen to avoid division by zero. According to Ladas and Lugo, there remain only special case of (3) for which the bounded character has not been established yet. This case is the system

$$x_{n+1} = \frac{x_n}{y_n}, \qquad y_{n+1} = x_n + \varepsilon y_n \quad n = 1, 2, 3, \ldots. \tag{4}$$

with positive parameters and nonnegative initial conditions. In [5] has been established that every solution of the system

$$x_{n+1} = \frac{x_n}{y_n}, \qquad y_{n+1} = \frac{\alpha + \gamma y_n}{x_n + y_n} \quad n = 1, 2, 3, \ldots.$$

is bounded.

In this paper, we prove that in all ranges of positive parameters every solution of the difference Equation (4) is not bounded.

2. Description of the Qualitative Behavior of Solutions

THEOREM 2.1 *The unique equilibrium of the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_{n-1}}, \qquad n = 1, 2, 3, \ldots$$

with positive parameters and nonnegative initial conditions is globally asymptotically stable.

Proof see theorem 1 of [5]. ■

THEOREM 2.2 *Every solution is bounded for the rational difference equation*

$$x_n = \left(\frac{1}{x_{n-1}x_{n-2}}\right)\frac{\alpha + \gamma x_{n-1}x_{n-2}}{1 + x_{n-1}}, \qquad n \geqslant 2$$

with positive parameters and nonnegative initial conditions .

Proof see theorem 2 of [5]. ■

Now we present the boundedness and unboundedness character of the following system

$$x_{n+1} = \frac{x_n}{y_n}, \qquad y_{n+1} = x_n + \varepsilon y_n \quad n = 1, 2, 3, \ldots$$

with nonnegative parameters and nonnegative initial conditions. It turns out that the x_n component of the system (4) can be reduced to the difference equation

$$x_n = (\frac{x_{n-1}}{x_{n-2}})\frac{x_{n-1}}{x_{n-1}+\varepsilon}, \qquad n \geqslant 2 \tag{5}$$

through algebraic identities. This reduction proceeds as follows, the first Equation of the system (4) gives us

$$y_n = \frac{x_n}{x_{n+1}}, \qquad n \geqslant 0. \tag{6}$$

Also the first Equation of the system (4) gives us

$$x_{n+2} = \frac{x_{n+1}}{x_n + \varepsilon y_n}, \qquad n \geqslant 0 \tag{7}$$

substituting (6) in (7) equation gives us

$$x_{n+2} = (\frac{x_{n+1}}{x_n})\frac{x_{n+1}}{x_{n+1}+\varepsilon}, \qquad n \geqslant 0$$

this yields the difference Equation (5). So from now on it suffices to show that every solution of (5) is not bounded.

THEOREM 2.3 *Every solution is not bounded for the rational difference Equation (5) with positive parameters and positive initial conditions.*

Proof Recall our difference Equation (5)

$$x_n = (\frac{x_{n-1}}{x_{n-2}})\frac{x_{n-1}}{x_{n-1}+\varepsilon}, \qquad n \geqslant 2. \tag{8}$$

For investigate proof first consider several case separately. Since $\varepsilon > 0$ and the initial conditions are nonnegative so

$$\frac{x_{n-1}}{x_{n-1}+\varepsilon} < 1, \qquad n \geqslant 2 \tag{9}$$

we're using (9) in (8) therefore

$$x_n < \frac{x_{n-1}}{x_{n-2}}, \qquad n \geqslant 2. \tag{10}$$

For Equation (10), we consider the following cases separately.
The first case

$$\frac{x_{n-1}}{x_{n-2}} < 1, \qquad n \geqslant 2$$

in this case it is clear that the sequence

$$x_n \longrightarrow 0.$$

The second case, if

$$\frac{x_{n-1}}{x_{n-2}} = 1, \qquad n \geqslant 2$$

then

$$x_n = \frac{x_{n-1}}{x_{n-1}+\varepsilon}, \qquad n \geqslant 2.$$

Also in this case it is clear that the sequence

$$x_n \longrightarrow a, \qquad 0 \leqslant a \leqslant 1.$$

The third case, if

$$\frac{x_{n-1}}{x_{n-2}} > 1, \qquad n \geqslant 2$$

then

$$x_n \longrightarrow \infty$$

therefore only in the first and second states solutions are bounded, in the third case the solutions are unbounded. ■

Corollary 2.4 *Every solution of the system* (5) *is not asymptotically stable therefore is not bounded.*

Proof Because

$$y_n = \frac{x_n}{x_{n+1}}, \qquad n \geqslant 0,$$

Thus with utilize proof of the previous theorem we can consider separately the following cases.
The first case

$$y_n \longrightarrow \infty.$$

The second case

$$y_n \longrightarrow \frac{1}{a}, \qquad 0 \leqslant a \leqslant 1.$$

The third case

$$y_n \longrightarrow 0.$$

According the previous theorem (2.3) and results above only in a case solutions of system (4) are bounded and stable, but are not asymptotically stable and in other cases every solution of system (4) similar hyperbolic behavior around the fix point.
■

3. Conclusion

In [5] has been shown that the unique equilibrium of the difference Equation (2) is globally asymptotically stable. We have shown that every solution of the difference Equation (5) is not stable and also every solution of the system difference equation is not bounded. We leave the reader with three crucial conjectures pertaining to four special cases. The special cases that are the two subcases of the following second order rational difference equation

$$x_{n+1} = \frac{\alpha + x_n}{A + x_n + Cx_{n-1}}, \qquad n = 1, 2, 3, \ldots \tag{11}$$

with $A \geqslant 0$, all other parameters positive, and nonnegative initial conditions. It is conjectured in [4] that the unique positive equilibrium is globally asymptotically stable for this the two subcases of the second order rational difference Equation (11). this the two subcases of the second order rational difference Equation (11) are now the only second-order rational difference equations for which the qualitative behavior has not been established yet. The special case

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-2} + Cx_{n-1}}, \qquad n = 1, 2, 3, \ldots \tag{12}$$

with positive parameters and nonnegative initial conditions is the only remaining third-order rational difference equation whose boundedness character is yet to be determined. It is conjectured in [3] that there exist unbounded solutions for some choice of nonnegative initial conditions for the difference Equation (12).

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