International Journal of
Mathematical Modelling & Computations
Vol. 03, No. 01, 2013, 59- 70

Application of Fuzzy Expansion Methods for Solving Fuzzy Fredholm- Volterra Integral Equations of the First Kind

Sh. S. Behzadi*, T. Allahviranloo and S. Abbasbandy
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran .

Received: 16 January 2013; Accepted: 18 March 2013.

Abstract. In this paper we intend to offer new numerical methods to solve the fuzzy Fredholm- Volterra integral equations of the first kind $(FVFIE-1)$ base on collocation and Galerkin methods. Some examples are investigated to verify convergence results and to illustrate the efficiently of the methods.

Keywords: Airfoil polynomials, Jacobi polynomials, Fuzzy Collocation method, Fuzzy Galerkin method, Fuzzy integral equations, Volterra and Fredholm integral equations, Zero fuzzy singleton.

Index to information contained in this paper

1. Introduction

The fuzzy differential and integral equations are important part of the fuzzy analysis theory and they have the valued application in mechanics, electrical engineering, the theory of automatic control, medicine and biology. Recently, some mathematicians have studied solution of fuzzy integral equation and fuzzy integro-differential equation by numerical methods [1, 13].

In this work, we develope the Jacobi polynomials and the Airfoil polynomials fuzzy collocation methods and the fuzzy Galerkin method to solve the fuzzy Fredholm- Volterra integral equation of the first kind as follows:

$$\widetilde{f}(s) = \mu_1 \int_a^b k_1(s,t)\widetilde{x}(t)\ dt + \mu_2 \int_a^s k_2(s,t)\widetilde{x}(t)\ dt, \tag{1}$$

* Corresponding author. Email: shadan_behzadi@yahoo.com

where μ_1 and μ_2 are crisp constant values, $k_1(x,t)$ and $k_2(x,t)$ are crisp functions that have derivatives on an interval $a \leqslant s \leqslant t \leqslant b$ and $\widetilde{f}(s)$ is fuzzy function.

Here is an outline of the paper. In section 2, the basic notations and definitions in fuzzy calculus are briefly presented . Section 3 describes how to find an approximate solution of the given fuzzy Fredholm- Volterra integral equations of the first kind by using proposed methods. Finally in section 4, we apply the proposed methods by examples to show the simplicity and efficiency of the methods, and a brief conclusion is given in Section 5.

2. Basic Concepts

In this section, some basic definitions of a fuzzy number are given [10, 12].

Definition 2.1 *An arbitrary fuzzy number $\widetilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:*

(i) $\overline{u} : r \to u_r^- \in \mathbb{R}$ is a bounded left-continuous non-decreasing function over $[0,1]$,

(ii) $\underline{u} : r \to u_r^+ \in \mathbb{R}$ is a bounded left-continuous non-increasing function over $[0,1]$,

(iii) $\underline{u} \leqslant \overline{u}, \quad 0 \leqslant r \leqslant 1$.

Definition 2.2 *For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E$, we use the distance (Hausdorff metric) [7]*

$D(u(r), v(r)) = \max\{\sup_{r\in[0,1]} |\underline{u}(r) - \underline{v}(r)|, \sup \ |\overline{u}(r) - \overline{v}(r)|\}$,

and it is shown [7] that (E , D) is a complete metric space and the following properties are well known:

$$D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\ \widetilde{u}, \widetilde{v} \in E,$$
$$D(k\widetilde{u}, k\widetilde{v}) = \mid k \mid D(\widetilde{u}, \widetilde{v}), \forall\ k \in \mathbb{R}, \widetilde{u}, \widetilde{v} \in E,$$
$$D(\widetilde{u}+\widetilde{v}, \widetilde{w}+\widetilde{e}) \leqslant D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}), \forall\ \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E.$$

Definition 2.3 *A triangular fuzzy number is defined as a fuzzy set in E, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leqslant b \leqslant c$ such that $[u]^r = [u_-^r, u_+^r]$ are the endpoints of r-level sets for all $r \in [0,1]$, where $u_-^r = a + (b-a)r$ and $u_+^r = c - (c-b)r$. Here, $u_-^0 = a, u_+^0 = c, u_-^1 = u_+^1 = b$, which is denoted by u^1. The set of triangular fuzzy numbers will be denoted by E.*

Definition 2.4 *A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L(for left), R(for right) and scalar $\alpha \geqslant 0, \beta \geqslant 0$ with*

$$\tilde{\mu}_A(x) = \begin{cases} L(\frac{a-x}{\alpha}) & x \leqslant a \\ R(\frac{x-b}{\beta}) & x \geqslant a \end{cases} \tag{2}$$

the mean value of $\tilde{A}, a$ is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by (a, α, β).

Definition 2.5 *Let $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR}$ and $\lambda \in \mathbb{R}^+$. Then,*

$(1) : \lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$

$(2) : -\lambda\tilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$

$(3) : \tilde{M} \oplus \tilde{N} = (m+n, \alpha+\gamma, \beta+\delta)_{LR}$

$$(4) : \tilde{M} \odot \tilde{N} \simeq \begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR} & \tilde{M}, \tilde{N} > 0 \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR} & \tilde{M} > 0, \tilde{N} < 0 \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR} & \tilde{M}, \tilde{N} < 0 \end{cases} \quad (3)$$

Definition 2.6 *The integral of a fuzzy function was defined in [12] by using the Riemann integral concept.*
Let $f : [a, b] \to E^1$, for each partition $P = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_i - 1, t_i]$, $1 \leqslant i \leqslant n$, suppose

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

$$\Delta := \max\{|t_i - t_{i-1}|, 1 \leqslant i \leqslant n\}.$$

The definite integral of f(t) over [a,b] is

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_p,$$

provided that this limit exists in the metric D.

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [12], and also,

$$\underline{(\int_a^b f(t, r)dt)} = \int_a^b \underline{f}(t, r)dt,$$

$$\overline{(\int_a^b f(t, r)dt)} = \int_a^b \overline{f}(t, r)dt.$$

3. Description of the Methods

In this section we are going to solve the equation.(1) by using the Jacobi polynomials and the Airfoli polynomials fuzzy collocation methods and the fuzzy fast Galerkin method.

3.1 *Description of the Jacobli Polynomials Fuzzy Collocation Method*

To obtain the approximation solution of equation.(1), according to the Jacobli polynomials method [16] we can write:

$$\widetilde{x}_n(s) = w(s) \sum_{i=0}^{n} \widetilde{a}_i p_i^{\alpha,\beta}(s), \quad \alpha, \beta > -1, \quad (4)$$

where,

$$w(s) = \frac{(1-s)^\alpha}{(1+s)^\beta}, \quad p_i^{\alpha,\beta}(s) = \frac{(1-s)^{-\alpha}(1+s)^{-\beta}}{(-2)^n\ n!}\frac{d^n}{ds^n}[(1-s)^{n+\alpha}(1+s)^{n+\beta}], \tag{5}$$

and a_i are fuzzy coefficients.
From equation. (1)

$$\mu_1\sum_{i=0}^{n}\widetilde{a}_i\int_a^b w(t)k_1(s,t)\ p_i^{\alpha,\beta}(t)\ dt+\mu_2\sum_{i=0}^{n}\widetilde{a}_i\int_a^s w(t)k_2(s,t)\ p_i^{\alpha,\beta}(t)\ dt = \widetilde{f}(s)+\widetilde{R}_n(s). \tag{6}$$

So, we have

$$\mu_1\sum_{i=0}^{n}\widetilde{a}_i\int_a^b w(t)k_1(s,t)\ p_i^{\alpha,\beta}(t)\ dt+\mu_2\sum_{i=0}^{n}\widetilde{a}_i\int_a^s w(t)k_2(s,t)\ p_i^{\alpha,\beta}(t)\ dt\ominus\widetilde{f}(s) = \widetilde{R}_n(s). \tag{7}$$

$$\widetilde{R}_n(s_j) = \widetilde{0}. \tag{8}$$

The zero is fuzzy singleton.
It means,

$$\underline{R}_n^r(s_j) = 0, \quad \overline{R}_n^r(s_j) = 0, \quad \forall r\in[0,1].$$

Where s_j $(j=1,...,n)$ are collocation points.
Therefore we can write,

$$\mu_1\textstyle\sum_{i=0}^{n}\widetilde{a}_i\int_a^b w(t)k_1(s_j,t)\ p_i^{\alpha,\beta}(t)\ dt+\mu_2\sum_{i=0}^{n}\widetilde{a}_i\int_a^{s_j} w(t)k_2(s_j,t)\ p_i^{\alpha,\beta}(t)\ dt\ominus\widetilde{f}(s_j) = \widetilde{0}. \tag{9}$$

equation.(9) can be written in the following operator form

$$A\widetilde{a}\oplus L\widetilde{a}\ominus\widetilde{F} = \widetilde{0}. \tag{10}$$

$$\sum_{a_{ij}\geqslant 0} a_{ij}\widetilde{a}_j\oplus\sum_{a_{ij}<0} a_{ij}\widetilde{a}_j\oplus\sum_{l_{ij}\geqslant 0} l_{ij}\widetilde{a}_j\oplus\sum_{l_{ij}<0} l_{ij}\widetilde{a}_j\ominus\widetilde{F}_j = \widetilde{0}.$$

Where,

$$(\widetilde{F})_j = \widetilde{f}(s_j), \quad (A)_{ij} = \mu_1\textstyle\int_a^b w(t)k_1(s_j,t)\ p_i^{\alpha,\beta}(t)\ dt, \quad (L)_{ij} = \mu_2\int_a^{s_j} w(t)k_2(s_j,t)\ p_i^{\alpha,\beta}(t)\ dt. \tag{11}$$

3.2 *Description of the Fuzzy Galerkin Method*

The Galerikn condition is as follows:

$$\int_{-1}^{1} \frac{\widetilde{R}_n(s)T_i(s)}{\sqrt{1-s^2}}\, ds = \widetilde{0}. \tag{12}$$

It means,

$$\begin{array}{l} \int_{-1}^{1} \frac{\underline{R}_n^r(s)T_i(s)}{\sqrt{1-s^2}}\, ds = 0, \\ \int_{-1}^{1} \frac{\overline{R}_n^r(s)T_i(s)}{\sqrt{1-s^2}}\, ds = 0, \quad \forall r \in [0,1]. \end{array}$$

Where $T_j(s)$ $(j = 0, 1, ..., n)$ are Chebyshev polynomials. The Chebyshev polynomials are orthogonal on $[-1, 1]$ with weight function $\frac{1}{\sqrt{(1-s^2)}}$. Also, we have [8]

$$\int_{-1}^{1} \frac{T_i(s)\ T_j(s)}{\sqrt{(1-s^2)}}\, ds = \begin{cases} \pi, & i = j = 0, \\ \frac{\pi}{2}, & i = j > 0, \\ 0, & i \neq j. \end{cases}$$

$$\begin{array}{l} [a, b] \to [-1, 1] \\ t_1 = \frac{b+a}{2} + \frac{b-a}{2} t. \end{array}$$

Therefore we can write,

$$\begin{array}{l} \sum_{j=0}^{n} \widetilde{a}_j \int_{-1}^{1}\int_{-1}^{1} \mu_1 \frac{k_1(s,t)\ T_i(s)\ T_j(t)}{\sqrt{1-s^2}}\, dt\ ds + \\ \sum_{j=0}^{n} \widetilde{a}_j \int_{-1}^{1} \mu_2 \int_{-1}^{1} \frac{k_2(s,t)\ T_i(s)\ T_j(t)}{\sqrt{1-s^2}}\, dt\ ds \ominus \int_{-1}^{1} \frac{f(s)\ T_i(s)}{\sqrt{1-s^2}}\, ds = \widetilde{0}. \end{array} \tag{13}$$

equation.(13) can be written in the following operator form

$$A_1\widetilde{a} + L_1\widetilde{a} \ominus \widetilde{F}_1 = \widetilde{0}, \tag{14}$$

where,

$$\begin{array}{l} (\widetilde{f}_1)_i(s) = \int_{-1}^{1} \frac{\widetilde{f}(s)\ T_i(s)}{\sqrt{1-s^2}}\, ds, \\ (A_1)_{ij} = \int_{-1}^{1}\int_{-1}^{1} \mu_1 \frac{k_1(s,t)\ T_i(s)\ T_j(t)}{\sqrt{1-s^2}}\, dt\ ds, \\ (L_1)_{ij} = \int_{-1}^{1}\int_{-1}^{1} \mu_2 \frac{k_2(s,t)\ T_i(s)\ T_j(t)}{\sqrt{1-s^2}}\, dt\ ds. \end{array} \tag{15}$$

3.3 *Description of the Airfoil Polynomials Fuzzy Collocation Method*

To obtain the approximation solution of equation.(1), according to the airfoil polynomials method of the first kind [9] we can write:

$$\widetilde{x}_n(s) = w(s)\sum_{i=0}^{n} \widetilde{a}_i t_i(s), \tag{16}$$

where

$$\begin{aligned} & w(s) = \sqrt{\tfrac{1+s}{1-s}}, \\ & t_i(s) = \frac{\cos[(\frac{1}{2})\arccos s]}{\cos(\frac{1}{2}\arccos s)}, \\ & u_i(s) = \frac{\sin[(\frac{1}{2})\arcsin s]}{\cos(\frac{1}{2}\arcsin s)}, \\ & (1+s)t_i^{'}(s) = (i+\tfrac{1}{2})u_i(s) - \tfrac{1}{2}t_i(s). \end{aligned} \tag{17}$$

We can write equation. (1) as follows:

$$\mu_1 \sum_{i=0}^{n} \widetilde{a}_i \int_a^b w(t)k_1(s,t)\ t_i(t)\ dt + \mu_2 \sum_{i=0}^{n} \widetilde{a}_i \int_a^s w(t)k_2(s,t)\ t_i(t)\ dt = \widetilde{f}(s) + \widetilde{R}_n(s). \tag{18}$$

So, we have

$$\mu_1 \sum_{i=0}^{n} \widetilde{a}_i \int_a^b w(t)k_1(s,t)\ t_i(t)\ dt + \mu_2 \sum_{i=0}^{n} \widetilde{a}_i \int_a^s w(t)k_2(s,t)\ t_i(t)\ dt \ominus \widetilde{f}(s) = \widetilde{R}_n(s). \tag{19}$$

$$\widetilde{R}_n(s_j) = \widetilde{0}. \tag{20}$$

It means,

$$\underline{R}_n^r(s_j) = 0, \quad \overline{R}_n^r(s_j) = 0, \quad \forall r \in [0,1].$$

Where s_j $(j = 1, ..., n)$ are collocation points.

$$s_j = -\cos\frac{2j-1}{2n+3}\pi,\ \ j = 0, 1, ..., n.$$

Therefore we can write,

$$\mu_1 \textstyle\sum_{i=0}^{n} \widetilde{a}_i \int_a^b w(t)k_1(s_j,t)\ t_i(t)\ dt + \mu_2 \sum_{i=0}^{n} \widetilde{a}_i \int_a^{s_j} w(t)k_2(s_j,t)\ t_i(t)\ dt \ominus \widetilde{f}(s_j) = \widetilde{0}. \tag{21}$$

equation .(21) can be written in the following operator form

$$A_2\widetilde{a} \oplus L_2\widetilde{a} \ominus \widetilde{F}_2 = \widetilde{0}. \tag{22}$$

$$\sum_{a_{2ij}\geqslant 0} a_{2ij}\widetilde{a}_j \oplus \sum_{a_{2ij}<0} a_{2ij}\widetilde{a}_j \oplus \sum_{l_{2ij}\geqslant 0} l_{2ij}\widetilde{a}_j \oplus \sum_{l_{2ij}<0} l_{2ij}\widetilde{a}_j \ominus \widetilde{F}_{2j} = \widetilde{0}.$$

Where,

$$\begin{array}{l} (\widetilde{F_2})_j = \widetilde{f}(s_j), \\ (A_2)_{ij} = \mu_1 \int_a^b w(t)k_1(s_j,t)\ t_i(t)\ dt, \\ (L_2)_{ij} = \mu_2 \int_a^{s_j} w(t)k_2(s_j,t)\ t_i(t)\ dt. \end{array} \tag{23}$$

4. Numerical Example

In this section, we solve the fuzzy Fredholm -Volterra integral equation of the first kind by using the Jacobli polynomials and Airfoil polynomials fuzzy collocation methods and fuzzy Galerkin method. The program has been provided with Mathematica 6.

Algorithm :
Step 1. Set $n \leftarrow 0$.
Step 2. Solve the systems (10), (14) and (22).
Step 3. If $D(\widetilde{x}_{n+1}(s), \widetilde{x}_n(s)) < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $\widetilde{x}_n(s)$ as the approximation of the exact solution.

Example 4.1
Consider the fuzzy Fredholm- Volterra integral equation as follows:

$$\widetilde{f}(s) = \int_0^{0.6} (\frac{s^2}{2} + 3t)\ \widetilde{x}(t)\ dt + \int_0^s (s+t)\ \widetilde{x}(t)\ dt,$$

where,

$$\widetilde{f}(s) = (s^3 + 0.01, s^3 + 0.03, s^3 + 0.06).$$

$$\varepsilon = 10^{-4}.$$

$$\alpha = \frac{-1}{5},$$

$$\beta = \frac{-1}{2}.$$

Table 1 Numerical results for Example 4.1 by using the Airfoli polynomail fuzzy collocation method

r	$(\underline{u}, n = 6, s = 0.43)$	$(\overline{u}, n = 6, s = 0.43)$
0.0	0.3631245	0.6442618
0.1	0.3725855	0.6339714
0.2	0.3837482	0.6237159
0.3	0.4053144	0.5929352
0.4	0.4246097	0.5933355
0.5	0.4346512	0.5728573
0.6	0.4554643	0.5565578
0.7	0.4672641	0.5466493
0.8	0.4969823	0.5248704
0.9	0.5158822	0.5037657
1.0	0.5255268	0.5255268

$$D(E_n(s), \widetilde{0}) \leqslant 0.000849$$

.

Table 2 Numerical results for Example 4.1 by using the Jacobi polynomail fuzzy collocation method

r	$(\underline{u}, n = 5, s = 0.43)$	$(\overline{u}, n = 5, s = 0.43)$
0.0	0.4531245	0.7542618
0.1	0.4634509	0.7422315
0.2	0.4738752	0.7317609
0.3	0.4945266	0.7047854
0.4	0.5139122	0.6964821
0.5	0.5247759	0.6844657
0.6	0.5466553	0.6682127
0.7	0.5576559	0.6469543
0.8	0.5875427	0.6342396
0.9	0.6048283	0.6261548
1.0	0.6125213	0.6125213

$$D(E_n(s), \widetilde{0}) \leqslant 0.000825$$

.

Table 3 Numerical results for Example 4.1 by using the fast fuzzy Galerkin method

r	$(\underline{u}, n = 7, s = 0.43)$	$(\overline{u}, n = 7, s = 0.43)$
0.0	0.3624618	0.6535473
0.1	0.3563766	0.6441427
0.2	0.3675285	0.6344128
0.3	0.3848764	0.6027423
0.4	0.4042252	0.6138628
0.5	0.4133324	0.5925766
0.6	0.4360441	0.5759637
0.7	0.4483275	0.5661893
0.8	0.4767472	0.5571436
0.9	0.4956548	0.5416229
1.0	0.5049746	0.5049746

$$D(E_n(s), \widetilde{0}) \leqslant 0.000856$$

.

Example 4.2
Consider the fuzzy Fredholm- Volterra integral equation of the first kind as follows:

$$\widetilde{f}(s) = \int_0^{0.8} \sqrt{s+t}\ \widetilde{x}(t)\ dt + \int_0^{s} (s^2+2t)\ \widetilde{x}(t)\ dt,$$

where,

$$\underline{f}(s,r) = rs + \tfrac{1}{8} - \tfrac{1}{8}r - \tfrac{4}{17}s^2 + \tfrac{4}{17}s^2 r,$$
$$\overline{f}(s,r) = 2r - rs + \tfrac{1}{8}r - \tfrac{1}{8} + \tfrac{4}{17}s^2 r - \tfrac{4}{17}s^2.$$

$$\varepsilon = 10^{-4}.$$

$$\alpha = \frac{-1}{4},$$

$$\beta = \frac{-1}{5}.$$

Table 4 Numerical results for Example 4.2 by using the Airfoli polynomail fuzzy collocation method

r	$(\underline{u}, n = 7, s = 0.25)$	$(\overline{u}, n = 7, s = 0.25)$
0.0	0.3536454	0.7525317
0.1	0.3739474	0.7357419
0.2	0.3838278	0.7243321
0.3	0.4029266	0.7019884
0.4	0.4236241	0.6922446
0.5	0.4362346	0.6776322
0.6	0.45452487	0.6562742
0.7	0.4636722	0.6476634
0.8	0.50592819	0.6223654
0.9	0.51727508	0.6025857
1.0	0.52672622	0.5267262

$$D(E_n(s), \widetilde{0}) \leqslant 0.000803$$

.

Table 5 Numerical results for Example 4.2 by using the Jacobi polynomail fuzzy collocation method

r	$(\underline{u}, n = 6, s = 0.25)$	$(\overline{u}, n = 6, s = 0.25)$
0.0	0.4529479	0.8432675
0.1	0.4724382	0.8341378
0.2	0.4827677	0.8246705
0.3	0.5035533	0.8044826
0.4	0.5258682	0.7926488
0.5	0.5367692	0.7755653
0.6	0.5547509	0.7582505
0.7	0.5647325	0.7466912
0.8	0.5818413	0.7225719
0.9	0.6077328	0.7044217
1.0	0.6258249	0.6258249

$$D(E_n(s), \widetilde{0}) \leqslant 0.000798$$

.

Table 6 Numerical results for Example 4.2 by using the fast fuzzy Galerkin method

r	$(\underline{u}, n = 7, s = 0.25)$	$(\overline{u}, n = 7, s = 0.25)$
0.0	0.3836454	0.7525317
0.1	0.4039474	0.7657419
0.2	0.4138278	0.7443321
0.3	0.4329266	0.7219884
0.4	0.4536241	0.6822446
0.5	0.4662346	0.6676322
0.6	0.4845248	0.6545369
0.7	0.5022493	0.6432617
0.8	0.5337426	0.6252722
0.9	0.5482287	0.6038639
1.0	0.5661833	0.5661833

$$D(E_n(s), \widetilde{0}) \leqslant 0.000811$$

.

5. Conclusion

In this study, the Jacobi polynomials and the Airfoil polynomials fuzzy collocation methods and the fuzzy Galerkin method have been presented to solve the fuzzy Fredholm- Volterra integral equations of the first kind. These methods have been successfully employed to obtain the approximate solution of the fuzzy Fredholm-Volterra integral equations of the first kind. We can use these methods to solve another nonlinear fuzzy problems such as fuzzy partial differential equations and fuzzy integral equations.

References

[1] Allahviranloo, T., Hashemzehi, S., The homotopy perturbation method for fuzzy Fredholm integral equations., J. Appl. Math. IAU., **19** (2008) 1-13.

[2] Allahviranloo, T., Khalilzadeh, N., Khezerloo, S., Solving linear Fredholm fuzzy integral equations of the second kind by modified trapezoidal method, J. Appl. Math, Islamic Azad University of Lahijan. **7** (2011) 25-37.

[3] Allahviranloo, T., Khezerloo, M., Ghanbari, M., Khezerloo, S., The homotopy perturbation method for fuzzy Volterra integral equations, Int. J. Comput. Cognition. **8**(2010) 31-37.

[4] Alikhani, R., Bahrami, F., Jabbari, A., Existence of global solutions to nonlinear fuzzy Volterra integro-differential equations, Nonlinear Analysis., **75** (2012) 1810-1821.

[5] Babolian, E., Sadeghi Goghary, H., Abbasbandy, S., Numerical solution of linear fredholm fuzzy integral equations of the second kind by Adomian method, Appl. Math. Comput., **161** (2005) 733-744.

[6] Behzadi, Sh. S., Solving fuzzy nonlinear Volterra-Fredholm integral equations by using homotopy analysis and Adomian decomposition methods, Journal of Fuzzy Set Valued Analysis. (2011) doi: 10.5899/ 2011/ jfsva-00067 1-13.

[7] Bede, B., Gal, S. G., Generalizations of differentiability of fuzzy-number valued function with application to fuzzy differential equations, Fuzzy Sets Syst., **151** (2005) 581-599.

[8] Delves, L.M., Mohamed, J.L., Computation method for integral equations, Cambridge University Press (1983).

[9] Desmarais, R. N., Bland, S. R., Tables of properties of airfoil polynomials, Nasa Reference Publication 1343, September (1995).

[10] Dubois, D., H. Prade, Theory and application, Academic Press, (1980).

[11] Friedman, M., Ma, M., Kandel, A., Numerical methods for calculating the fuzzy integral, Fuzzy Sets Syst., **83** (1996) 57-62.

[12] Goetschel, R., Voxman, W., Elementary calculus. Fuzzy Sets Syst.**18** (1986) 31-43.

[13] Jahantigh, M., Allahviranloo, T., Otadi, M., Numerical solution of fuzzy integral equation, Appl. Math. Sci. **2** (2008) 33-46.

[14] Khorasany, M., Khezerloo, S., Yildirim, A., Numerical method for solving fuzzy Abel integral equations, World. Appl. Sci. J., **13** (2011) 2350-2354.

[15] Kauffman, A., Gupta, M.M., Introduction to Fuzzy Arithmetic, Theory and Application. Van Nostrand Reinhold, New York, (1991).

[16] Li, X., TANG,T., Convergence analysis of Jacobi spectral collocation methods for Abel-Volterra integral equations of second kind, Front. Math. China., **7** (2012) 69-84.

[17] Mosleh, M., Otadi, M., Numerical solution of fuzzy integral equations using Bernstein polynomials, Aust. J. Basic Appl. Sci, **5**(2011) 724-728.

[18] Mikaeilv, N., Khakrangin, S., Allahviranloo, T., Solving fuzzy Volterra integro-differential equation by fuzzy differential transform method, EUSFLAT- LFA. (2011), 891-896.

[19] Molabahrami, A., Shidfar, A., Ghyasi, A., An analytical method for solving linear Fredholm fuzzy integral equations of the second kind, Comput .Math with Appl. **61** (2011) 2754-2761.

[20] Puri, M.L., Ralescu, D., Fuzzy random variables. J. Math. Anal. Appl., **114** (1986) 409-422.

[21] Sugeno, M., Theory of Fuzzy Integrals and Its Application, Ph.D. thesis, Tokyo Institute of Technology. (1974).

[22] Zadeh, L.A., Fuzzy Sets. Information and Control, **8** (1965) 338-353.