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Numerical Approach to Solve Singular Integral Equations Using BPFs and Taylor Series Expansion

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Abstract. In this paper, we give a numerical approach for approximating the solution of second kind Volterra integral equation with Logarithmic kernel using Block Pulse Functions (BPFs) and Taylor series expansion. Also, error analysis shows efficiency and applicability of the presented method. Finally, some numerical examples with exact solution are given.

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1. Introduction

In this paper we present a method based on the use of Taylor series expansion and BPFs for solving a special case of singular Volterra integral equations of the second kind with logarithmic singularities defined as follows

$$y(x) = f(x) - \int_0^x k(x,t)\, y(t) \ln|x-t| dt, \quad 0 \leqslant t < x \leqslant 1, \tag{1}$$

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where, $f(x)$ and $k(x,t)$ are known and $y(x)$ unknown functions. We assume that Eq. (1) has a unique solution to be determined. Singular integral equations are usually difficult to solve analytically so it is necessary to obtain the approximate solution. Several numerical methods have been proposed for approximating the solution of singular integral equations. Discrete Galerkin method for first kind integral equations with logarithmic kernel is presented in [2]. A Galerkin solution for regularized Cauchy singular integro differential equation is discussed in [5]. Radial basis functions are applied for numerical solution of weakly singular Volterra equations [6]. Taylor series expansion method for second kind Volterra integral equations with convolution kernel was introduced in [8]. Taylor series expansion method for Integro-Differential-Difference Equation was used in [1]. Fractional calculus for solving system of generalized Abel integral equations was proposed in [9]. A numerical method based on BPFs and Taylor series expansion are considered for approximating the solution of weakly singular Volterra integral equations in [11]. Legendre wavelets were used for Abel integral equations in [12]. This paper is organized as follows:

In section 2, we introduce BPFs and properties of these functions and we show how to approximate with BPFs. Also Taylor series expansion of a function is given. In section 3, BPFs together with the Taylor series expansion are applied to approximate the singular integral equations. In section 4, error in BPFs approximation is presented. Finally, in section 5, two numerical examples are solved by this method. Section 6 concludes this paper with a brief summary.

2. Materials and Methods

In this section, we define an m-set of BPFs over the interval $[0,T)$ as:

$$B_{i,m}(t) = \begin{cases} 1, & \frac{(i-1)T}{m} \leq t < \frac{iT}{m} \; for\ all\ i = 1,2,\ldots,m \\ 0, & elsewhere \end{cases} \tag{2}$$

with a positive integer value for m. Also, $B_{i,m}$ is the i-th BPF. In this paper, it is assumed that $T = 1$, so BPFs are defined over $[0,1]$. BPFs, a set of orthogonal functions with piecewise constant values, are studied and applied extensively as a useful tool in the analysis, synthesis, identification and other problems of control and systems science. In comparison with other basis functions, BPFs can lead more easily to recursive computations to solve concrete problems [7], also BPFs has proved to be the most fundamental [3, 4]. The complete details for BPFs are given in [7, 10]. There are some properties for BPFs, the most important of which are disjointness, orthogonality, and completeness. The disjointness property can be clearly obtained from the definition of BPFs:

$$B_{i,m}(t).B_{j,m}(t) = \begin{cases} 0, & i \neq j \\ B_{i,m}(t), & i = j \end{cases} \tag{3}$$

where "." refers to the ordinary multiplication of real numbers and $i,j = 1,2,\ldots,m$. The other property is orthogonality:

$$< B_{i,m}(t), B_{j,m}(t) > = \begin{cases} 0, & i \neq j \\ \frac{1}{m}, & i = j \end{cases} \tag{4}$$

The third property, completeness, will be given after proposition 2.2

PROPOSITION 2.1 *For each continuous function $y(t)$ defined over the interval $[0,1)$ we have*

$$y(t) = \lim_{m\to\infty} \sum_{i=1}^{m} b_{i,m} B_{i,m}(t),$$

where the convergence is in $L^2[0,1)$ and

$$b_{i,m} = m < y(t), B_{i,m}(t) >= m \int_0^1 y\,(t)\, B_{i,m}\,(t)\, dt.$$

Proof We have to show that for each $\varepsilon > 0$

$$||y(t) - \sum_{i=1}^{m} b_{i,m} B_{i,m}(t)|| < \varepsilon,$$

for sufficiently large m. We recall that for $i < j$ we have $\left[\frac{i-1}{m}, \frac{i}{m}\right) \cap \left[\frac{j-1}{m}, \frac{j}{m}\right) = \{\}$, so $\int_0^1 B_{i,m}(t) B_{j,m}(t) dt = 0$. Now by using (3) and (4) we may proceed as follows:

$$\begin{aligned}
||y(t) - \sum_{i=1}^{m} b_{i,m} B_{i,m}(t)||^2 &=< y(t) - \sum_{i=1}^{m} b_{i,m} B_{i,m}(t), y(t) - \sum_{i=1}^{m} b_{i,m} B_{i,m}(t) > \\
&= \int_0^1 y^2(t) dt + \sum_{i=1}^{m} b_{i,m}^2 \int_0^1 B_{i,m}^2(t) dt - 2 \sum_{i=1}^{m} b_{i,m} \int_0^1 y\,(t)\, B_{i,m}\,(t)\, dt \\
&+ 2 \sum_{1 \leqslant i < j \leqslant m} b_{i,m} b_{j,m} \int_0^1 B_{i,m}(t) B_{j,m}(t) dt \\
&= \int_0^1 y^2(t) dt + \sum_{i=1}^{m} b_{i,m}^2 \int_0^1 B_{i,m}(t) dt - 2 \sum_{i=1}^{m} b_{i,m} \left(\frac{b_{i,m}}{m}\right) + 2(0) \\
&= \int_0^1 y^2(t) dt + \sum_{i=1}^{m} \frac{b_{i,m}^2}{m} - 2 \sum_{i=1}^{m} \frac{b_{i,m}^2}{m} \\
&= \int_0^1 y^2(t) dt - \frac{1}{m} \sum_{i=1}^{m} b_{i,m}^2. \qquad (5)
\end{aligned}$$

Now using the mean value theorem for integrals, we have

$$\begin{aligned} b_{i,m}^2 &= m^2 \left(\int_0^1 y\,(t)\, B_{i,m}\,(t)\, dt \right)^2 \\ &= m^2 \left(\int_{\frac{i-1}{m}}^{\frac{i}{m}} y\,(t)\, dt \right)^2 \\ &= m^2 \left(\frac{i}{m} - \frac{i-1}{m} \right)^2 y^2(t_i) \\ &= y^2(t_i), \quad \left(\frac{i-1}{m} \leqslant t_i \leqslant \frac{i}{m} \right). \end{aligned} \tag{6}$$

By substituting (6) into (5) we obtain

$$||y(t) - \sum_{i=1}^{m} b_{i,m} B_{i,m}(t)||^2 = \int_0^1 y^2(t)dt - \frac{1}{m} \sum_{i=1}^{m} y^2(t_i). \tag{7}$$

But from elementary calculus, we know that for each integrable function f on $[0, 1]$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} f(t_i) = \int_0^1 f(t)dt,$$

for arbitrarily chosen $t_i \in [\frac{i-1}{m}, \frac{i}{m}]$, which means

$$|\int_0^1 f(t)dt - \frac{1}{m} \sum_{i=1}^{m} f(t_i)| < \varepsilon^2,$$

for sufficiently large m. So, replacing f by y^2 and from (7), we obtain

$$||y(t) - \sum_{i=1}^{m} b_{i,m} B_{i,m}(t)|| < \varepsilon.$$

■

Proposition 2.2 *For each continuous function $k(x, t)$ defined over the interval $[0, 1] \times [0, 1]$ we have*

$$\lim_{m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} k_{i,j,m}\ B_{i,m}(x)\ B_{j,m}(t) = k(x, t)$$

where the convergence is in $L^2[0, 1)^2$ and

$$k_{i,j,m} = m^2 \int_0^1 \int_0^1 k\,(x, t)\, B_{i,m}\,(x)\, B_{j,m}\,(t)\, dxdt.$$

Proof Similarly to the proof of proposition 2.1 we may proceed as follows:

$$||\sum_{i=1}^{m}\sum_{j=1}^{m}k_{i,j,m}B_{i,m}(x)B_{j,m}(t)-k(x,t)||^2 \tag{8}$$

$$=<\sum_{i=1}^{m}\sum_{j=1}^{m}k_{i,j,m}B_{i,m}(x)B_{j,m}(t)-k(x,t),\sum_{i=1}^{m}\sum_{j=1}^{m}k_{i,j,m}B_{i,m}(x)B_{j,m}(t)-k(x,t)>$$

$$=<k(x,t),k(x,t)>+\sum_{i=1}^{m}\sum_{j=1}^{m}k_{i,j,m}^2\int_0^1\int_0^1 B_{i,m}^2(x)\,B_{j,m}^2(t)\,dxdt$$

$$-2\sum_{i=1}^{m}\sum_{j=1}^{m}k_{i,j,m}\int_0^1\int_0^1 k(x,t)\,B_{i,m}(x)\,B_{j,m}(t)\,dxdt$$

$$+\sum_{(i,j)\neq(i',j')}k_{i,j,m}k_{i',j',m}\int_0^1\int_0^1 B_{i,m}(x)B_{j,m}(t)B_{i',m}(x)B_{j',m}(t)dxdt$$

$$=<k(x,t),k(x,t)>+\sum_{i=1}^{m}\sum_{j=1}^{m}k_{i,j,m}^2\int_0^1\int_0^1 B_{i,m}(x)\,B_{j,m}(t)\,dxdt$$

$$-2\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{k_{i,j,m}^2}{m^2}$$

$$=\int_0^1\int_0^1 k^2(x,t)dxdt+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{k_{i,j,m}^2}{m^2}-2\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{k_{i,j,m}^2}{m^2}$$

$$=\int_0^1\int_0^1 k^2(x,t)dxdt-\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{k_{i,j,m}^2}{m^2}$$

$$=\int_0^1\int_0^1 k^2(x,t)dxdt-\sum_{i=1}^{m}\sum_{j=1}^{m}m^2\left(\int_0^1\int_0^1 k(x,t)B_{i,m}(x)\,B_{j,m}(t)\,dxdt\right)^2$$

$$=\int_0^1\int_0^1 k^2(x,t)dxdt-\sum_{i=1}^{m}\sum_{j=1}^{m}m^2\left(\int_{\frac{j-1}{m}}^{\frac{j}{m}}\int_{\frac{i-1}{m}}^{\frac{i}{m}} k(x,t)dxdt\right)^2$$

$$=\int_0^1\int_0^1 k^2(x,t)dxdt-\sum_{i=1}^{m}\sum_{j=1}^{m}m^2\frac{k^2(x_{ij},t_{ij})}{m^4}$$

$$=\int_0^1\int_0^1 k^2(x,t)dxdt-\frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}k^2(x_{ij},t_{ij}), \tag{9}$$

where $(x_{ij},t_{ij})\in[\frac{i-1}{m},\frac{i}{m}]\times[\frac{j-1}{m},\frac{j}{m}]$. In the above arguments, we used the mean value theorem for double integrals. But for an integrable function f on $[0,1]\times[0,1]$ we have

$$\lim_{m\to\infty}\frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}f(x_{ij},t_{ij})=\int_0^1\int_0^1 f(x,t)dxdt,$$

for arbitrarily chosen $(x_{ij}, t_{ij}) \in [\frac{i-1}{m}, \frac{i}{m}] \times [\frac{j-1}{m}, \frac{j}{m}]$ which means

$$|\int_0^1 \int_0^1 f(x,t)dxdt - \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} f(x_{ij}, t_{ij})| < \varepsilon^2,$$

for sufficiently large m. So, replacing f by k^2 and from (9), we obtain

$$||\sum_{i=1}^{m}\sum_{j=1}^{m} k_{i,j,m}B_{i,m}(x)B_{j,m}(t) - k(x,t)|| < \varepsilon.$$

■

Now we give the third property for BPFs, completeness
for every $y \in L^2[0,1]$, when m approaches ∞, a Parseval-like identity holds

$$\int_0^1 y^2(t)\, dt = \lim_{m\to\infty} \sum_{i=1}^{m} b_{i,m}^2 \, \|B_{i,m}(t)\|^2,$$

where, $b_{i,m} = m < y(t), B_{i,m}(t) > = m \int_0^1 y(t)\, B_{i,m}(t)\, dt$. The proof of this equality is similar to the proof of proposition 2.1 and is omitted. Now using materials mentioned in proposition 2.1 and proposition 2.2, we can expand functions $y(t)$ and $k(x,t)$ in terms of BPFs. So, a function $y(t)$ defined and continuous over the interval $[0,1]$ may be expanded as

$$y(t) = \sum_{i=1}^{\infty} b_{i,m}B_{i,m}(t). \tag{10}$$

In practice, only m terms of (10) are considered, where m is a power of 2, that is

$$y(t) \simeq y_m(t) = \sum_{i=1}^{m} b_{i,m}B_{i,m}(t), \tag{11}$$

or in matrix form

$$y(t) \simeq y_m(t) = \mathbf{b}^t\mathbf{B}(t), \tag{12}$$

where, $\mathbf{b} = [b_{1,m}, b_{2,m}, \ldots, b_{m,m}]^t$ and $\mathbf{B}(t) = [B_{1,m}(t), B_{2,m}(t), \ldots, B_{m,m}(t)]^t$. Also $k(x,t) \in L^2[0,1)^2$ may be approximated as

$$k(x,t) \simeq \sum_{i=1}^{m}\sum_{j=1}^{m} k_{i,j,m}B_{i,m}(x)B_{j,m}(t),$$

or in matrix form

$$k(x,t) \simeq \mathbf{B}^t(x)\, \mathbf{k}\mathbf{B}(t), \tag{13}$$

where, $\mathbf{k} = [k_{i,j,m}]_{1\leqslant i,j\leq m}$ and $k_{i,j,m} = m^2 \int_0^1 \int_0^1 k(x,t)\, B_{i,m}(x)\, B_{j,m}(t)\, dxdt$. Now we define the function

$$g(t) = \ln|x-t|, \quad 0 \leq t < x \leq 1.$$

Taylor series expansion of $g(t)$ based on expansion about the point $t = 0$ leads to

$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\, t^n}{n!}. \tag{14}$$

It is straight forward to show that

$$g^{(n)}(t) = \frac{-(n-1)!}{(x-t)^n}, \quad n = 1, 2, \ldots,$$

now,

$$g^{(n)}(0) = \frac{-(n-1)!}{x^n}, \quad n = 1, 2, \ldots, \tag{15}$$

and substituting (15) into (14) leads to

$$\begin{aligned} g(t) &= \frac{g(0)\, t^0}{0!} + \sum_{n=1}^{\infty} \frac{g^{(n)}(0)\, t^n}{n!} \\ &= \ln|x| - \sum_{n=1}^{\infty} \frac{t^n}{n x^n}. \end{aligned} \tag{16}$$

3. Singular Integral Equations

Now we consider the following second kind singular Volterra integral equation with logarithmic singularities

$$y(x) = f(x) - \int_0^x k(x,t)\ y(t)\ \ln|x-t|\, dt, \quad 0 \leqslant t < x \leqslant 1. \tag{17}$$

By substituting (12)-(13) and (16) into (17) we obtain:

$$\begin{aligned} \mathbf{b}^t\mathbf{B}(x) &= f(x) - \int_0^x \mathbf{B}^t(x)\,\mathbf{k}\mathbf{B}(t)\,\mathbf{B}^t(t)\left[\ln|x| - \sum_{n=1}^{\infty} \frac{t^n}{n x^n}\right] dt\mathbf{b} \\ &= f(x) - \mathbf{B}^t(x)\mathbf{k}\ln|x| \int_0^x \mathbf{B}(t)\mathbf{B}^t(t) dt\mathbf{b} + \sum_{n=1}^{\infty} \frac{1}{n x^n} \mathbf{B}^t(x)\,\mathbf{k} \int_0^x \mathbf{B}(t)\,\mathbf{B}^t(t)\, t^n dt\mathbf{b} \\ &= f(x) - \mathbf{B}^t(x)\,\mathbf{k}\ln|x|\,\mathbf{m}_0(x)\,\mathbf{b} + \sum_{n=1}^{\infty} \frac{1}{n x^n}\ \mathbf{B}^t(x)\,\mathbf{k}\ \mathbf{m}_n(x)\,\mathbf{b}, \end{aligned} \tag{18}$$

(The reason for changing the order of integration and the infinite sum will be given in the appendix) where

$$\mathbf{m}_n(x) = \int_0^x \mathbf{B}(t)\,\mathbf{B}^t(t)\ \ t^n dt,$$

consequently

$$\mathbf{m}_0(x) = \int_0^x \mathbf{B}(t)\,\mathbf{B}^t(t)\,dt.$$

By evaluating (18) at the collocation points $x_{j,m} = \frac{j}{m}$ $\quad j = 1, 2, \dots, m$, we obtain

$$\mathbf{b}^t\mathbf{B}(x_{j,m}) = f(x_{j,m}) - \mathbf{B}^t(x_{j,m})\,\mathbf{k}\ln|x_{j,m}|\,\mathbf{m}_0(x_{j,m})\,\mathbf{b} + \sum_{n=1}^{\infty}\frac{1}{nx_{j,m}^n}\mathbf{B}^t(x_{j,m})\,\mathbf{k}\mathbf{m}_n(x_{j,m})\,\mathbf{b}, \tag{19}$$

but $\mathbf{B}(x_{j,m}) = \mathbf{e}_{j,m}$, where $\mathbf{e}_{j,m}$ is the j-th column of the identity matrix of order m, so from (18) we have

$$b_{j,m} = f(x_{j,m}) - \mathbf{e}_{j,m}^t\mathbf{k}\ln|x_{j,m}|\,\mathbf{m}_0(x_{j,m})\,\mathbf{b} + \sum_{n=1}^{\infty}\frac{1}{nx_{j,m}^n}\mathbf{e}_{j,m}^t\mathbf{k}\mathbf{m}_n(x_{j,m})\ \mathbf{b},\ j = 1, 2, \dots, m. \tag{20}$$

Using (3) gives

$$\mathbf{B}(t)\mathbf{B}^t(t) = diag[B_{1,m}(t), B_{2,m}(t), ..., B_{m,m}(t)],$$

and by using the definition of BPFs we have
$0 \le t < \frac{1}{m}$ implies that $B_{1,m}(t) = 1$ and $B_{i,m}(t) = 0$ for $i = 2, \dots, m$
$\frac{1}{m} \le t < \frac{2}{m}$ implies that $B_{2,m}(t) = 1$ and $B_{i,m}(t) = 0$ for $i = 1, \dots, m, i \ne 2$
$\vdots$
$\frac{m-1}{m} \le t < 1$ implies that $B_{m,m}(t) = 1$ and $B_{i,m}(t) = 0$ for $i = 1, \dots, m-1$.
Therefore for evaluating $\mathbf{m}_n(x)$ and $\mathbf{m}_0(x)$ at the collocation points $x_{j,m} = \frac{j}{m}$, $\quad j = 1, 2, \dots, m$, we may proceed as follows

$$\begin{aligned}
\mathbf{m}_n(x_{j,m}) &= \int_0^{x_{j,m}} \mathbf{B}(t)\mathbf{B}^t(t)t^n dt \\
&= \int_0^{\frac{j}{m}} \mathbf{B}(t)\mathbf{B}^t(t)t^n dt \\
&= \sum_{i=1}^{j}\int_{\frac{i-1}{m}}^{\frac{i}{m}} diag[B_{1,m}(t), B_{2,m}(t), ..., B_{m,m}(t)]t^n dt \\
&= \frac{1}{(n+1)m^{n+1}}diag[1, 2^{n+1}-1, 3^{n+1}-2^{n+1}, \dots, j^{n+1}-(j-1)^{n+1}, 0, \dots, 0] \\
&= \frac{1}{(n+1)m^{n+1}}\mathbf{d}^{(j,n)},
\end{aligned} \tag{21}$$

where the diagonal matrix $\mathbf{d}^{(j,n)}$, $\quad j = 1, 2, ..., m$ is defined as:

$$\mathbf{d}_{pq}^{(j,n)} = \begin{cases} p^{n+1} - (p-1)^{n+1}, & p = q = 1, 2, \dots, j, \\ 0, & p = q = j+1, \dots, m, \\ 0, & p \ne q. \end{cases}$$

Consequently,

$$\begin{aligned}\mathbf{m}_0(x_{j,m}) &= \int_0^{x_{j,m}} \mathbf{B}(t)\mathbf{B}^t(t)dt \\ &= \int_0^{\frac{j}{m}} \mathbf{B}(t)\mathbf{B}^t(t)dt \\ &= \frac{1}{m}\mathbf{d}^{(j,0)}, \end{aligned} \tag{22}$$

where the definition of the diagonal matrix $\mathbf{d}^{(j,0)}$, $j = 1, 2, ..., m$, is the same definition of the diagonal matrix $\mathbf{d}^{(j,n)}$, $j = 1, 2, ..., m$, with $n = 0$. Substituting (21)-(22) into (20) and using the fact that $x_{j,m} = \frac{j}{m}$ we obtain

$$b_{j,m} = f(x_{j,m}) - \frac{1}{m}\ln|x_{j,m}|\mathbf{e}^t_{j,m}\mathbf{k}\,\mathbf{d}^{(j,0)}\mathbf{b} + \frac{1}{m}\mathbf{e}^t_{j,m}\,\mathbf{k}\sum_{n=1}^{\infty}\frac{1}{n(n+1)j^n}\mathbf{d}^{(j,n)}\mathbf{b}, \quad j = 1, 2, \ldots, m. \tag{23}$$

If we define

$$\mathbf{s}^j = \sum_{n=1}^{\infty}\frac{1}{n(n+1)j^n}\mathbf{d}^{(j,n)},$$

by definition of the diagonal matrix $\mathbf{d}^{(j,n)}$ we can write

$$\begin{aligned}\mathbf{s}^j &= \sum_{n=1}^{\infty}\frac{1}{n(n+1)j^n}\mathbf{d}^{(j,n)} \\ &= \sum_{n=1}^{\infty}\frac{1}{n(n+1)j^n} diag[1, 2^{n+1} - 1, 3^{n+1} - 2^{n+1}, \ldots, j^{n+1} - (j-1)^{n+1}, 0, \ldots, 0] \\ &= diag[\sum_{n=1}^{\infty}\frac{1}{n(n+1)j^n}, \sum_{n=1}^{\infty}\frac{2^{n+1}-1}{n(n+1)j^n}, \ldots, \sum_{n=1}^{\infty}\frac{j^{n+1}-(j-1)^{n+1}}{n(n+1)j^n}, 0, \ldots, 0] \\ &= diag[(j-1)\ln(\frac{j-1}{j}) + 1, (j-2)\ln(\frac{j-2}{j}) - (j-1)\ln(\frac{j-1}{j}) + 1, \ldots, \\ &\quad (j-i)\ln(\frac{j-i}{j}) - (j-i+1)\ln(\frac{j-i+1}{j}) + 1, 0, \ldots, 0],\end{aligned}$$

because for $i = 1, 2, \ldots, j$, we have

$$\sum_{n=1}^{\infty}\frac{i^{n+1}-(i-1)^{n+1}}{n(n+1)j^n} = (j-i)\ln(\frac{j-i}{j}) - (j-i+1)\ln(\frac{j-i+1}{j}) + 1.$$

Substituting $\mathbf{s}^j$ into (23) implies that

$$b_{j,m} = f(x_{j,m}) - \frac{1}{m}\ln|x_{j,m}|\mathbf{e}^t_{j,m}\mathbf{k}\,\mathbf{d}^{(j,0)}\mathbf{b} + \frac{1}{m}\mathbf{e}^t_{j,m}\,\mathbf{k}\mathbf{s}^j\mathbf{b}, \quad j = 1, 2, \ldots, m. \tag{24}$$

Equation (24) is a linear system of algebraic equations and can be solved for $b_{1,m}, b_{2,m}, \ldots, b_{m,m}$, so the desired approximation for $y(t)$ may be obtained by $y_m(t)$

as

$$y\,(t) \simeq y_m\,(t) = \sum_{i=1}^{m} b_{i,m}\; B_{i,m}\,(t).$$

4. Error in BPFs Approximation

If we assume that $y(t)$ is a differentiable function with bounded first derivative on (0,1), that is,

$$\exists M > 0; \quad \forall t \in (0,1): \quad |y'(t)| \leqslant M,$$

the representation error when $y(t)$ is represented in a series of BPFs over every subinterval $[\frac{i-1}{m}, \frac{i}{m})$, is

$$\begin{aligned} e_{i,m}(t) &= b_{i,m}B_{i,m}(t) - y(t) \\ &= b_{i,m} - y(t). \end{aligned}$$

It can be shown that

$$\|e_{i,m}\|^2 \leqslant \frac{1}{m^3}M^2,$$

this leads to

$$\|e\|^2 \leqslant \frac{1}{m^2}M^2,$$

where, $e(t) = y_m(t) - y(t)$. For more details see [11].
Now for estimating M, we assume that y' is continuous on $[0,1]$. Since $y'(t)$ is continuous and bounded on $[0,1]$, $y'(t) \in L^2[0,1)$, thus $y'(t)$ may be approximated as

$$y'(t) \simeq \sum_{n=1}^{m} c_{n,m}B_{n,m}(t) \tag{25}$$

or in matrix form

$$y'(t) \simeq \mathbf{c}^t\mathbf{B}(t), \tag{26}$$

where, $\mathbf{c} = [c_{1,m}c_{2,m}, \ldots, c_{m,m}]^t$ and $c_{n,m} = m < y'(t), B_{n,m}(t) >, n = 1, 2, \ldots, m$. Integrating (25) leads to

$$y(t) \simeq \sum_{n=1}^{m} c_{n,m}p_{n,m}(t) + y(0), \tag{27}$$

where, $p_{n,m}(t) = \int_0^t B_{n,m}(x)dx$, $t \in [0,1]$. Now we define

$$t_{j,m} = \frac{j - 0.01}{m}, \quad j = 1, 2, \ldots, m,$$

by evaluating (27) at the points $t_{j,m}$ we have

$$y(t_{j,m}) - y(0) \simeq \sum_{n=1}^{m} c_{n,m} p_{n,m}(t_{j,m}), \quad j = 1, 2, \ldots, m, \tag{28}$$

where, $p_{n,m}(t_{j,m}) = \int_0^{t_{j,m}} B_{n,m}(x)dx$. If we write (28) in the matrix form we get

$$\mathbf{y} - \mathbf{y}(0) \simeq \mathbf{Pc}, \tag{29}$$

where, $\mathbf{P} = [p_{n,m}(t_{j,m})]_{1 \leqslant n \leqslant m, 1 \leqslant j \leqslant m}$, $\mathbf{c} = [c_{1,m}, c_{2,m}, \ldots, c_{m,m}]^t$, $\mathbf{y} = [y(t_{1,m}), y(t_{2,m}), \ldots, y(t_{m,m})]^t$ and $\mathbf{y}(0) = [y(0), y(0), \ldots, y(0)]^t_{1 \times m}$. By solving the linear system (29) we can find vector $\mathbf{c}$, so from (26), $y'(t)$ can be calculated for each $t \in [0, 1]$. Now let $x_{i,m} \in [0, 1]$, $i = 1, \ldots, l$ be l equidistant points and calculate $y'(x_{i,m})$ for $i = 1, 2, \ldots, l$, then $\varepsilon + \max_{1 \leqslant i \leqslant l} |y'(x_{i,m})|$ may be considered as an estimation for M. Clearly, the estimation would become more precise if l increases and ε can be chosen, for example, equal to 1.

5. Numerical Examples

In this section we consider two examples for which the exact solution is unknown. Also approximated solution $y_m(x)$, as an approximation of exact solution $y(x)$, is obtained at the points $x = x_i = 0.1i, \quad i = 1, 2, ..., 9$ for $m = 8, 16, 32, 64$ and the values of $r_1 = |y_{16}(x) - y_8(x)|$, $r_2 = |y_{32}(x) - y_{16}(x)|$ and $r_3 = |y_{64}(x) - y_{32}(x)|$ are computed at the points x_i.

Example 1.

$$y(x) = f(x) - \int_0^x k(x,t)\, y(t) \ln|x-t|dt, \quad 0 \leqslant t < x \leqslant 1,$$

where $k(x,t) = 1$ and

$$f(x) = -\frac{1}{12}x\left(-12\ln(x) - 6x\ln\left(\frac{1}{x}\right)^2 - 12x - 12x\ln\left(\frac{1}{x}\right) - 6x\ln(x)\ln\left(\frac{1}{x}\right) - 6x\ln(x)^2 + \pi x^2\right).$$

The results for example 1. are seen in Table 1.

Table 1.
Computational results for example 1.

x	m =8	m =16	m =32	m =64	r_1	r_2	r_3
0.1	−.31034	−.28725	−.27434	−.24907	0.02309	0.01290	0.02527
0.2	−.41370	−.38231	−.35121	−.33275	0.03139	0.03110	0.01761
0.3	−.43932	−.40099	−.38298	−.37347	0.03832	0.01801	0.00950
0.4	−.41382	−.39938	−.38573	−.37608	0.01443	0.01364	0.00965
0.5	−.34926	−.35781	−.35457	−.35319	0.00855	0.00323	0.00138
0.6	−.34926	−.32487	−.31026	−.31057	0.02438	0.01460	0.00030
0.7	−.25244	−.23831	−.25094	−.25502	0.01413	0.01262	0.00408
0.8	−.12770	−.18576	−.17846	−.17393	0.05805	0.00729	0.00453
0.9	−.02195	−.06412	−.09423	−.09212	0.08608	0.03014	0.00214

Example 2.

$$y(x) = f(x) - \int_0^x k(x,t)\, y(t) \ln|x-t| dt, \quad 0 \leqslant t < x \leqslant 1,$$

where $k(x,t) = 1$ and

$$f(x) = 1 - e^{-x} + \frac{e^{-x}}{6x^2}\left(6xe^x - 6x^2e^x\ln(x) + 6e^x - 4x^3e^x + 6x^3e^x\ln(x) - 6 - 12x - 9x^2 + 6x^2\ln(x)\right).$$

The results for example 2. are seen in Table 2.

Table 2.
Computational results for example 2.

x	m =8	m =16	m =32	m =64	r_1	r_2	r_3
0.1	.15458	.13809	.13029	.11089	0.01649	0.00779	0.01940
0.2	.30550	.27293	.24593	.20399	0.03256	0.02700	0.02193
0.3	.45976	.34097	.32211	.31300	0.11879	0.01885	0.00911
0.4	.62151	.48062	.42035	.40842	0.14089	0.06026	0.01192
0.5	.79402	.62772	.55679	.52387	0.16630	0.07092	0.03292
0.6	.79402	.70496	.66476	.62770	0.08905	0.04020	0.03705
0.7	.98002	.86841	.77875	.73716	0.11160	0.08966	0.04158
0.8	1.1816	.95517	.89971	.87326	0.22650	0.05546	0.02644
0.9	1.4006	1.1399	1.0284	.99810	0.26067	0.11148	0.03039

6. Conclusion

In this paper, a computational approach for solving singular Volterra integral equations with logarithmic singularities was introduced. Taylor series expansion together with the BPFs are used to reduce the problem to the solution of linear algebraic equations. Error analysis and numerical examples suggest that more accurate approximate solutions may be obtained by using larger m.

6.1 *Appendix*

PROPOSITION 6.1 *For each* $0 < x \leqslant 1$ *we have*

$$\int_0^x \left[\mathbf{B}(t)\,\mathbf{B}^t(t)\sum_{n=1}^{\infty}\frac{t^n}{nx^n}\right]dt = \sum_{n=1}^{\infty}\int_0^x\left[\mathbf{B}(t)\,\mathbf{B}^t(t)\,\frac{t^n}{nx^n}\right]dt. \tag{30}$$

Before proceeding to prove proposition 6.1, we remind from analysis that if a series $\sum_{n=1}^{\infty} f_n$ of Reimann integrable functions is uniformly convergent over an interval $[a,b]$, then

$$\int_a^b \sum_{n=1}^{\infty} f_n(t)dt = \sum_{n=1}^{\infty}\int_a^b f_n(t)dt,$$

and if the convergence is over (a,b) and each f_n has a real limit when $t \longrightarrow b^-$, then

$$\lim_{t\to b^-}\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty}\lim_{t\to b^-} f_n(t).$$

Proof Integrals of matrices are defined and computed entry-wise and we have to show that the corresponding entries from the both sides of (30) are equal. First

$$\mathbf{B}(t)\mathbf{B}^t(t) = diag[B_{1,m}(t), B_{2,m}(t), ..., B_{m,m}(t)],$$

so

$$\int_0^x \left[\mathbf{B}\,(t)\,\mathbf{B}^t\,(t)\sum_{n=1}^{\infty}\frac{t^n}{nx^n}\right]dt$$
$$= diag\left[\int_0^x\left[\sum_{n=1}^{\infty}\frac{t^n}{nx^n}B_{1,m}(t)\right]dt, \int_0^x\left[\sum_{n=1}^{\infty}\frac{t^n}{nx^n}B_{2,m}(t)\right]dt, ..., \int_0^x\left[\sum_{n=1}^{\infty}\frac{t^n}{nx^n}B_{m,m}(t)\right]dt\right],$$

and

$$\sum_{n=1}^{\infty}\int_0^x\left[\mathbf{B}\,(t)\,\mathbf{B}^t\,(t)\,\frac{t^n}{nx^n}\right]dt$$
$$= diag\left[\sum_{n=1}^{\infty}\int_0^x\frac{t^n}{nx^n}B_{1,m}(t)dt, \sum_{n=1}^{\infty}\int_0^x\frac{t^n}{nx^n}B_{2,m}(t)dt, ..., \sum_{n=1}^{\infty}\int_0^x\frac{t^n}{nx^n}B_{m,m}(t)dt\right].$$

So we should show that for each $0 < x \leqslant 1$ and each $j = 1, 2, ..., m$:

$$\int_0^x\sum_{n=1}^{\infty}\frac{t^n}{nx^n}B_{j,m}(t)dt = \sum_{n=1}^{\infty}\int_0^x\frac{t^n}{nx^n}B_{j,m}(t)dt.$$

Note that all the integrals we are dealing with, are improper ones and for example the true form of (17) would be the following

$$y\,(x) = f\,(x) - \lim_{z\to x^-}\int_0^z k\,(x,t)\;\; y\,(t)\;\; \ln|x-t|\,dt, \quad 0 \leqslant t < x \leqslant 1.$$

For the rest of the proof, we assume $0 < x \leqslant 1$ is an arbitrary but fix number. First off, for each a and b where $0 \leqslant a < b < x$ we have $|\frac{t^n}{nx^n}B_{j,m}(t)| \leqslant \frac{b^n}{nx^n} \leqslant (\frac{b}{x})^n$ for $t \in [a, b]$ and the geometric series $\sum_{n=1}^{\infty}(\frac{b}{x})^n$ is convergent. Hence the Weierstrass M-test says that $\sum_{n=1}^{\infty}\frac{t^n}{nx^n}B_{j,m}(t)$ is uniformly convergent and we can write

$$\int_a^b\sum_{n=1}^{\infty}\frac{t^n}{nx^n}B_{j,m}(t)dt = \sum_{n=1}^{\infty}\int_a^b\frac{t^n}{nx^n}B_{j,m}(t)dt.$$

Next we consider three cases based on the value of j.

1) $\frac{j}{m} < x$. In this case for each z, where $\frac{j}{m} \leqslant z < x$, we would have

$$\int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt = \int_{\frac{j-1}{m}}^{\frac{j}{m}} \frac{t^n}{nx^n} B_{j,m}(t)dt = c_{j,n},$$

and

$$\int_0^x \frac{t^n}{nx^n} B_{j,m}(t)dt = \lim_{z\to x^-} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt = c_{j,n},$$

which imply that

$$\int_0^z \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt = \sum_{n=1}^{\infty} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt = \sum_{n=1}^{\infty} c_{j,n},$$

and then

$$\begin{aligned}\int_0^x \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt &= \lim_{z\to x^-} \int_0^z \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt \\ &= \sum_{n=1}^{\infty} c_{j,n} \\ &= \sum_{n=1}^{\infty} \int_0^x \frac{t^n}{nx^n} B_{j,m}(t)dt.\end{aligned}$$

2) $x \leqslant \frac{j-1}{m}$. In this case, for each $z < x$, we will have $B_{j,m}(t) = 0$ for $t \in [0, z]$ and

$$\int_0^z \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt = 0 = \sum_{n=1}^{\infty} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt,$$

and taking left limits as before, will yield

$$\int_0^x \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt = \sum_{n=1}^{\infty} \int_0^x \frac{t^n}{nx^n} B_{j,m}(t)dt = 0.$$

3) $\frac{j-1}{m} < x \leqslant \frac{j}{m}$. For each z, where $\frac{j-1}{m} < z < x$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt &= \sum_{n=1}^{\infty} \int_{\frac{j-1}{m}}^z \frac{t^n}{nx^n} dt \\ &= \sum_{n=1}^{\infty} \frac{1}{nx^n} \frac{z^{n+1} - (\frac{j-1}{m})^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \left(\frac{z^{n+1}}{n(n+1)x^n} - \frac{(\frac{j-1}{m})^{n+1}}{n(n+1)x^n} \right),\end{aligned}$$

$\sum_{n=1}^{\infty} \frac{(\frac{j-1}{m})^{n+1}}{n(n+1)x^n}$ is convergent by comparing it to the geometric series $\sum_{n=1}^{\infty} (\frac{\frac{j-1}{m}}{x})^n$ and the functional series $\sum_{n=1}^{\infty} \frac{z^{n+1}}{n(n+1)x^n}$ is uniformly convergent over the interval

$(0, x)$ according to Weierstrass M-test, because

$$|\frac{z^{n+1}}{n(n+1)x^n}| \leqslant \frac{x^{n+1}}{n(n+1)x^n} = \frac{x}{n(n+1)},$$

for each $z \in (0, x)$ and the numerical series $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is convergent. Hence the functional series $\sum_{n=1}^{\infty} \left(\frac{z^{n+1}}{n(n+1)x^n} - \frac{(\frac{j-1}{m})^{n+1}}{n(n+1)x^n} \right)$ or equivalently $\sum_{n=1}^{\infty} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt$ is uniformly convergent and we would have

$$\begin{aligned} \lim_{z \to x^-} \sum_{n=1}^{\infty} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt &= \sum_{n=1}^{\infty} \lim_{z \to x^-} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt \\ &= \sum_{n=1}^{\infty} \int_0^x \frac{t^n}{nx^n} B_{j,m}(t)dt. \end{aligned}$$

So

$$\begin{aligned} \int_0^x \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt &= \lim_{z \to x^-} \int_0^z \sum_{n=1}^{\infty} \frac{t^n}{nx^n} B_{j,m}(t)dt \\ &= \lim_{z \to x^-} \sum_{n=1}^{\infty} \int_0^z \frac{t^n}{nx^n} B_{j,m}(t)dt \\ &= \sum_{n=1}^{\infty} \int_0^x \frac{t^n}{nx^n} B_{j,m}(t)dt. \end{aligned}$$

■

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