International Journal of
Mathematical Modelling & Computations
Vol. 06, No. 01, 2016, 83- 103

An Application of Fuzzy Numbers to the Assessment of Mathematical Modelling Skills

Michael Gr. Voskoglou* - Igor Ya. Subbotin**

* *Department of Applied Mathematics, Faculty of Technological Applications, Graduate Technological Educational Institute of Western Greece, Patras, Greece.*

** *Department of Mathematics and Natural Sciences, College of Letters and Sciences, National University, Los Angeles, USA.*

Abstract. In this paper we use the Triangular and Trapezoidal Fuzzy Numbers as tools for assessing student Mathematical Modelling (MM) skills. Fuzzy Numbers play a fundamental role in fuzzy mathematics analogous to the role played by the ordinary numbers in classical mathematics, On the other hand, MM appears today as a dynamic tool for teaching and learning mathematics, because it connects mathematics with our everyday life giving the possibility to students to understand its usefulness in practice thus increasing their interest about mathematics. Therefore, the subject of the present paper appears to be of great interest. Our results are illustrated by three examples, through which the effectiveness of our new fuzzy assessment approach is validated compared to already established by the present authors in earlier works assessment methods of the classical (calculation of the means, GPA index) and the fuzzy logic (Trapezoidal Model).

Received: 4 August 2015; Revised: 20 October 2015; Accepted: 11 November 2015.

Keywords: Mathematical Modelling (MM), GPA index, Fuzzy assessment methods, Trapezoidal Fuzzy Assessment Model (TpFAM), Fuzzy Numbers (FNs), Triangular Fuzzy Numbers (TFNs), Trapezoidal Fuzzy Numbers (TpFNs).

Index to information contained in this paper

*Corresponding author. Email: : mvosk@hol.gr , voskoglou@teiwest.gr.

1. Introduction

The *systems' modelling* is a basic principle in engineering, in natural and in social sciences. When we face a problem concerning a system's operation (e.g. maximizing the productivity of an organization, minimizing the functional costs of a company, etc) a *model* is required to describe and represent the system's multiple views. The model is a simplified representation of the basic characteristics of the real system including only its entities and features under concern. The construction of a model usually involves a preliminary deep abstracting process on identifying the system's dominant variables and the relationships governing them. The resulting structure of this action is known as the *assumed real system*. The model, being a further abstraction of the assumed real system, identifies and simplifies the relationships among these variables in a form amenable to analysis.

There are several types of models in use according to the form of the corresponding problem ([25], Section 1.3.1). The representation of a system's operation through the use of a *mathematical model* is achieved by a set of mathematical expressions (equalities, inequalities, etc) and functions properly related to each other. The solutions provided by a mathematical model are more general and accurate than those provided by the other types of models. However, in cases where a system's operation is too complicated to be described in mathematical terms (e.g. biological systems), or the corresponding mathematical relations are too difficult to deal with for providing the problem's solution, a *simulation model* can be used, which is usually constructed with the help of computers.

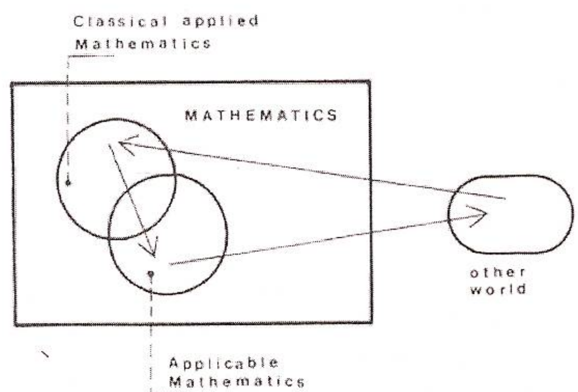

Figure 1: The circle of modelling

Until the middle of 1970's Mathematical Modelling (MM) was mainly a tool in hands of scientists and engineers for solving the real world problems related to their disciplines (physics, industry, constructions, economics, etc). One of the first who described the process of MM in such a way that it could be used for teaching mathematics was Pollak [14]. He

represented the interaction between mathematics and the real world with the scheme shown in Figure 1, which is known as the *circle of modelling*.
The most important feature of Pollak's scheme is the direction of the arrows, representing a looping between the other (real) world (including all the other sciences and the human activities of everyday life) and the "universe" of mathematics: Starting from a real problem of the other world we transfer to the other part of the scheme, where we use or develop suitable mathematics for its solution. Then we return to the other world interpreting and testing on the real situation the mathematical results obtained. If these results are not giving a satisfactory solution to the real problem, then we repeat the same circle again one or more times.
From the time that Pollak presented this scheme in ICME-3 (Karlsruhe, 1976) until nowadays much effort has been placed to analyze in detail the process of MM [1, 2, 3, 5, 8] , etc. A brief but comprehensive account of the variation of models used for the description of the MM process can be found in Haines & Crouch [9] including Voskoglou's *stochastic model* [28] in which the MM circle is treated as a *Markov chain process* dependent upon the transition between the successive discrete stages of the MM process. The arrows in Figure 2 below are shown the possible transitions between stages which are:

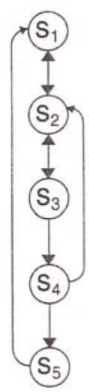

Figure 2: The flow-diagram of Voskoglou's Markov chain model for the MM process

S_1: *Analysis* of the problem (understanding the statement and recognizing the restrictions and requirements of the real system.
S_2: *Mathematization* (formulation of the problem and construction of the model).
S_3: *Solution* of the model by proper mathematical manipulation
S_4: *Validation* (control) of the model, which is usually achieved by reproducing, through the model, the behaviour of the real system under the conditions existing before the solution of the model and by comparing it to the existing, from the previous "history" of the corresponding real system, real data (in cases of systems having no past history, an extra simulation model can be used for the validation of the initial mathematical model).
S_5: *Interpretation* of the final mathematical results and implementation of them to the real

system, in order to give the “answer” to the real world problem.
In concluding, MM appears today as a dynamic tool for teaching and learning mathematics, because it connects mathematics with our everyday life giving the possibility to students to understand its usefulness in practice and therefore increasing their interest about mathematics. In other words, according to the Polya’s [15] terminology, MM works as a *best motivation* for learning mathematics. Therefore, the assessment of student MM skills is a very important task that enables the instructor to obtain a concentrating view of his/her students’ progress on MM and thus to readapt his/her teaching methods and practices in order to succeed the best possible, under the circumstances, result.
Our target in this paper is to use the *Triangular* (TFN) and the *Trapezoidal* (TpFN) *Fuzzy Numbers* (FNs) as a tool for assessing student MM skills. Notice that, there exist strong logical pro arguments for employing this approach. In fact, roughly speaking a TFN (*a, b, c*), with *a*, *b* and *c* real numbers such that $a< b < c$, means *“approximately equal to b”* or, if you prefer, *“the value of b lies in the real interval [a, c]”*. Furthermore, a TpFN (*a, b, c, d*), with *a*, *b*, *c* and *d* real numbers such that $a< b < c < d$, actually means *“approximately in the interval [b, c]”*. Obviously the above expressions constitute the basis for a fuzzy assessment.
The rest of the paper is formulated as follows: In Section 2 we give a summary of our previous researches on using principles of Fuzzy Logic (FL) for the MM process. In Section 3 we present the FNs and basic properties of them, while in Section 4 we discuss TFNs and TpFNs, which are two of the simplest forms of TFNs. In Section 5 we describe the use of TFNs/TpFNs for assessing MM skills and finally in our last Section 6 we state our conclusions and we discuss the perspectives of further research on the subject

2. Fuzzy Logic in MM: A summary of our previous researches

Models for the MM process like all those mentioned in the previous section are useful in understanding what is termed by Haines & Crouch [9] as the *ideal behaviour*, in which the modellers proceed effortlessly from a real world problem through a mathematical model to acceptable solutions and report on them. However, life in classroom (and probably amongst modellers in science, industry and elsewhere) is not like that. More recent researches [4, 6, 7], report that students in school take *individual routes* when tackling MM problems, associated with their individual learning styles and the level of their cognition, which utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teachers’ point of view there usually exists a degree of vagueness about their students’ way of thinking in each of the stages of the MM process, when tackling such kind of problems.
All these gave us the impulsion to introduce principles of FL for treating in a more realistic way the process of MM in classroom. In fact, Voskoglou [29] *represented the main stages of the MM process as fuzzy sets in a set of linguistic labels (grades) characterizing the students’ performance* and he has used the concept of a system’s *uncertainty*, which emerges naturally within the broad framework of fuzzy sets theory, for obtaining a measure of students’ MM skills [29].
Subbotin et al. [18] introduced the idea of applying the commonly used in FL *Center of Gravity (COG) defuzzification technique* (e.g. see [27]) for assessing students’ performance.

According to the COG technique the defuzzification of a fuzzy situation's data is succeeded through the calculation of the coordinates of the COG of the level's section contained between the graph of the membership function associated with this situation and the X- axis. Subbotin and Voskoglou, either collaborating or independently to each other, have adapted several times in the past the COG technique for assessing students' skills in a variety of different (mainly mathematical) tasks [18, 19, 22, 31, 32, 33], etc, for testing the effectiveness of a CBR system [20], for assessing Bridge players' performance [35], etc. For this, they represented the group under assessment as a fuzzy set on the set of the linguistic grades U = {A, B, C, D. F} characterizing its members' performance, where A = excellent (85-100%), B = very good (75-84%), C = good (60-74%), D = fair (50-59%) and F = unsatisfactory (0-49%). We emphasize that the scores assigned to each of the above grades are *indicative only*, since they may slightly differ from case to case. For example, one in a more strict ranking could take A=90-100%, B=89-80%, C=79-70%, D=69-60% , F=59-0%, etc

Recently two variations of the COG technique, initiated by Subbotin [21], have been developed treating better the ambiguous assessment cases, which are at the boundaries between two successive linguist grades (e.g. something like 84-85% being at the boundaries between A and B): The *Triangular* (TFAM) [23] and the *Trapezoidal* (TpFAM) [24] *Fuzzy Assessment Models*, which have been proved to be *equivalent to each other*, in the sense that they obtain exactly the same assessment results. Therefore, it is logical to focus here on one of the TFAM/TpFAM models, instead of the COG technique.

According to the TpFAM [24], for example, the coordinates (X_c, Y_c) of the COG of the resulting scheme are calculated by the formulas: $X_c = (7\sum_{i=1}^{5} i y_i) - 2$, $Y_c = \frac{3}{7}\sum_{i=1}^{5} {y_i}^2$ (1),

where x_1=F, x_2=D, x_3=C, x_4=B, x_5=A and y_i is the percentage of students who obtained the grade x_i, for i=1, 2, 3, 4, 5. Further, between two groups, *the group with the greater value for X_c demonstrates the better performance*. Also, *if the two groups have the same value for X_c then: a) If $X_c \geq 19$, the group with the greater value of Y_c demonstrates the better performance, b) If $X_c < 19$, the group with the smaller value of Y_c demonstrates the better performance.*

The TpFAM, as well as the TFAM and the COG technique, measures the *quality performance* of a group, since, as it turns out from the first of formulas (1), it assigns greater coefficients (weights) to the higher scores.

Notice that an analogous method of the classical logic measuring the group's quality performance is the very popular in USA and some other Western countries *Grade Point Average (GPA) index*. Using the same as above notation the GPA index is calculated by the formula GPA = $0y_1 + y_2 + 2y_3 + 3y_4 + 4y_5$ (2) ; e.g. see Section 4.1 of [24]. In the ideal case ($y_1 = y_2 = y_3 = y_4 = 0$, $y_5 = 1$) equation (2) gives that GPA = 4, while in the worst case ($y_2 = y_3 = y_4 = y_5 = 0$, $y_1 = 1$) it gives that GPA = 0. Therefore, we have in general that $0 \leq$ GPA ≤ 4. On comparing (2) with the first of formulas (1) one observes that *the TpFAM is more sensitive to the higher scores than the GPA index*, since it assigns greater coefficients to them.

3. Fuzzy Numbers

3.1 Definitions

A *Fuzzy Number* (FN) is a special form of fuzzy set [36] on the set ***R*** of real numbers. For those not familiar to the subject we recall that *a fuzzy set A on the universal set U*, is a set of ordered pairs of the form $A = \{(x, m_A(x)): x \in U\}$, defined in terms of a *membership function* $m_A : U \rightarrow [0,1]$ that assigns to each element of U a real value from the interval [0,1]. For general facts on fuzzy sets, which find nowadays applications to almost all sectors of human activities (e.g. see [10, 11, 17, 20, 22, 31, 32, 33], etc), we refer to the book of Klir & Folger [12].

FNs play a fundamental role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics. For general facts on FNs we refer to Chapter 3 of the book of Theodorou [26], which is written in Greek language, and also to the classical on the subject book of Kaufmann and Gupta [12]. For introducing the notion of a FN, it becomes necessary to give first the following three introductory definitions:

Definition 1: A fuzzy set A on U with membership function $y = m(x)$ is said to be *normal*, if there exists x in U, such that $m(x) = 1$.

Definition 2: Let A be a fuzzy set in U, and let x be a real number of the interval [0, 1]. Then the x-*cut* of A, denoted by A^x, is defined to be the set $A^x = \{y \in U: m(y) \geq x\}$.

Definition 3: A fuzzy set A on ***R*** is said to be *convex,* if its x-cuts A^x are ordinary closed real intervals, for all x in [0, 1].

For example, for the fuzzy set A whose membership function's graph is represented in Figure 3, we observe that $A^{0.4} = [5, 8.5] \cup [11, 13]$ and therefore A is not a convex fuzzy set.

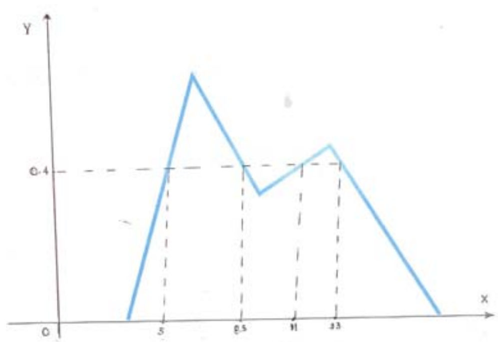

Figure 3: Graph of a non convex fuzzy set

We are ready now to give the definition of a FN:

Definition 4: A FN is a normal and convex fuzzy set A on ***R*** with a piecewise continuous membership function.

Figure 4 represents the graph of a FN expressing the fuzzy concept: *"The real number x is* *approximately equal to 5".* We observe that the membership function of this FN takes constantly the value 0 outside the interval [0, 10], while its graph in [0, 1] is a parabola.

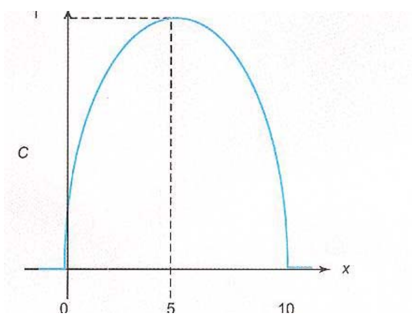

Figure 4: Graph of a fuzzy number

Since the x-cuts A^x of a FN A are closed real intervals, we can write $A^x = [A_l^x, A_r^x]$ for each x in [0, 1], where A_l^x, A_r^x are real numbers depending on x. The following statement defines a *partial order* in the set of all FNs:

Definition 5: Given the FNs A and B we write $A \le B$ (or $\ge$) if, and only if, $A_l^x \le B_l^x$ and $A_r^x \le B_r^x$ (or $\ge$) for all x in [0, 1]. Two FNs for which the above relations hold are called *comparable*, otherwise they are called *non comparable*.

3.2 Arithmetic operations on FNs

The basic arithmetic operations on FNs are defined in general in two alternative ways, which are *equivalent* to each other:

(i) *With the help of their x-cuts and the Representation-Decomposition Theorem* for fuzzy sets For this, we recall first that the Representation-Decomposition Theorem of Ralesscou-Negoita ([16], Theorem 2.1, p.16) states that a fuzzy set A can be completely and uniquely expressed by the family of its x-cuts in the form $A = \sum_{x\in[0,1]} xA^x$.

Now, if *A* and *B* are given FNs, and "*" denotes an arithmetic operation (addition, subtraction, multiplication or division) between them, applying the above theorem for the fuzzy set *A* * *B* we find that $A * B = \sum_{x\in[0,1]} x(A*B)^x$. But the x-cuts of the FNs are ordinary closed real intervals, therefore, if we define that $(A * B)^x = A^x * B^x$ (where, for reasons of simplicity, "*" in the second term of the last equation denotes the corresponding operation between closed real intervals), the *fuzzy arithmetic is turned to the well known arithmetic of the closed real intervals* (we recall that an arithmetic operation "*" between closed real

intervals is defined by the general rule [a, b] * [a_1, b_1] = {x * y: x, y $\in$ ***R***, a $\leq$ x $\leq$ a_1, b $\leq$ y $\leq$ b_1} [12]).
(ii) By applying the *Zadeh's extension principle* ([13], Section 1.4, p.20), which provides the means for any function f mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.
In practice the above two general methods of the fuzzy arithmetic, requiring laborious calculations, are rarely used in applications, where the utilization of simpler forms of FNs is preferred.

4. Triangular and Trapezoidal Fuzzy Numbers

4.1 Definition and Basic Properties of Triangular Fuzzy Numbers (TFNs)

The membership function's graph of a TFN (*a, b, c*), where *a*< *b* < *c* are given real numbers, is represented in Figure 5. We observe that the membership function *y=m(x)* of it takes constantly the value 0, if x is outside the interval [*a, c*], while its graph in the interval [*a, c*] is the union of two straight line segments forming a triangle with the X-axis.

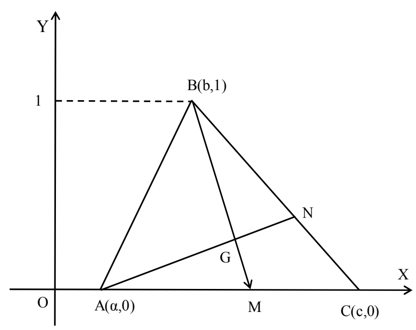

Figure 5: Graph of the TFN (*a, b, c*)

Therefore, the definition of a TFN is given as follows:
Definition 6: Let *a, b* and *c* be real numbers with *a* < *b* < *c*. Then the *Triangular Fuzzy Number (TFN)* A = (*a*, *b*, *c*) is a FN with membership function:

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a} \quad , & x \in [a,b] \\ \dfrac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \ or \ x > c \end{cases}$$

In the above definition we obviously have that *m(b)=1*, while *b* need not be in the "middle" of *a* and *c*.

The following two Propositions refer to basic properties of TFNs that we are going to use later in this paper:

Proposition 1: The x-cuts A^x of a TFN A = (*a, b, c*), $x \in [0, 1]$, are calculated by the formula $A^x = [A_l^x, A_r^x] = [a + x(b - a), c - x(c - b)]$.

Proof: Since $A^x = \{y \in \boldsymbol{R}: m(y \geq x\}$, Definition 6 gives for the case of A_l^x that $\frac{y-a}{b-a} = x \Leftrightarrow y = a + x(b-a)$. Similarly for the case of A_r^x we have that $\frac{c-y}{c-b} = x$ $\Leftrightarrow y = c - x(c - b)$.

Proposition 2 (Defuzzification of a TFN): The coordinates (*X*, *Y*) of the COG of the graph of the TFN (*a*, *b*, *c)* are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.

Proof: The graph of the TFN (*a*, *b*, *c)* is the triangle ABC of Figure 5, with A (*a*, 0), B (*b*, 1) and C (*c*, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM, where N ($\frac{b+c}{2}$, $\frac{1}{2}$) and M ($\frac{a+c}{2}$, 0). Therefore the equation of the straight line on which AN lies is $\dfrac{x-a}{\frac{b+c}{2}-a} = \dfrac{y}{\frac{1}{2}}$, or $x + (2a - b - c)y = a$ (3). In the same way one finds that the equation of the straight line on which BM lies is $2x + (a + c - 2b)y = a + c$ (4).

Since D = $\begin{vmatrix} 2 & a+c-2b \\ 1 & 2a-b-c \end{vmatrix} = 3(a-c) \neq 0$, the linear system of (3) and (4) has a unique solution with the respect to the variables *x* and *y* determining the coordinates of the triangle's COG.

The proof of the Proposition is completed by observing that

$$D_x = \begin{vmatrix} a+c & a+c-2b \\ a & 2a-b-c \end{vmatrix} = a^2 - c^2 + ba - bc = (a+c)(a-c) + b(a-c)$$

$= (a-c)(a+c+b)$ and $D_y = \begin{vmatrix} 2 & a+c \\ 1 & a \end{vmatrix} = a - c$.

4.2 Arithmetic operations on TFNs

It can be shown that the two general methods for defining arithmetic operations on FNs presented in Section 3 lead to the following simple rules for the *addition* and *subtraction* of TFNs:
Let A = (a, b, c) and B = (a_1, b_1, c_1) be two TFNs. Then

- The sum A + B = $(a+a_1, b+b_1, c+c_1)$.
- The difference A - B = A + (-B) = $(a-c_1, b-b_1, c-a_1)$, where –B = $(-c_1, -b_1, -a_1)$ is defined to be the *opposite* of B.

Obviously A + (-A) = (a-c, 0, c-a) $\neq$ O = (0, 0, 0), where the TFN O is defined by O(x) = 1, if x = 0 and O(x)=0, if x $\neq$ 0.
In other words, the opposite of a TFN, as well as the sum and the difference of two TFNs are also TFNs. On the contrary, the product and the quotient of two TFNs, although they are FNs, *they are not always TFNs*. However, in the special case where a, b, c, a_1, b_1, c_1 are in $\boldsymbol{R}^+$, it can be shown that the fuzzy operations of *multiplication* and *division* of TFNs can be *approximately performed* by the rules:

- The product A . B = (aa_1, bb_1, cc_1).
- The quotient A : B = A . B^{-1} = $(\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1})$, where B^{-1} = $(\frac{1}{a_1}, \frac{1}{b_1}, \frac{1}{c_1})$ is defined to be the *inverse* of B.

In other words, in $\boldsymbol{R}^+$ the inverse of a TFN, as well as the product and the division of two TFNs can be approximately considered to be TFNs too.
Further, one can define the following two *scalar operations*:

- k + A= (k+a, k+b, k+c), k $\in \boldsymbol{R}$
- kA = (ka, kb, kc), if k>0 and kA = (kc, kb, ka), if k<0.

4.3 Definition and defuzzification of a Trapezoidal Fuzzy Number (TpFN)

The membership function's graph of the TpFN (a, b, c, d), where $a< b < c < d$ are given real numbers, is represented in Figure 6 below:

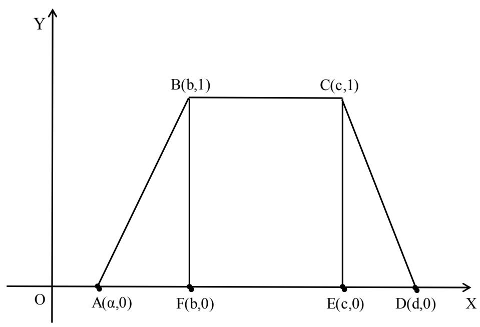

Figure 6: *Graph of the TpFN (a, b, c, d)*

We observe that its membership function *y=m(x)* is constantly 0 outside the interval [*a, d*], while its graph in this interval [*a, d*] is the union of three straight line segments forming a trapezoid with the X-axis. Therefore, its analytic definition is given as follows:

Definition 7: Let $a < b < c < d$ be given real numbers. Then the TpFN (*a, b, c, d*) is the FN with membership function:

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a} \quad , & x \in [a,b] \\ x = 1, \quad , & x \in [b,c] \\ \dfrac{d-x}{d-c}, & x \in [c,d] \\ 0, & x < a \;\; and \;\; x > d \end{cases}$$

Obviously the TFN (*a, b, c*) can be considered as a special case of the TpFN *(a, b, c, d)* with *b=c.*

The following proposition provides a *defuzzification* of a given TpFN with the COG technique:

Proposition 3: The coordinates (*X*, *Y*) of the COG of the graph of the TpFN (*a, b, c, d*) are calculated by the formulas $X = \dfrac{c^2+d^2-a^2-b^2+dc-ba}{3(c+d-a-b)}$, $Y = \dfrac{2c+d-a-2b}{3(c+d-a-b)}$.

Proof: We divide the trapezoid forming the graph of the TpFN (*a, b, c, d*) in three parts, two triangles and one rectangle (Figure 6). The coordinates of the three vertices of the triangle ABE are (*a,* 0), (*b,* 1) and (*b,* 0) respectively, therefore, by Proposition 2, the COG of this triangle is the point C_1 ($\dfrac{a+2b}{3}$, $\dfrac{1}{3}$). Similarly one finds that the COG of the triangle FCD is the point C_2 ($\frac{d+2c}{3}$, $\frac{1}{3}$). Also, it is easy to check that the COG of the rectangle BCFE is the point C_3 ($\frac{b+c}{2}$, $\frac{1}{2}$). Further, the areas of the two triangles are equal to $S_1 = \frac{b-a}{2}$ and $S_2 = \frac{d-c}{2}$ respectively, while the area of the rectangle is equal to S_3 = c- b.

It is well known then (e.g. see [37]) that the coordinates of the COG of the trapezoid, being the resultant of the COGs C_i (x_i, y_i), for i=1, 2, 3, are calculated by the formulas $X = \frac{1}{S}\sum_{i=1}^{3} S_i x_i$,

$Y = \frac{1}{S}\sum_{i=1}^{3} S_i y_i$ (5), where $S = S_1 + S_2 + S_3 = \frac{c+d-b-a}{2}$ is the area of the trapezoid.

The proof of the Proposition is completed by replacing the above values of S, S_i, x_i and y_i, i = 1, 2, 3, in formulas (5) and by performing the corresponding algebraic operations.

4.4 Arithmetic operations on TpFNs

It can be shown that the two general methods for performing the basic arithmetic operations between FNs (see Section 3) lead to the following simple rules for the *addition* and *subtraction* of TpFNs:

Let A = (a_1, a_2, a_3, a_4) and B = ($b_1, b_2, b_3, b4$) be two TFNs. Then

- The sum A + B = ($a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4$).
- The difference A - B = A + (-B) = ($a_1-b_4, a_2-b_3, a_3-b_2, a_4-b_1$) , where –B = ($-b_4, -b_3, -b_2, -b_1$) is defined to be the *opposite* of B.

In other words, the opposite of a TpFN, as well as the sum and the difference of two TpFNs are also TpFNs.

On the contrary, the product and the quotient of two TFNs, although they are FNs, *they are not always TpFNs*, apart from some special cases, or in terms of suitable approximating formulas.

Further, one can define the following two *scalar operations*:

- k + A= (k+a_1, k+a_2, k+a_3, $k+a_4$), $k \in \boldsymbol{R}$
- kA = (ka_1, ka_2, ka_3, ka_4), if k>0 and kA = (ka_4, ka_3, ka_2, ka_1), if k<0.

We close this section with the following definition, which will be used later in this paper for assessing MM skills with the help of TpFNs (TFNs):

Definition 8: Let A_i, i = 1, 2, …, n be TpFNs (TFNs), where n is a non negative integer, $n \geq 2$. Then we define the *mean value* of the above TpFNs (TFNs) to be the TpFN (TFN)

$$A = \frac{1}{n}(A_1 + A_2 + \ldots + A_n).$$

5. Assessing students' model building skills

In this section we use the TFNs and TpFNs as a tool for assessing MM skills. This new fuzzy assessment approach is validated by comparing the results obtained with the corresponding results of other assessment methods of the traditional (mean values, GPA index) and of the fuzzy logic (TpFAM) already utilized by the present authors in earlier works. All these are materialized through the following three examples:

5.1 Example

Three MM problems (see Appendix) were given for solution to the students of two different Departments of the School of Management and Economics of the Graduate Technological Educational Institute (T. E. I.) of Western Greece at their common progress exam of the course “Mathematics for Economists I”. The students achieved the following scores (in a climax from 0 to 100):

First Department (D_1): 100(2 times), 99(3), 98(5), 95(8), 94(7), 93(1), 92 (6), 90(5), 89(3), 88(7), 85(13), 82(6), 80(14), 79(8), 78(6), 76(3), 75(3), 74(3), 73(1), 72(5), 70(4), 68(2), 63(2), 60(3), 59(5), 58(1), 57(2), 56(3), 55(4), 54(2), 53(1), 52(2), 51(2), 50(8), 48(7), 45(8), 42(1), 40(3), 35(1).

Second Department (D_2) *:* 100(1), 99(2), 98(3), 97(4), 95(9), 92(4), 91(2), 90(3), 88(6), 85(26), 82(18), 80(29), 78(11), 75(32), 70(17), 64(12), 60(16), 58(19), 56(3), 55(6), 50(17), 45(9), 40(6).

The *assessment* of the above data will be performed with the following methods:

I) *Mean values:* Calculating the means of the above scores in the classical way one approximately finds that $\frac{12314}{170} \approx 72.44$ for D_1 and $\frac{18369}{255} \approx 72.04$ for D_2 respectively, showing that D_1 demonstrated a slightly better *mean performance* than D_2.

II) *GPA index:* Summarizing the student scores with respect to the grades A, B, C, D and F, one forms the following Table:

Table 1: Students’ performance in terms of the linguistic grades

Grade	D_1	D_2
A	60	60
B	40	90
C	20	45
D	30	45
F	20	15
Total	170	255

From Table 1 one easily calculates the percentages of the students of D_1 who obtained the grades F, D, C, B and A respectively, which are: $y_1 = y_3 = \frac{2}{17}$, $y_2 = \frac{3}{17}$, $y_4 = \frac{4}{17}$, $y_5 = \frac{6}{17}$.

Replacing these values in formula (2) of Section 2 one finds that the GPA index for D_1 is

GPA = $\frac{30+2*20+3*40+4*60}{170}=\frac{430}{170}\approx 2.529$ and similarly the same value for D_2. This means that both Departments demonstrated the same *quality performance*, which can be characterized as more than satisfactory, since the value 2.529 found for the GPA index is greater than the half of its maximal possible value (4:2=2).

III) *Application of the TpFAM:* Replacing the student percentages of D_1 in the first of formulas (1) of Section 2 one finds that the x-coordinate of the COG of the TpFAM's scheme for D_1 is equal to

$X_c = 7(\frac{2+2*3+3*2+4*4+5*6}{17})-2=\frac{386}{17}\approx$ 22.7. Working similarly one also finds the same value of X_c for D_2. Therefore, in order to compare the two Departments' performance one must also calculate the y-coordinates Y_c of the corresponding COGs. For this, replacing the values of the y_i, for i=1, 2, 3, 4, 5, in the second of formulas (1) one finds for D_1 that

$Y_c = \frac{3}{7}[(\frac{2}{17})^2+(\frac{3}{17})^2+(\frac{2}{17})^2+(\frac{4}{17})^2+(\frac{6}{17})^2]=\frac{207}{2023}$. In the same way one finds for D_2 the value

$Y_c = \frac{213}{2023}$. But $X_c \approx 22.7 > 19$, therefore, according to the comparison criterion for the TpFAM stated above, D_2 demonstrated a slightly better *quality performance* than D_1.

Notice also that in case of the ideal performance ($y_5 = 1$, $y_1 = y_2 = y_3 = y_4 = 0$) the first of formulas (1) -give that $X_c = 33$. Therefore, since the value of $X_c \approx 22.7$ found for both Departments is greater than the half of its value corresponding to the ideal performance (33:2 = 16.5), the quality performance of the two Departments can be characterized as more than satisfactory.

Finally we observe that, although according to the GPA index the two Departments demonstrated the same quality performance, the application of the TpFAM have shown that D_2 demonstrated a slightly better than D_1 quality performance. This is due to the fact that, as we have seen above, the TpFAM is more sensitive than the GPA index to the higher scores.

IV) *Use of the TFNs:* We assign to each linguistic grade a TFN (denoted, for simplicity, by the same letter) as follows: A= (85, 92.5, 100), B = (75, 79.5, 84), C = (60, 67, 74), D= (50, 54.5, 59) and F = (0, 24.5, 49). Namely, the left entry of each TFN is equal to the lower bound of the student scores assigned to the corresponding grade, its middle entry is equal to the mean value of these scores and its right entry is equal to the upper bound of these scores. In this way a TFN corresponds to each student assessing his (her) *individual performance*. The replacement of the linguistic grades by TFNs for the individual student assessment *has the advantage of determining numerically the scores assigned to each grade*, which, as we have already seen, are not standard, since they may slightly differ from case to case.

It is of worth to notice here that in an earlier work [34] an assessment of the student individual performance in problem solving was attempted by assigning to each student an *ordered triple of linguistic grades* characterizing his (her) performance in the three main steps of the problem solving process. In the same work it was shown that this approach is equivalent to the A. Jones method [11] of assessing a student's knowledge in terms of his (her) *fuzzy deviation with respect to the teacher*. The same approach can be also applied here for assessing the individual student MM skills. For example, the ordered triple (A, B, C) could be assigned to a student who demonstrated an excellent performance at the stage S_2 of mathematization, a very good performance at the stage S_3 of the solution of the model and a good performance at the stage S_4 of validation (see the Voskoglou's model for the MM processd presented in Section 1). However, in this way the overall performances of two different students are not always comparable. For example this happens with two students with profiles (A, B, C) and (B, B, B) respectively. Mathematically speaking, this approach defines a *partial order* only on the student individual performances; e.g. a student with profile (A, B, C) demonstrates a better performance than one with profile (B, B, D), etc. Further, this approach is laborious requiring an independent evaluation of the student performance at *each stage* of the MM process, which could not be practically possible, since the boundaries between these stages are not always clear.

After this parenthesis, let us return to the TFNs. We observe that in Table 1 we actually have 170 TFNs representing the individual performance of the students of D_1 and 255 TFNs representing the individual performance of the students of D_2. Therefore, it is logical to accept that the overall performance of each Department can be represented by the corresponding mean values of the above TFNs (see Definition 8). For simplifying our notation, let us denote the above means by the letter of the corresponding Department. Then, making the required straightforward calculations, one finds that

$$D_1 = \frac{1}{170} . (60A+40B+20C+30D+20F) \approx (63.53, 71.74, 83.47) \text{ and}$$

$$D_2 = \frac{1}{255} . (60A+90B+45C+45D+15F) \approx (65.88, 72.63, 79.53).$$

The above TFNs (mean values) give us the following information:

(i) The overall performance of D_1 is characterized numerically by a score lying in the interval [63.53, 83.47], i.e. from good (C) to very good (B). Similarly, the performance of D_2 is characterized by a score lying in the interval [65.88, 79.53].

(ii) The middle entries 71.74 and 72.63 of the two TFNs give a *rough approximation* (C=good) of the scores characterizing numerically the performance of D_1 and D_2 respectively.

But, why we have characterized the values of the middle entries of the TFNs D_1 and D_2 as been rough approximations of the corresponding scores? We observe first that these values *do not correspond to the mean performances* of the two Departments. In fact, calculating the means of the student scores in the classical way we found above (case I) the values 72.44 and 72.04 respectively, demonstrating a slightly better mean performance for D_1. Let us now go back to the definition of the TFNs A, B, C, D and F. The middle entries of these TFNs were chosen to be equal to the means of the scores assigned to each of the corresponding linguistic grades. Therefore the middle entries of the TFNS D_1 and D_2 are actually *equal to the mean values of these means*, which justifies completely the characterization "rough" given to them. Thus, the question is how one can compare the overall performances of the two Departments.

If the TFNS D_1 and D_2 are comparable (see Definition 5), the answer to this question is easy. For example, if $D_1 < D_2$, then D_2 demonstrates a better performance than D_1. Therefore, it becomes necessary to check if the TFNs D_1 and D_2 obtained above are comparable or not.
For this, by Proposition 1 one finds that the x-cuts of the two TFNs are
D_1^x = [63.53+8.21x, 83.47-11.73x] and D_2^x = [65.88+6.75x, 79.53-6.9x] respectively for all x in [0, 1]. Further, we have that $63.53+8.21x \leq 65.88+6.75x \Leftrightarrow 1.46x \leq 2.35 \Leftrightarrow x \leq 1.61$, which is true for all x in [0, 1]. But, $83.47\text{-}11.73x \leq 79.53\text{-}6.9x \Leftrightarrow 3.94 \leq 4.83x \Leftrightarrow 0.82 \leq x$, which does not hold for all x in [0, 1]. Therefore, according to Definition 5, the TFNs D_1 and D_2 are not comparable, which means that one *can not immediately decide which of the two Departments demonstrates the better performance*.
A good way to overcome this difficulty is to *defuzzify* the TFNs D_1 and D_2. For this, we apply the COG defuzzification technique. In fact, by Proposition 2, the COGs of the triangles forming the graphs of the TFNs D_1 and D_2 have x-coordinates equal to
$X = \frac{63.53+71.74+83.47}{3} \approx 72.91$ and $X' = \frac{65.88+72.63+79.53}{3} \approx 72.68$ respectively.
Observe now that the GOGs of the graphs of D_1 and D_2 lie in a rectangle with sides of length 100 units on the X-axis (student scores from 0 to 100) and one unit on the Y-axis (normal fuzzy sets). Therefore, *the nearer the x-coordinate of the COG to 100, the better the corresponding Department's performance*, Thus, since $X > X'$, D_1 demonstrates a better overall performance than D_2.

5.2 Example

Six different mathematics teachers train a group of five students of the Upper Secondary Education, who won at the final stage of the National Mathematical Competition, in order to participate in the International Mathematical Olympiad. In a preparatory test (solving of MM problems) during their training the students received the following scores (from 0-100) by their teachers: $\mathbf{S_1}$ (Student 1)**:** 43, 48, 49, 49, 50, 52, $\mathbf{S_2}$**:** 81, 83. 85, 88, 91, 95, $\mathbf{S_3}$**:** 76, 82, 89, 95, 95, 98, $\mathbf{S_4}$**:** 86, 86, 87, 87, 87, 88 and $\mathbf{S_5}$**:** 35, 40, 44, 52, 59, 62. The students' performance is characterized by the linguistic grades A, B, C, D and F introduced in Section 2.
In this example we shall apply the same methods for *assessing the student performance* with Example 5.1 and for the same purpose we shall also use the TpFNs:
I) *Mean values:* Calculating the mean values of the above scores separately for each student, one finds approximately the following *individual mean performances* for them (in parentheses we give the corresponding qualitative characterizations): $\mathbf{S_1}$**:** 48.5 (F), $\mathbf{S_2}$**:** 87.17 (A), $\mathbf{S_3}$**:** 89.17 (A), $\mathbf{S_4}$**:** 86.83 (A) and $\mathbf{S_5}$**:** 48.67(F). Also, calculating the mean value of the above individual means on finds that the student *overall mean performance* is approximately equal to 72.05, i.e. it can be characterized as very good (B).
II) *GPA index:* Inspecting the n=5*6=30 in total student scores one finds that 14 of them are characterized as excellent (A), 4 as very good (B), 1 as good (C), 4 as fair (D) and 7 are characterized as unsatisfactory. Therefore the corresponding percentages are $y_1 = \frac{7}{30}$,

$y_2 = \frac{4}{30}$, $y_3 = \frac{1}{30}$, $y_4 = \frac{4}{30}$ and $y_5 = \frac{14}{30}$. Replacing these values in formula (2) of Section 2 one finds that GPA = $\frac{74}{30} \approx 2.47$, i.e. the GPA index is greater than the half of its maximal value (4:2=2). Therefore the student *overall quality performance* is characterized as more than satisfactory.

III) *Application of the TpFAM:* Replacing the values of the y_i, for i = 1, 2, 3, 4, 5, in the first of formulas (1) of Section 2 one finds that the x-coordinate of the COG of the TpFAM's scheme is equal to X_c = 22.27, which is greater than the half of its value in the case of the ideal performance (33:2 = 16.5; see case III of Example 5.1)). Therefore the student *overall quality performance* is characterized, according to the TpFAM this time, as more than satisfactory.

IV) *Use of the TFNs:* We consider the TFNs A, B, C, D and F defined in case IV of Example 5.1. Observing the 5*6 = 30 in total student scores one finds that in the present Example we have 14 TFNs equal to A, 4 equal to B, 1 equal to C, 4 equal to D and 7 TFNs equal to F characterizing the student performance. The mean value of the above TFNs (Definition 8)is equal to M = $\frac{1}{30}$(14A + 4B + C + 4D + 7F) $\approx$(60.33, 68.98, 79.63). Therefore, the student overall performance lies in the interval [60.33, 79.63], i.e. it can be characterized from good (C) to very good (B). Further, a rough approximation of this performance is given by the score 68.98 (good)

(V) *Use of the TpFNs:* We assign to each student $\mathbf{S_i}$ a TpFN (denoted, for simplicity, with the same letter) as follows: $\mathbf{S_1}$ = (0, 43, 52, 59), $\mathbf{S_2}$ = (75, 81, 95, 100), $\mathbf{S_3}$ = (75, 76, 98, 100), $\mathbf{S_4}$ = (85, 86, 88, 100) and $\mathbf{S_5}$ = (0, 35, 62, 74).

Each of the above TpFNs characterizes the individual performance of the corresponding student in the form (*a, b, c, d*), where *a* is the lower bound of his/her performance with respect to the linguistic grades defined above, *b* and *c* are the lower and higher scores respectively assigned to the student by the teachers and *d* is the upper bound of his/her performance with respect to the linguistic grades.

Next, for assessing the overall players' performance in terms of the TpFNs, we calculate the mean value of the TpFNs $\mathbf{S_i}$, i =1, 2, 3, 4, 5 (Definition 8), which is equal to the TpFN

S = $\frac{1}{5}\sum_{i=1}^{5} S_i$ = (47, 64.2, 79, 86.6).

The above TbFN **S** gives us the following information:

(i) The students' performance, according to the scores assigned to them by their teachers, was fluctuated from unsatisfactory (a_1=47) to excellent (a_4=86.6).

(ii) The overall student mean performance is lying in the interval [a_2, a_3]=[64.2, 79], i.e. it can be characterized from good (C) to very good (B).

Example 5.3

Reconsider Example 5.2 and assume that the same six teachers marked the papers of a

second group of five students examined on the same test. Assume further that the overall performance of the second group was assessed as in Example 5.2 using the TPFNs (case V) and that the mean value of the corresponding TpFNs was found to be equal to
S΄ = (47.8, 65.3, 78.1, 85.9).
We shall use the COG technique for *comparing the two student group performances*. For this, applying Proposition 3 one finds that the x-coordinate of the COC of the trapezoid constituting the graph of the TpFN S is equal to

$$X = \frac{79^2 + (86.6)^2 - (64.2)^2 - 47^2 + 79*86.6 - 47*(64.2)}{3(79 + 86.6 - 47 - 64.2)} \approx 68.84 .$$

In the same way one finds that the x-coordinate of the graph of S΄ is approximately equal to 68.13. Therefore, using the same argument as that at the end of Example 5.1, one finds that the first group demonstrates a better overall performance.

6. Conclusions and discussion

In the present paper we used the TFNs/TpFNs as a tool for student assessment. The main advantage of this approach is that in case of *individual assessment* it leads numerical results, which are more indicative than the qualitative results obtained in earlier works by applying alternative fuzzy assessment methods. On the contrary, in case of *group assessment* this approach *initially leads to a linguistic characterization of the corresponding group's overall performance, which is not always sufficient for comparing the performances of two different groups*, as our fuzzy assessment methods applied in earlier works do. This is due to the fact that the inequality between TFNs/TpFNs defines on them a relation of partial order only. Therefore, in cases where our outputs are non comparable TFNs or TpFNs (in general) *some extra calculations are needed* in order to obtain the required comparison by defuzzifying these fuzzy outputs. This could be considered a disadvantage of this approach, although the extra calculations needed are very simple.
Further, our new method of using the TFNs/TpFNs for the assessment of MM skills is of general character, which means that it could be utilized for assessing other human (or machine) activities too. This is one of the main targets of our future research on the subject.

References

[1] Berry, J. and Davies, A. (1996), Written reports, Mathematics Learning and Assessment: Sharing Innovative Practices, in Haines, C. R. & Dunthornr (Eds.), London, Arnold.

[2] Blomhψj, M. and Jensen, T. H. (2003), Developing mathematical modelling competence: Conceptual clarification and educational planning, *Teaching Mathematics and its Applications*, 22, 123-129.

[3] Blum, W. & Leii, D. (2007), How do students and teachers deal with modelling problems? In C. Haines , P. Galbraith , W. Blum & S. Khan (Eds.): *Mathematical Modelling (ICTMA 12): Education, Engineering and Economics*, Chichester, Horwood Publishing, 222-231.

[4] Borroneo Ferri, R. (2007), Modelling problems from a cognitive perspective. In C. Chaines , P. Galbraith , W. Blum & S. Khan (Eds.): *Mathematical Modelling (ICTMA 12): Education, Engineering and Economics*, Chichester,

Horwood Publishing, 260-270.

[5] Edwards D. & Hamson M. J. (1996), *Mathematical Modelling Skills*, London, Macmillan.

[6] Doer, H.M. (2007), What knowledge do teachers need for teaching mathematics through applications and modeling? In W. Blum et al. (Eds.): *Modelling and Applications in Mathematics Education*, 69-78, NY: Springer.

[7] Galbraith, P.L. & Stillman, G. (2001), Assumptions and context: Pursuing their role in modeling activity. In J.F. Matos et al. (Eds.): *Modelling and Mathematics Education: Applications in Science and Technology (ICTMA 9)*, 300-310, Chichester: Horwood Publishing.

[8] Greefrath, G. (2007), *Modellieren lernen mitoffenen realitatsnahen Aufgaben*, Koln, Aulis Verlag.

[9] Haines, C. & Crouch, R. (2010), Remarks on a Modelling Cycle and Interpretation of Behaviours. In R.A. Lesh et al. (Eds.): *Modelling Students' Mathematical Modelling Competencies* (ICTMA 13), 145-154, Springer, US.

[10] Jelodar, Sh. S., Allahviranloo, T. Abbasbandy, Application of Fuzzy Expansion Methods for Solving Fuzzy Fredhom-Voltera Integral Equations of the First Kind, International Journal of Mathematical Modelling & Computations, 1(2013), 59-70

[11] Jones, A., Kaufmman, A. & Zimmerman, H. J. (1986), *Fuzzy Sets: Theory and Applications*, NATO ASI Series, Series C: Mathematical and Physical Sciences, Vol. 177, Dordrecht (Holland), Reidel Publishing Company.

[12] Kaufmann, A. & Gupta, M. (1991), *Introduction to Fuzzy Arithmetic*, New York, Van Nostrand Reinhold Company.

[13] Klir, G. J. & Folger, T. A. (1988), *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London.

[14] Pollak H. O. (1979), The interaction between Mathematics and other school subjects, *New Trends in Mathematics Teaching*, Volume IV, Paris: UNESKO.

[15] Polya, G. (1963), On learning, teaching and learning teaching, *American MathematicalMonthly,*70, 605-619.

[16] Sakawa, M. (1993), *Fuzzy Sets and Interactive Multiobjective Optimization,* Plenum press, NY and London.

[17] Sajadi, S., Amirfakhrian, M., Fuzzy K-Nearest Neighbor Method to Classify Data in A Closed Area, International Journal of Mathematical Modelling & Computations, 2(2013), 109-114

[18] Subbotin, I. Ya., Badkoobehi, H., Bilotckii, N. N. (2004), Application of fuzzy logic to learning assessment. *Didactics of Mathematics: Problems and Investigations*, 22, 38-41, Donetsk.

[19] Subbotin, I., Mossovar-Rahmani, F. & Bilotskii, N. (2011), Fuzzy logic and the concept of the Zone of Proximate Development, *Didactics of Mathematics: Problems and Investigations*, 36, 101-108.

[20] Subbotin, I. Ya, Voskoglou, M. Gr. (2011), Applications of fuzzy logic to Case-Based Reasoning, *International Journal of Applications of Fuzzy Sets and Artificial intelligence*, 1, 7-18.

[21] Subbotin, I. Ya., Bilotskii, N. N. (2014), Triangular fuzzy logic model for learning assessment, *Didactics of Mathematics: Problems and Investigations,* 41, 84-88, Donetsk.

[22] Subbotin, I. Ya, Voskoglou, M. Gr. (2014), Language, mathematics and critical thinking: The cross influence and cross enrichment, *Didactics of Mathematics: Problems and Investigations,* 41, 89- 94.

[23] Subbotin, I. Ya, Voskoglou, M. Gr. (2014), A Triangular Fuzzy Model for Assessing Critical Thinking Skills, *International Journal of Applications of Fuzzy Sets and Artificial intelligence*, 4, 173 -186.

[24] Subbotin, I. Ya. & Voskoglou, M. Gr. (2014), Fuzzy Assessment Methods, *Universal Journal of Applied Mathematics*, 2(9), 305-314.

[25] Taha, H. A. (1967), *Operations Research – An Introduction,* Second Edition, Collier Macmillan, N.Y.*,* London.

[26] Theodorou, J. (2010), *Introduction to Fuzzy Logic*, Thessaloniki (Greece), Tzolas Publications, (in Greek

language).

[27] van Broekhoven, E. & De Baets, B. (2006), Fast and accurate centre of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions, *Fuzzy Sets and Systems*, 157(7), 904-918.

[28] Voskoglou, M. Gr. (2007), A stochastic model for the modelling process, In C. Haines , P. Galbraith , W. Blum & S. Khan (Eds.): *Mathematical Modelling (ICTMA 12): Education, Engineering and Economics*, Chichester, Horwood Publishing, 149-157.

[29] Voskoglou, M. Gr. (2010), A Fuzzy Model for the Modelling Process. In Munteanu, V. et al (Eds.): *Recent Advances in Fuzzy Systems*, 44-49, WSEAS Press, Iasi, Romania.

[30] Voskoglou, M. Gr. (2010), Use of total possibilistic uncertainty as a measure of students' modeling capacities, *International Journal of Mathematical Education in Science and Technology*, 41(8), 1051-1060.

[31] Voskoglou, M. Gr. & Subbotin, I. Ya. (2012), Fuzzy Models for Analogical Reasoning, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, Vol. 2, 1-38.

[32] Voskoglou, M. Gr. (2013), Problem Solving, Fuzzy Logic and Computational Thinking, *Egyptian Computer Science Journal*, 37(1), 131-145.

[33] Voskoglou, M. Gr. & Subbotin, I. Ya. (2013), Dealing with the Fuzziness of Human Reasoning, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, Vol. 3, 91-106.

[34] Voskoglou, M. Gr. (2013), Assessing Students Individual Problem Solving Skills, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, 3, 39-49.

[35] Voskoglou, M. Gr. (2014), Assessing the Players' Performance in the Game of Bridge: A Fuzzy Logic Approach, *American Journal of Applied Mathematics and Statistics*, 2(3), 115-120.

[36] Zadeh, L. A. (1965), Fuzzy Sets, *Information and Control*, 8, 338-353.

[37] Wikipedia (2014), *Center of mass: A system of particle,* retrieved on October 10, 2014 from: http://en.wikipedia.org/wiki/Center_of_mass#A_system_of_particles .

Acknowledgement: The first author wishes to thank his colleague Dr. J. Theodorou, Professor of Mathematics at the Graduate Technological Educational Institute of Central Greece, for his useful ideas on FNs.

Appendix: List of MM problems used in Example 5.1

Problem 1: A car dealer has a mean annual demand of 250 cars, while he receives 30 new cars per month. The annual cost of storing a car is 100 euros and each time he makes a new order he pays an extra amount of 2200 euros for general expenses (transportation, insurance etc). The first cars of a new order arrive at the time when the last car of the previous order has been sold. How many cars must he order for achieving the minimum total cost?

Problem 2: The demand function $P(Q_d)=25-Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s)=2Q_s+1$ represents the prices at which different quantities Q_s of the same good will be supplied. When the market's equilibrium occurs at (Q_0, P_0), the producers who would supply at lower price than P_0 gain. Find the total gain to producers'.

Problem 3: Among all cylindrical towers having a total surface of 180π m^2, which one has the maximal volume?