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Certain Sufficient Conditions for Close-to-Convexity of Analytic Functions

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Abstract. The object of this paper is to derive certain sufficient conditions for close-to-convexity of certain analytic functions defined on the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.

Keywords: Discrete Markovian service process; N threshold policy; Finite buffer;Queue; Supplementary variable

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1. Introduction

Let $\mathcal{H}(\Delta)$ be the class of analytic functions in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. We denote $\mathcal{H} = \mathcal{H}[1, 1]$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$. Thus, the class $\mathcal{A}$ consists of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

A function $p(z) = 1+p_1 z+p_2 z^2+\dots$ is said to be in the class $\mathcal{P}$ if $\operatorname{Re} p(z) > 0$. For two analytic functions f and g, we say that f is *subordinate* to g or g is *superordinate* to f, denoted by $f \prec g$, if there is a Schwarz function w with $|w(z)| \leqslant |z|$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$. A function $f \in \mathcal{A}$ is starlike if $f(\Delta)$ is starlike domain with respect to 0, and a function $f \in \mathcal{A}$ is convex if $f(\Delta)$ is a convex domain. Analytically, the

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prerequisites are equivalent to the following conditions $\frac{zf'(z)}{f(z)} \in \mathcal{P}$ and $1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}$, respectively. The class of starlike and convex functions of order $\alpha, (0 \leqslant \alpha < 1)$ is defined as follows:

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$$

and

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

These classes are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. The class of close to convex functions is defined by

$$\mathcal{C}(\alpha) := \{f : f \in \mathcal{A}; \text{and Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, z \in \delta, 0 \leqslant \alpha < 1; g \in \mathcal{K}\}.$$

It is well known [1] that $f \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha)$. Thus, if $f \in \mathcal{S}^*(\alpha)$, then $f \in \mathcal{C}(\alpha)$.

The following Lemma is needed in the present investigation:

Lemma 1.1 *[2, 3] Let the function $w(z)$ be analytic in δ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| < 1$ at a point $z_0 \in \Delta$, then $z_0 w'(z_0) = cw(z_0)$, where $c \geqslant 1$.*

2. Main Results

Theorem 2.1 *Let $c \geqslant 1$ and one of the following conditions holds*

(1) $A = 1, 0 < B < 1$

(2) $0 < A < 1, 0 \leqslant B < A$.

If the function $f \in \mathcal{A}$ satisfies the inequality

$$\text{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) > 1 + \frac{Ac}{1+A} + \frac{(1+A)Bc}{(1+B)^2} \ (z \in \Delta),$$

then

$$|f'(z) - 1| < |A - Bf'(z)|.$$

Proof Let the function w be defined as

$$f'(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \neq -\frac{1}{B}. \tag{2}$$

Then, clearly w is analytic in the unit disk Δ with $w(0) = 0$. From (2), by a simple computation , we get

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{Azw'(z)}{(1 + Aw(z))} - \frac{Bzw'(z)}{(1 + Bw(z))}. \tag{3}$$

Suppose that there is a point z_0 in the unit disk Δ with the properties $|w(z_0)| = 1$ and $|w(z)| < 1$, whenever $|z| < |z_0|$. Now, from the Lemma 1.1, we have

$$z_0 w(z_0) = cw(z_0), (c \geqslant 1, w(z_0) = e^{i\theta}, \theta \in \mathbb{R}). \tag{4}$$

From (3) and (4), we obtain

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \frac{Ac(\cos(\theta) + A)}{1 + A^2 + 2A\cos(\theta)} - \frac{Bc(\cos(\theta) + B)}{1 + B^2 + 2B\cos(\theta)} := u(\theta).$$

A simple calculation shows that $u(\theta)$ attains its maximum at $\theta = 0$ and

$$\max_{\theta \in \mathbb{R}}\{u(\theta)\} = 1 + \frac{Ac}{1 + A} + \frac{(1 + A)Bc}{(1 + B)^2}.$$

Which is a contradiction to our hypothesis. Thus, $|w(z)| < 1,\ z \in \Delta$ which implies that $|f'(z) - 1| < |A - Bf'(z)|$. This completes the proof. ∎

If we set $B = 0$ in the Theorem 2.1, then we have:

Corollary 2.2 *Let $c \geqslant 1$ and $0 < A < 1$. If the function $f \in \mathcal{A}$ satisfies the inequality*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) > 1 + \frac{Ac}{1 + A}\ (z \in \Delta),$$

then

$$\mathrm{Re}(f'(z)) > 1 - A.$$

Which equivalently can be written as $f \in \mathcal{C}(1 - A)$.

If we set $A = 1/2$ and $c = 1$ in the Corollary 2.2, then we have:

Corollary 2.3 *If the function $f \in \mathcal{A}$ satisfies the inequality*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) > 2.33 \quad (z \in \Delta),$$

then

$$\mathrm{Re}(f'(z)) > 1/2.$$

Which equivalently can be written as $f \in \mathcal{C}(1/2)$.

Setting $A = 1$ in the Theorem 2.1, we have:

Corollary 2.4 *Let $c \geqslant 1$ and $0 < B < 1$. If the function $f \in \mathcal{A}$ satisfies the inequality*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) > 1 + \frac{c}{2} + \frac{2Bc}{(1 + B)^2}\ (z \in \Delta),$$

then

$$|f'(z)| < \frac{2}{1 - B}.$$

Setting $B = 1/2$ and $c = 2$ in the above corollary, we have:

COROLLARY 2.5 *If the function $f \in \mathcal{A}$ satisfies the inequality*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) > 2.88 \quad (z \in \Delta),$$

then

$$|f'(z)| < 4.$$

3. Conclusion

In this paper several sufficient conditions for close-to-convexity of analytic functions are obtained. Further this paper leaves a scope to the researchers to discuss more general results in this direction using differential subordination.

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