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Numerical solution of Boussinesq equation using modified Adomian decomposition and homotopy analysis methods

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Abstract. In this paper, a Boussinesq equation is solved by using the Adomian's decomposition method, modified Adomian's decomposition method and homotopy analysis method. The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved in detail. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords: Boussinesq equation, Adomian decomposition method (ADM), Modified Adomian decomposition method (MADM), Homotopy analysis method (HAM).

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1. Introduction

Boussinesq equation describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice [1-4]. In recent years some works have been done in order to find the numerical solution of this equation, for example [5-12]. In this work, we develop the ADM, MADM and HAM to solve the Boussinesq equation as follows:

$$u_{tt} = \alpha u_{xx} + \beta (u^2)_{xx} + u_{xxxx}, \tag{1}$$

where α and β are arbitrary constants and the initial conditions are:

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$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned} \tag{2}$$

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1). In section 3 we prove the existence and uniqueness of the solution and convergence of the proposed methods. Finally, a numerical example is solved in section 4.

In order to obtain an approximate solution of Eq.(1), let us integrate two times from Eq.(1) with respect to t using the initial conditions we obtain,

$$\begin{aligned} u(x,t) =& z(x,t) + \alpha \int_0^t \int_0^t D^2(u(x,t))\ dt\ dt + \beta \int_0^t \int_0^t F(u(x,t))\ dt\ dt \\ &+ \int_0^t \int_0^t D^4(u(x,t))\ dt\ dt, \end{aligned} \tag{3}$$

where,

$$\begin{aligned} D^i(u(x,t)) &= \frac{\partial^i u(x,t)}{\partial x^i}, \quad i = 2,4, \\ F(u(x,t)) &= \frac{\partial^2 u^2(x,t)}{\partial x^2}, \\ z(x,t) &= f(x) + tg(x). \end{aligned}$$

The double integrals in (3) can be written as [16]:

$$\begin{aligned} \int_0^t \int_0^t (D^2(x,t))\ dt\ dt &= \int_0^t (x-t)\ D^2(u(x,t))\ dt, \\ \int_0^t \int_0^t F(u(x,t))\ dt\ dt &= \int_0^t (x-t)\ F(u(x,t))\ dt, \\ \int_0^t \int_0^t (D^4(x,t))\ dt\ dt &= \int_0^t (x-t)\ D^4(u(x,t))\ dt. \end{aligned}$$

So, we can write Eq.(3) as follows:

$$\begin{aligned} u(x,t) =& z(x,t) + \alpha \int_0^t (x-t)\ D^2(u(x,t))\ dt + \beta \int_0^t (x-t)\ F(u(x,t))\ dt \\ &+ \int_0^t (x-t)\ D^4(u(x,t))\ dt. \end{aligned} \tag{4}$$

In Eq.(4), we assume $z(x,t)$ is bounded for all x, t in $J = [a, T](a, T \in \mathbb{R})$. The terms $D^i(u(x,t))$ $(i=2,4)$ and $F(u(x,t))$ are Lipschitz continuous with

$$|D^2(u) - D^2(u^*)| \leqslant L_1|u - u^*|,$$
$$|F(u) - F(u^*)| \leqslant L_2|u - u^*|,$$
$$|D^4(u) - D^4(u^*)| \leqslant L_3|u - u^*|.$$

also,

$$|x - t| \leqslant M, \quad \forall\ a \leqslant x, \quad t \leqslant T, \quad M \in \mathbb{R}^+.$$

2. The iterative methods

2.1 *Description of the MADM and ADM*

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = g_1, \tag{5}$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g_1 is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(5), and using the given conditions we obtain

$$u(x,t) = f_1(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{6}$$

where the function $f_1(x)$ represents the terms arising from integrating the source term g_1. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where A_n, $n \geqslant 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]\right]_{\lambda=0}. \tag{8}$$

The first Adomian polynomials are [13-15]:

$$\begin{aligned}
A_0 &= G_1(u_0),\\
A_1 &= u_1 G_1'(u_0),\\
A_2 &= u_2 G_1'(u_0) + \frac{1}{2!}u_1^2 G_1''(u_0),\\
A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!}u_1^3 G_1'''(u_0), \ldots
\end{aligned} \tag{9}$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in (5) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{10}$$

where, the components $u_0, u_1, \ldots$ which can be determined recursively

$$\begin{aligned}
u_0 &= z(x,t),\\
u_1 &= \alpha \int_0^t (x-t)\ A_0(x,t)\ dt + \beta \int_0^t (x-t)\ B_0(x,t)\ dt + \int_0^t (x-t)\ L_0(x,t)\ dt,
\end{aligned} \tag{11}$$

$$\vdots$$

$$u_{n+1} = \alpha \int_0^t (x-t)\ A_n(x,t)\ dt + \beta \int_0^t (x-t)\ B_n(x,t)\ dt + \int_0^t (x-t)\ L_n(x,t)\ dt, n \geqslant 0.$$

Substituting (11) into (9) leads to the determination of the components of u.

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [16]. The modified forms was established on the assumption that the function $z(x,t)$ can be divided into two parts, namely $z_1(x,t)$ and $z_2(x,t)$. Under this assumption we set

$$z(x,t) = z_1(x,t) + z_2(x,t). \tag{12}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part z_1 be assigned to the zeroth component u_0, whereas the remaining part z_2 be combined with the other terms given in (12) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned}
u_0 &= z_1(x,t),\\
u_1 &= z_2(x,t) - L^{-1}(Ru_0) - L^{-1}(A_0),\\
&\vdots\\
u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geqslant 1,
\end{aligned} \tag{13}$$

was developed.

To obtain the approximation solution of Eq.(1), according to the MADM, we can write the iterative formula (13) as follows:

$$
\begin{aligned}
&u_0 = z_1(x,t),\\
&u_1 = z_2(x,t) + \alpha\int_0^t (x-t)\ A_0(x,t)\ dt + \beta\int_0^t (x-t)\ B_0(x,t)\ dt + \int_0^t (x-t)\ L_0(x,t)\ dt\\
&\vdots\\
&u_{n+1} = \alpha\int_0^t (x-t)\ A_n(x,t)\ dt + \beta\int_0^t (x-t)\ B_n(x,t)\ dt + \int_0^t (x-t)\ L_n(x,t)\ dt, \quad n \geqslant 1.
\end{aligned}
\tag{14}
$$

The operators $D^j(u)$ $(j = 2, 4)$ and $F(u)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$
\begin{aligned}
D^2(u) &= \sum_{i=0}^{\infty} A_i,\\
F(u) &= \sum_{i=0}^{\infty} B_i,\\
D^4(u) &= \sum_{i=0}^{\infty} L_i
\end{aligned}
$$

where A_i, B_i, and L_i are the Adomian polynomials.
Also, we can use the following formula for the Adomian polynomials [17]:

$$
\begin{aligned}
A_n &= D^2(s_n) - \sum_{i=0}^{n-1} A_i,\\
B_n &= F(s_n) - \sum_{i=0}^{n-1} B_i,\\
L_n &= D^4(s_n) - \sum_{i=0}^{n-1} L_i
\end{aligned}
\tag{15}
$$

where $s_n = \sum_{i=0}^{n} u_i(x,t)$ is the partial sum.

2.2 *Description of the HAM*

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x is an independent variable. Let $u_0(x,t)$ denotes an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$\begin{aligned}&(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] = \\ &\hat{H}[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q].\end{aligned} \tag{16}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (16) to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q); u_0(x,t), H_1(x,t), h, q] = 0, \tag{17}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \tag{18}$$

When $q = 0$, the zero-order deformation Eq. (18) becomes

$$\phi(x;0) = u_0(x,t), \tag{19}$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq. (18) is equivalent to

$$\phi(x,t;1) = u(x,t). \tag{20}$$

Thus, according to (19) and (20), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [18-21].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{21}$$

where,

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}.$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (21) of $\phi(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{22}$$

From Eq.(21), we can write Eq.(18) as follows

$$(1-q)L[\phi(x,t,q) - u_0(x,t)] = (1-q)L[\sum_{m=1}^{\infty} u_m(x,t)\ q^m] = q\ h\ H_1(x,t)N[\phi(x,t,q)]$$

$$\Rightarrow L[\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m] = q\ h\ H_1(x,t)N[\phi(x,t,q)] \tag{23}$$

By differentiating (23) m times with respect to q, we obtain

$$\begin{aligned}
&\{L[\sum_{m=1}^{\infty} u_m(x,t)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m]\}^{(m)} \\
&= \{q\ h\ H_1(x,t)N[\phi(x,t,q)]\}^{(m)} \\
&= m!\ L[u_m(x,t) - u_{m-1}(x,t)] \\
&= h\ H_1(x,t)\ m\ \frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .
\end{aligned}$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)\Re_m(u_{m-1}(x,t)), \tag{24}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{25}$$

and

$$\chi_m = \begin{cases} 0,\ m \leqslant 1 \\ 1,\ m > 1 \end{cases}$$

Note that the high-order deformation Eq. (24) is governing the linear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (25) for any nonlinear operator N.

To obtain the approximation solution of Eq. (1), according to HAM, let

$$
\begin{aligned}
N[u(x,t)] =& u(x,t) - z(x,t) - \alpha \int_0^t (x-t)\ D^2(u(x,t))\ dt - \\
& \beta \int_0^t (x-t)\ F(u(x,t))\ dt - \int_0^t (x-t)\ D^4(u(x,t))\ dt,
\end{aligned}
$$

so,

$$
\begin{aligned}
\Re_m(u_{m-1}(x,t)) =& u_{m-1}(x,t) - z(x,t) - \alpha \int_0^t (x-t)\ D^2(u_{m-1}(x,t))\ dt - \\
& \beta \int_0^t (x-t)\ F(u_{m-1}(x,t))\ dt - \int_0^t (x-t)\ D^4(u_{m-1}(x,t))\ dt.
\end{aligned} \tag{26}
$$

Substituting (26) into (24)

$$
\begin{aligned}
L[u_m(x,t) - \chi_m u_{m-1}(x,t)] &= hH_1(x,t)[u_{m-1}(x,t) - \alpha \int_0^t (x-t)\ D^2(u_{m-1}(x,t))\ dt \\
&- \beta \int_0^t (x-t)\ F(u_{m-1}(x,t))\ dt - \int_0^t (x-t)\ D^4(u_{m-1}(x,t))\ dt + (1-\chi_m) z(x,t)(x)].
\end{aligned} \tag{27}
$$

We take an initial guess $u_0(x,t) = z(x,t)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into (27) to give the recurrence relation

$$
\begin{aligned}
u_0(x,t) &= z(x,t), \\
u_{n+1}(x,t) &= \alpha \int_0^t (x-t)\ D^2(u_n(x,t))\ dt + \beta \int_0^t (x-t)\ F(u_n(x,t))\ dt \\
&+ \int_0^t (x-t)\ D^4(u_n(x,t))\ dt, \quad n \geqslant 0.
\end{aligned} \tag{28}
$$

Therefore, the solution $u(x,t)$ becomes

$$
\begin{aligned}
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = z(x,t) + \sum_{n=1}^{\infty} \Big(\alpha \int_0^t (x-t)\ D^2(u_n(x,t))\ dt \\
+ \beta \int_0^t (x-t)\ F(u_n(x,t))\ dt + \int_0^t (x-t)\ D^4(u_n(x,t))\ dt \Big).
\end{aligned} \tag{29}
$$

3. Existence and convergence of iterative methods

In the following theorem we consider,

$$\alpha_1 := TM(|\alpha|L_1 + |\beta|L_2 + L_3).$$

Theorem 3.1.
Let $0 < \alpha_1 < 1$, then Boussinesq equation (1), has a unique solution.
Proof. Let u and u^* be two different solutions of (3) then

$$\begin{aligned}
|u-u^*| = \Big|\alpha \int_0^t (x-t)[D^2(u(x,t)) - D^2(u^*(x,t))]dt + \beta \int_0^t (x-t)[F(u(x,t)) - F(u^*(x,t))]dt \\
+ \int_0^t (x-t)[D^4(u(x,t)) - D^4(u^*(x,t))]dt\Big| \\
\leqslant |\alpha| \int_0^t |x-t||D^2(u(x,t)) - D^2(u^*(x,t))|dt + |\beta| \int_0^t |x-t||F(u(x,t)) - F(u^*(x,t))|dt \\
+ \int_0^t |x-t||D^4(u(x,t)) - D^4(u^*(x,t))|dt \\
\leqslant \mathrm{TM}(|\alpha|L_1 + |\beta|L_2 + L_3)|u-u^*| = \alpha_1|u-u^*|.
\end{aligned}$$

From which we get $(1-\alpha_1)|u-u^*| \leqslant 0$. Since $0 < \alpha_1 < 1$, then $|u-u^*| = 0$. Implies $u = u^*$ and completes the proof.
Theorem 3.2.
The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of equation (1) using ADM (or MADM) is convergent when

$$0 < \alpha_1 < 1, |u_1(x,t)| < \infty.$$

Proof. Denote as $(C[J], \|\ .\ \|)$ the Banach space of all continuous functions on J with the norm $\| f(t) \| = \max |f(t)|$, for all t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geqslant m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\| s_n - s_m \| = \max_{\forall t \in J} |s_n - s_m| = \max_{\forall t \in J} \sum_{i=m+1}^{n} u_i(x,t)| \\
= \max_{\forall t \in J} \Big|\alpha \int_0^t (x-t)(\sum_{i=m}^{n-1} A_i)dt + \beta \int_0^t (x-t)(\sum_{i=m}^{n-1} B_i)dt + \int_0^t (x-t)(\sum_{i=m}^{n-1} L_i)dt\Big|.
\end{aligned}$$

From [18], we have

$$\sum_{i=m}^{n-1} A_i = D^2(s_{n-1}) - D^2(s_{m-1}),$$
$$\sum_{i=m}^{n-1} B_i = F(s_{n-1}) - F(s_{m-1}),$$
$$\sum_{i=m}^{n-1} L_i = D^4(s_{n-1}) - D^4(s_{m-1}).$$

So,

$$\begin{aligned}
\| s_n - s_m \| &= \max_{\forall t \in J} \Big| \alpha \int_0^t (x-t)[D^2(s_{n-1}) - D^2(s_{m-1})]dt \\
&\quad + \beta \int_0^t (x-t)[F(s_{n-1}) - F(s_{m-1})]dt + \int_0^t (x-t)[D^4(s_{n-1}) - D^4(s_{m-1})]dt \Big| \\
&\leqslant |\alpha| \int_0^t |x-t||D^2(s_{n-1}) - D^2(s_{m-1})dt \\
&\quad + |\beta| \int_0^t |x-t||F(s_{n-1}) - F(s_{m-1})|dt + \int_0^t |x-t||D^4(s_{n-1}) - D^4(s_{m-1})dt \\
&\leqslant \alpha_1 \| s_n - s_m \| .
\end{aligned}$$

Let $n = m+1$, then

$$\| s_n - s_m \| \leqslant \alpha_1 \| s_m - s_{m-1} \| \leqslant \alpha_1^2 \| s_{m-1} - s_{m-2} \| \leqslant \ldots \leqslant \alpha_1^m \| s_1 - s_0 \| .$$

From the triangle inequality we have

$$\begin{aligned}
\| s_n - s_m \| &\leqslant \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + \cdots + \| s_n - s_{n-1} \| \\
&\leqslant [\alpha_1^m + \alpha_1^{m+1} + \cdots + \alpha_1^{n-m-1}] \| s_1 - s_0 \| \\
&\leqslant \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-m-1}] \| s_1 - s_0 \| \\
&\leqslant \alpha_1^m \left[\frac{1-\alpha_1^{n-m}}{1-\alpha_1} \right] \| u_1(x,t) \| .
\end{aligned}$$

Since $0 < \alpha_1 < 1$, we have $(1 - \alpha_1^{n-m}) < 1$, then

$$\| s_n - s_m \| \leqslant \frac{\alpha_1^m}{1-\alpha_1} \max_{\forall t \in J} |u_1(x,t)|. \tag{30}$$

But $|u_1(x,t)| < \infty$, so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergent and the proof is complete.

Theorem 3.3.

If the series solution (28) of problem (1) using HAM is convergent then it converges to the exact solution of the problem (1).

Proof. We assume:

$$\begin{aligned} u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t), \\ \widehat{D}^2(u(x,t)) &= \sum_{m=0}^{\infty} D^2(u_m(x,t)), \\ \widehat{F}(u(x,t)) &= \sum_{m=0}^{\infty} F(u_m(x,t)), \\ \widehat{D}^4(u(x,t)) &= \sum_{m=0}^{\infty} D^4(u_m(x,t)), \end{aligned}$$

where,

$$\lim_{m\to\infty} u_m(x,t) = 0.$$

We can write,

$$\begin{aligned} \sum_{m=1}^{n} [u_m(x,t) - \chi_m u_{m-1}(x,t)] &= u_1 + (u_2 - u_1) + \cdots + (u_n - u_{n-1}) \\ &= u_n(x,t). \end{aligned} \tag{31}$$

Hence, from (31),

$$\lim_{n\to\infty} u_n(x,t) = 0. \tag{32}$$

So, using (32) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = L\left[\sum_{m=1}^{\infty} [u_m(x,t) - \chi_m u_{m-1}(x,t)]\right] = 0.$$

Therefore from (31), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t) \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) = 0. \tag{33}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (33) and simplifying it, we have

$$\begin{aligned}\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) &= \sum_{m=1}^{\infty}[u_{m-1}(x,t) - \alpha \int_0^t (x-t)D^2(u_{m-1}(x,t))dt \\ &- \beta \int_0^t (x-t)F(u_{m-1}(x,t))dt - \int_0^t (x-t)D^4(u_{m-1}(x,t))dt \\ &+ (1-\chi_m)z(x,t)] \\ &= u(x,t) - z(x,t) - \alpha \int_0^t (x-t)\widehat{D}^2(u(x,t))dt \\ &- \beta \int_0^t (x-t)\widehat{F}(u(x,t))dt - \int_0^t (x-t)\widehat{D}^4(u(x,t))dt.\end{aligned} \tag{34}$$

From (33) and (34), we have

$$u(x,t) = z(x,t) + \alpha \int_0^t (x-t)\ \widehat{D}^2(u(x,t))\ dt + \beta \int_0^t (x-t)\ \widehat{F}(u(x,t))\ dt + \int_0^t (x-t)\ \widehat{D}^4(u(x,t))\ dt.$$

Therefore, $u(x,t)$ must be the exact solution.

4. Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm :

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relations (11) for ADM, (14) for MADM and (28) for HAM.

Step 3. If $|u_{n+1} - u_n| < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Example 4.1

Consider the Boussinesq equation as follows:

$$u_{tt} = u_{xx} - 6(u^2)_{xx} + u_{xxxx},$$

subject to the initial conditions:

$$\begin{aligned} u(x,0) &= \frac{1}{x^2}, \\ u_t(x,0) &= \frac{-2}{x^2}. \end{aligned}$$

Table 1. Numerical results of the Example 4.1

(x,t)	Errors		
	ADM(n=15)	MADM(n=12)	HAM(n=4)
$(0.1, 0.12)$	0.070513	0.062725	0.023378
$(0.2, 0.15)$	0.071572	0.063668	0.023706
$(0.3, 0.22)$	0.072673	0.064235	0.024458
$(0.4, 0.26)$	0.072785	0.064788	0.024842
$(0.5, 0.30)$	0.073472	0.065325	0.025173
$(0.7, 0.33)$	0.074185	0.065864	0.025663

Table 1, shows that the approximate solution of the Boussinesq equation is convergent with 4 iterations by using the HAM. By comparing the results of Table 1, one can observe that the HAM is more rapid convergent than the ADM and MADM.

5. Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the Boussinesq equation. For this purpose, we showed that the HAM has more rapid convergence than the ADM and MADM. Also, the example shows that the number of computations in HAM is less than the number of computations in ADM and MADM.

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