A Sixth Order Method for Solving Nonlinear Equations

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Abstract. In this paper, we present a new iterative method with order of convergence sixth for solving nonlinear equations. This method is developed by extending a fourth order method of Ostrowski. Per iteration this method requires three evaluations of the function and one evaluation of its first derivative. A general error analysis providing the sixth order of convergence is given. Several numerical examples are given to illustrate the efficiency and performance of the new method.

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1. Introduction

Numerical methods for solving nonlinear equations is a popular and important research topic in numerical analysis. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x) = 0$, where $f : D \subset R \to R$ for an open interval D is a scalar function.

Newton method is an important and basic approach for solving nonlinear equations [8,10], and its formulation is given by

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This method convergence quadratically.

Newton method has been modified in a number of ways to avoid the use of derivatives without affecting the order of convergence. For example, on replacing in (1) the derivative by the forward approximation

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)},$$

Newton method becomes

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}.$$

Which is called Steffensen method [6]. This method still has quadratic convergence, in spite of being derivative free and using only two functional evaluations per step.

The rest of this paper is organized as follows. In Section 2, we suggest a new iterative method with the conjectured sixth-order convergence. In Section 3, we establish the convergence order of this method. Finally, in Section 4, we compare it with related methods for solving nonlinear equations.

2.   Description of the Method

The well-known Ostrowski method is given by:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\[2mm] x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \end{cases}, \tag{2}$$

Which has fourth convergence [7].
For a given x_0 compute the approximate solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \tag{4}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{5}$$

Now using the linear interpolation on two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$ we get

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n),$$

Thus an approximation to $f'(z_n)$ is given by

$$f'(z_n) \approx \frac{z_n - x_n}{y_n - x_n} f'(y_n) + \frac{z_n - y_n}{x_n - y_n} f'(x_n),$$

and based on Ostrowski method we have

$$f'(y_n) = \frac{f(x_n) - 2f(y_n)}{x_n - y_n}. \tag{6}$$

Now using (3), (4) and (6), we get

$$f'(z_n) \approx \frac{f'(x_n)[4f(y_n)f(x_n) - 2f(y_n)^2 - f(x_n)^2]}{f(x_n)(2f(y_n) - f(x_n))}, \tag{7}$$

Now replacing $f'(z_n)$ in (5) by (7), our proposed method can be described as given below

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\[2ex] z_n = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \\[2ex] x_{n+1} = z_n - \frac{f(z_n)f(x_n)(2f(y_n) - f(x_n))}{f'(x_n)[4f(y_n)f(x_n) - 2f(y_n)^2 - f(x_n)^2]} \end{cases}. \tag{8}$$

Where relation (8) is an optimal sixth-order Ostrowski-type method. In the next section, we state and prove the convergence theorem for the method (8).

3. Analysis of Convergence

In this section we analyze the order of convergence of the method described previously.

THEOREM 3.1 *Let $r \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to r, then iterative scheme (8) has sixth-order convergence.*

Proof Let $e_n = x_n - r$ be the error in the iterate x_n. Using Taylor series expansion, we get:

$$f(x_n) = f'(r)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)], \qquad (9)$$

$$f'(x_n) = f'(r)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6)], \qquad (10)$$

where

$$c_k = \frac{1}{k!} \frac{f^{(k)}(r)}{f'(r)}, \quad k = 2, 3, \dots .$$

From (9) and (10), we have,

$$y_n = r + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + (-6c_3^2 + 20c_3 c_2^2 - 10c_2 c_4 + 4c_5 - 8c_2^4)e_n^5$$

$$+ (-17c_4 c_3 + 28c_4 c_2^2 - 13c_2 c_5 + 5c_6 + 33c_2 c_3^2 - 52c_3 c_2^3 + 16c_2^5)e_n^6 + (-22c_5 c_3 + 36c_5 c_2^2$$

$$+ 6c_7 - 16c_2 c_6 - 12c_4^2 + 92c_4 c_2 c_3 - 72c_4 c_2^3 + 18c_3^3 - 126c_3^2 c_2^2 + 128c_3 c_2^4 - 32c_2^6)e_n^7 + O(e_n^8).$$
$$(11)$$

Using Taylor series, we have

$$f(y_n) = f'(r)[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 5c_2^3)e_n^4 + (-6c_3^2 + 24c_3 c_2^2 - 10c_2 c_4 + 4c_5 - 12c_2^4)e_n^5$$

$$+ (-17c_4 c_3 + 34c_4 c_2^2 - 13c_2 c_5 + 5c_6 + 37c_2 c_3^2 - 73c_3 c_2^3 + 28c_2^5)e_n^6 + (-22c_5 c_3 + 44c_5 c_2^2 + 6c_7$$

$$- 16c_2 c_6 - 12c_4^2 + 104c_4 c_2 c_3 - 104c_4 c_2^3 + 18c_3^3 - 160c_3^2 c_2^2 + 206c_3 c_2^4 - 64c_2^6)e_n^7 + O(e_n^8)].$$
$$(12)$$

Using Eq. (8), we obtain

$$e_{n+1} = c_2 c_3 (c_3 - c_2^2)O(e_n^6) + O(e_n^7).$$

which shows that iterative method (8) is sixth-order convergence.

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This method requires three evaluations of the function, namely, $f(x_n)$, $f(y_n)$ and $f(z_n)$ and one evaluation of first derivative $f'(x_n)$. We consider the definition of efficiency index [2,3] as $\sqrt[w]{P}$, where P is the order of the method and w is the number of function evaluations per iteration required by the method. We have that the efficiency index of the method Equation (8) is $\sqrt[4]{6} \approx 1.565$.

4. Numerical Examples

We present some examples to illustrate the efficiency of the new iterative method in this paper. We compare the Newton method (NM), the method of Ostrowski (OM), the method of Mtinfar-Aminzadeh [5] (MAM) , the method of Fang-Chen-Tian-Sun-Chen [1] (FCTSCM) and Eq. (8), introduced in this paper.

MAM:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\ \\ z_n = y_n - \frac{f(y_n)}{f'(y_n - \frac{f(y_n)}{2P_f(x_n,y_n)})} \end{cases},$$

Where $P_f(x_n, y_n) = f'(y_n)$

$$P_f(x_n, y_n) = [\frac{2f(x_n) - 5f(y_n)}{2f(x_n) - f(y_n)}]f'(x_n).$$

FCTSCM:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{\alpha f(x_n) + f'(x_n)} \\ \\ z_n = y_n - \frac{f(y_n)}{\beta f(y_n) + f'(y_n)} \\ \\ x_{n+1} = z_n - \frac{f(z_n)}{\gamma f(z_n) + f'(y_n)} \end{cases},$$

Where $\alpha = \beta = \gamma = 1$.

We use the following functions, most of which are the same as in [4,9].

$$f_1(x) = x^3 + 4x^2 - 10, \qquad x* = 1.3652300134140969,$$
$$f_2(x) = x^2 - e^x - 3x + 2, \qquad x* = 0.25753028543986084,$$
$$f_3(x) = cos(x) - x, \qquad x* = 0.739085133215160,$$
$$f_4(x) = sin^2(x) - x^2 + 1, \qquad x* = 1.4044916482153411.$$

Displayed in Table 1 are the number of iterations (n) and the number of function evaluations (NFE) required such that $|f(x_n)| < 1.E - 14$.

From Table 1, it is clear that the new method (8) performs better than the other iterative methods suggested in this Table.

5. Conclusions

A sixth order method is proposed for finding real roots of nonlinear equations without evaluating the second order derivative of the given function. We prove that the order of convergence of this method is sixth. Our method has the efficiency index equal to $\sqrt[4]{6} \approx 1.565$ which is better then Newton method with efficiency

Table 1.   Comparison of various iterative methods.

	n	NFE		n	NFE
$f_1(x), x_0 = 1$			$f_3(x), x_0 = 0.2$		
NM	5	10		5	10
OM	3	9		3	9
MAM	2	8		2	8
FCTSCM	3	15		3	15
Eq. (8)	2	8		2	8
$f_1(x), x_0 = 2$			$f_3(x), x_0 = 1$		
NM	5	10		4	8
OM	3	9		2	6
MAM	2	8		2	8
FCTSCM	3	15		2	10
Eq. (8)	2	8		2	8
$f_2(x), x_0 = 0$			$f_4(x), x_0 = 1$		
NM	4	8		6	12
OM	2	6		3	9
MAM	2	8		2	8
FCTSCM	2	10		4	20
Eq. (8)	2	8		2	8
$f_2(x), x_0 = 1$			$f_4(x), x_0 = 2$		
NM	4	8		5	10
OM	2	6		3	9
MAM	2	8		2	8
FCTSCM	2	10		3	15
Eq. (8)	2	8		2	8

index equal to $\sqrt[2]{2} \approx 1.414$ and the Fang-Chen-Tian-Sun-Chen method [1] with efficiency index equal to $\sqrt[5]{6} \approx 1.430$. It is also comparable with other sixth order method as its efficiency index is also $\sqrt[4]{6} \approx 1.565$. The method is tested on a number of numerical examples. On comparing our results with those obtained by Newton method (NM) and the method of Ostrowski [7] (OM), it is found that our method is most effective as it convergence to the root much faster. When compared with the sixth order method of Matinfar-Aminzadeh [5] and Fang-Chen-Tian-Sun-Chen [1], our method behaves either similarly or better on the examples considered.

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