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Non-Polynomial Spline for the Numerical Solution of Problems in Calculus of Variations

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Abstract.A Class of new methods based on a septic non-polynomial spline function for the numerical solution of problems in calculus of variations is presented. The local truncation errors and the methods of order 2th, 4th, 6th, 8th, 10th, and 12th. are obtained. The inverse of some band matrixes are obtained which are required in proving the convergence analysis of the presented method. Convergence analysis of these methods is discussed. Numerical results are given to illustrate the efficiency of methods and compared with the methods in [23,32-34].

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Index to information contained in this paper

1. Introduction

In some problems arising in analysis, mechanics, geometry, etc., it is necessary to determine the maximal and minimal of a certain functional. Such problems are called variational problems. Many authors obtained analytical and numerical methods for the solution of the calculus of variations. The direct method of Ritz methods, Walsh series method, Bernstein direct method, Haar wavelet, orthogonal

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polynomials, Legendre wavelets, Adomian decomposition, and Galerkin method in solving variational problems has been of considerable concern and is well covered in many textbooks and papers see [2,6-14,17,26] and references therein. The simplest form of a variational problem can be considered as:

$$J[u_1(t), u_2(t), ..., u_n(t)] = \int_a^b H(t, u_1(t), ..., u_n(n), u'_1(t), ..., u'_n(n))dt, \tag{1}$$

with the given boundary conditions:

$$\begin{cases} u_1(a) = \alpha_1, u_2(a) = \alpha_2, ..., u_n(a) = \alpha_n, \\ u_1(b) = \beta_1, u_2(b) = \beta_2, ..., u_n(b) = \beta_n. \end{cases} \tag{2}$$

In this paper we consider special form of the variational problem in the following form:

$$J[u(t)] = \int_a^b H(t, u(t), u'(t))dt, \tag{3}$$

with boundary conditions

$$u(a) = \lambda_1, u(b) = \lambda_2, \tag{4}$$

We known that the function solution should satisfy in the following equation (Euler-Lagrange equation):

$$H_u - \frac{d}{dt} H_{u'} = 0, \tag{5}$$

with same boundary conditions. Some authors obtained numerical methods for the solution of boundary value problems using finite difference, spline and non-polynomial spline for example see [1,3-5,15,16,18-25,27-34]. The basic motivation of this paper is discussed convergence analysis of the non-polynomial spline for solutions of calculus of variations problems. The paper is organized in four sections. We use the consistency relation of non-polynomial septic spline for approximate the solution of (3)-(4). Section 2 is devoted to the description of the method and development of boundary conditions and also we obtain the methods of order 2th, 4th, 6th, 8th, 10th, and 12th. The new approach for convergence analysis is discussed in section 3. Finally, in section 4, numerical evidences are included to show the practical applicability and superiority of our method and compare with the other methods.

2. Description of the Method and Development of Boundary Conditions

Let us consider a mesh with nodal points $x_i = a + ih$ on $[a, b]$ such that:

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

where $h = \frac{b-a}{n}, x_i = a + ih$, for $i = 0(1)n$. For each segment $[x_i, x_{i+1}], i = 0, 1, 2, ..., n-1$ the non-polynomial spline $Q_i(x)$ has the following form [20]:

$$Q_i(x) = a_i \text{cos}k\,(x - x_i) + b_i \text{sin}k\,(x - x_i) + c_i\,(x - x_i)^5 + d_i\,(x - x_i)^4 + e_i\,(x - x_i)^3$$

$$+\kappa_i\left(x-x_i\right)^2+g_i\left(x-x_i\right)+r_i, i=0,...,n, \tag{6}$$

where $a_i, b_i, c_i, d_i, e_i, \kappa_i, g_i$, and r_i are constants and also k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Let $u(x)$ be the exact solution, and let S_i be an approximation to u_i obtained by the segment $Q_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). The non-polynomial spline is defined by the following relations:

$$\begin{cases} S(x)=Q_i(x), & x\in[x_i,x_{i+1}], i=0,...,n-1, \\ S(x)\in C^{\infty}[a,b]. \end{cases}$$

To derive the coefficients $a_i, b_i, c_i, d_i, e_i, \kappa_i, g_i$, and r_i, we first define:

$$\begin{cases} Q_i(x_i)=S_i, & Q_i'(x_i)=m_i, & Q_i^{(2)}(x_i)=M_i, & Q_i^{(6)}(x_i)=L_i, \\ Q_i(x_{i+1})=S_{i+1}, & Q_{i+1}'(x_{i+1})=m_{i+1}, & Q_i^{(2)}(x_{i+1})=M_{i+1}, & Q_i^{(6)}(x_{i+1})=L_{i+1}. \end{cases}$$

By algebraic manipulation we get the following expression where $\theta=kh$:

$$a_i=-\frac{L_i}{k^6}, b_i=-\frac{-\cot(\theta)L_i+\csc(\theta)L_{1+i}}{k^6}, \kappa_i=-\frac{L_i-k^4M_i}{2k^4}, r_i=\frac{L_i}{k^6}+u_i,$$

$$g_i=-\frac{\cot(\theta)L_i-\csc(\theta)L_{1+i}-k^5m_i}{k^5},$$

$$\begin{aligned} c_i=-\frac{1}{20h^5k^6}&\left(120L_i-10h^2k^2L_i-60hk\cot(\theta)L_i-60hk\cos(\theta)\cot(\theta)L_i\right.\\ &-60hk\sin(\theta)L_i-120L_{1+i}+10h^2k^2L_{1+i}+60hk\cot(\theta)L_{1+i}+60hk\csc(\theta)L_{1+i}\\ &\left.+60hk^6m_i+60hk^6m_{1+i}+10h^2k^6M_i-10h^2k^6M_{1+i}+120k^6u_i-120k^6u_{1+i}\right), \end{aligned}$$

$$\begin{aligned} d_i=-\frac{1}{4h^4k^6}&\left(-60L_i+6h^2k^2L_i+32hk\cot(\theta)L_i+28hk\cos(\theta)\cot(\theta)L_i\right.\\ &+28hk\sin(\theta)L_i+60L_{1+i}-4h^2k^2L_{1+i}-28hk\cot(\theta)L_{1+i}-32hk\csc(\theta)L_{1+i}\\ &\left.-32hk^6m_i-28hk^6m_{1+i}-6h^2k^6M_i+4h^2k^6M_{1+i}-60k^6u_i+60k^6u_{1+i}\right), \end{aligned}$$

$$\begin{aligned} e_i=-\frac{1}{2h^3k^6}&\left(20L_i-3h^2k^2L_i-12hk\cot(\theta)L_i-8hk\cos(\theta)\cot(\theta)L_i\right.\\ &-8hk\sin(\theta)L_i-20L_{1+i}+h^2k^2L_{1+i}+8hk\cot(\theta)L_{1+i}+12hk\csc(\theta)L_{1+i}\\ &\left.+12hk^6m_i+8hk^6m_{1+i}+3h^2k^6M_i-h^2k^6M_{1+i}+20k^6u_i-20k^6u_{1+i}\right). \end{aligned}$$

Using the continuity of the third, fourth and fifth derivatives, that is $Q_{i-1}^{(\rho)}(x)=Q_i^{(\rho)}(x), \rho=3,4,$ and 5, we obtain the following useful consistency relation in terms of second derivative of spline M_i and u_i.

$$\alpha_1(M_{i-3}+M_{i+3})+\alpha_2(M_{i-2}+M_{i+2})+\alpha_3(M_{i-1}+M_{i+1})+\alpha_4 M_i=[(u_{i+3}+u_{i-3})$$

$$+\beta_1(u_{i+2}+u_{i-2})+\beta_2(u_{i+1}+u_{i-1})+\beta_3 u_i]\frac{1}{h^2}, i=3,...,n-3, \qquad (7)$$

where
$\alpha_1=\frac{(120\theta-20\theta^3+\theta^5-120\sin(\theta))}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,
$\alpha_2=\frac{-2(240\theta+20\theta^3-13\theta^5+\theta(120-20\theta^2+\theta^4)\cos(\theta)-360\sin(\theta))}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,
$\alpha_3=\frac{(840\theta+100\theta^3+67\theta^5+(960\theta+80\theta^3-52\theta^5)\cos(\theta)1800\sin(\theta))}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,
$\alpha_4=-\frac{4(240\theta+20\theta^3-13\theta^5+3\theta(120+20\theta^2+11\theta^4)\cos[\theta]-600\sin(\theta))}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,
$\beta_1=-\frac{40\theta^2(-\theta(12+\theta^2)+\theta(-6+\theta^2)\cos(\theta)+18\sin(\theta))}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,
$\beta_2=-\frac{20\theta^2(42\theta+5\theta^3+4\theta(12+\theta^2)\cos(\theta)-90\sin(\theta))}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,
$\beta_3=\frac{80\theta^2(12\theta+\theta^3+3\theta(6+\theta^2)\cos(\theta)-30\sin[\theta])}{20\theta^2(-6\theta+\theta^3+6\sin(\theta))}$,

By expanding (7) in Taylor series about x_i, we obtain the following local truncation error t_i:

$$t_i=-(2+2\beta_1+2\beta_2+\beta_3)\,u_i+h^2u_i^{(2)}\,(-9+2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4-4\beta_1-\beta_2)$$

$$+\frac{1}{12}h^4u_i^{(4)}\,(-81+108\alpha_1+48\alpha_2+12\alpha_3-16\beta_1-\beta_2)$$

$$+\frac{1}{360}h^6u_i^{(6)}\,(-729+2430\alpha_1+480\alpha_2+30\alpha_3-64\beta_1-\beta_2)$$

$$+\frac{1}{20160}h^8u_i^{(8)}\,(-6561+40824\alpha_1+3584\alpha_2+56\alpha_3-256\beta_1-\beta_2)$$

$$+\frac{h^{10}}{1814400}u_i^{(10)}\,(-59049+590490\alpha_1+23040\alpha_2+90\alpha_3-1024\beta_1-\beta_2)$$

$$+\frac{h^{12}}{239500800}u_i^{(12)}\,(-531441+7794468\alpha_1+135168\alpha_2+132\alpha_3-4096\beta_1)$$

$$+\frac{h^{14}u_i^{(14)}}{43589145600}\,(-4782969+96722262\alpha_1+745472\alpha_2+182\alpha_3-16384\beta_1-\beta_2)$$

$$+O(h^{15}). \qquad (8)$$

By using the above truncation error to eliminate the coefficients of various powers h we can obtain classes of the methods. For different choices of parameters

$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ and β_3, we get the class of methods such as:

(I) *Second-order method.* For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = \alpha_4 = 0, \beta_1 = 24, \beta_2 = 15$ and $\beta_3 = -80$, we have

$$t_i = 450h^4 u_i^{(4)} + O(h^5).$$

(II) *Fourth-order method.* For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = -450, \alpha_4 = 900, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we have

$$t_i = 307h^6 u_i^{(6)} + O(h^7).$$

(III) *Sixth-order method.* For $\alpha_1 = \frac{1}{42}, \alpha_2 = \frac{20}{7}, \alpha_3 = \frac{397}{14}, \alpha_4 = \frac{1208}{21}, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we have

$$t_i = \frac{h^8 u_i^{(8)}}{252} + O(h^9).$$

(IV) *Eighth-order method.* For $\alpha_1 = \frac{337}{4927}, \alpha_2 = \frac{39712}{44343}, \alpha_3 = \frac{16285}{44343}, \alpha_4 = 0, \beta_1 = \frac{-33536}{14781}, \beta_2 = \frac{18755}{14781}$, and $\beta_3 = 0$, we have

$$t_i = \frac{65629h^{10} u_i^{(10)}}{27936090} + O(h^{11}).$$

(V) *Tenth-order method.* For $\alpha_1 = \frac{2867}{62559}, \alpha_2 = \frac{34180}{20853}, \alpha_3 = \frac{158339}{20853}, \alpha_4 = \frac{525032}{62559}, \beta_1 = \frac{6272}{993}, \beta_2 = -\frac{7265}{993}$, and $\beta_3 = 0$, we have

$$t_i = \frac{3319h^{12} u_i^{(12)}}{35390520} + O(h^{13}).$$

(VI) *Twelve-order method.* For $\alpha_1 = \frac{1857}{49483}, \alpha_2 = \frac{110322}{49483}, \alpha_3 = \frac{989739}{49483}, \alpha_4 = \frac{2175924}{49483}, \beta_1 = \frac{112266}{7069}, \beta_2 = \frac{112995}{7069}$, and $\beta_3 = -\frac{464660}{7069}$, we have

$$t_i = \frac{114669h^{14} u_i^{(14)}}{19812993200} + O(h^{15}).$$

we assume that

$$u_i'' = f(x_i, u_i) = f_i \equiv f(x_i, u(x_i)), \tag{9}$$

where u_i is the approximation of the exact value $u(x_i)$ and $S_i(x)$ is non-polynomial septic spline function [20]. By substituting (9) in the spline relation (7), we obtain the nonlinear equations in the following form.

$$\alpha_1(f_{i-3} + f_{i+3}) + \alpha_2(f_{i-2} + f_{i+2}) + \alpha_3(f_{i-1} + f_{i+1}) + \alpha_4 f_i$$

$$-\frac{1}{h^2}((u_{i+3} + u_{i-3}) + \beta_1(u_{i+2} + u_{i-2}) + \beta_2(u_{i+1} + u_{i-1}) + \beta_3 u_i) = 0,$$

$$i = 3, ..., n-3. \tag{10}$$

To obtain unique solution for the nonlinear system (10) we need four more equations. We define the following identities:

$$\begin{aligned}
&\sum_{k=0}^{4} \gamma_k u_k + h^2 \sum_{k=1}^{12} \eta_k u_k'' + t_1 h^{14} u_0^{(14)} = 0, && i = 1,\\
&\sum_{k=0}^{5} \mu_k u_k + h^2 \sum_{k=1}^{12} \sigma_k u_k'' + t_2 h^{14} u_0^{(14)} = 0, && i = 2,\\
&\sum_{k=0}^{5} \mu_k u_{n-k} + h^2 \sum_{k=1}^{12} \sigma_k u_{n-k}'' + t_{n-2} h^{14} u_0^{(14)} = 0, && i = n-2,\\
&\sum_{k=0}^{4} \gamma_k u_{n-k} + h^2 \sum_{k=1}^{12} \eta_k u_{n-k}'' + t_{n-1} h^{14} u_0^{(14)} = 0, && i = n-1,
\end{aligned} \tag{11}$$

In order that t_1, t_2, t_{n-2} and t_{n-1} are $O(h^{14})$, we find that the unknown coefficients in (11) as follows:

$$(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4) = (65, -104, 14, 24, 1), \quad t_1 = t_{n-1} = (\frac{1210210269217}{326918592000}),$$

$$(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9, \eta_{10}, \eta_{11}, \eta_{12}) =$$

$$(\frac{-2248215317}{19958400}, \frac{4539179}{17600}, \frac{-6055918291}{6652800}, \frac{9918918899}{4989600}, \frac{-892246279}{285120}, \frac{18019157507}{4989600},$$

$$\frac{-30650022317}{9979200}, \frac{2380569353}{1247400}, \frac{-16851712321}{19958400}, \frac{503717713}{1995840}, \frac{-130447723}{2851200}, \frac{18976637}{4989600}),$$

$$(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5,) = (26, 14, -80, 15, 24, 1), t_2 = t_{n-2} = (\frac{521085679991}{373621248000}),$$

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}) =$$

$$(\frac{-7712745923}{159667200}, \frac{259885933}{5702400}, \frac{-19222747601}{53222400}, \frac{8351791919}{11404800}, \frac{-3712133107}{3193344}, \frac{95883047}{71280},$$

$$\frac{-2614370381}{2280960}, \frac{28466192407}{39916800}, \frac{-7204033313}{22809600}, \frac{16761737}{177408}, \frac{-43435489}{2534400}, \frac{113795873}{79833600}).$$

3. Convergence Analysis

In this section, we investigate the convergence analysis of the tenth-order method and also in the same way we can prove the convergence analysis for any of the other methods. The equations (10) along with boundary condition (11) yields nonlinear system of equations, and may be written in a matrix form as

$$A_0 U^{(1)} + \lambda h^2 B \mathbf{f}^{(1)}(U^{(1)}) = R^{(1)}, \tag{12}$$

$$(\text{where } \mathbf{f}^{(1)}(U^{(1)}) = (f_1, ..., f_{n-1})^t,$$

the matrices A_0 and B are an $(n-1)\times(n-1)$-dimensional which have the following forms

$$A_0 = -(P_{n-1}(1,2,1))^3 + 30(P_{n-1}(1,2,1))^2 - 120 P_{n-1}(1,2,1), \tag{13}$$

where the matrix $P_{n-1}(x,z,y)$ has the following form:

$$P_{n-1}(x,z,y) = \begin{pmatrix} z & -y & & & \\ -x & z & -y & & \\ & \ddots & \ddots & \ddots & \\ & & -x & z & -y \\ & & & -x & z \end{pmatrix}, \tag{14}$$

$$B = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 & \eta_9 & \eta_{10} & \eta_{11} & \eta_{12} & & \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & & \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & & & & & & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & & & & \\ & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 & \alpha_2 & & & & \\ & & \sigma_{12} & \sigma_{11} & \sigma_{10} & \sigma_9 & \sigma_8 & \sigma_7 & \sigma_6 & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \\ & & \eta_{12} & \eta_{11} & \eta_{10} & \eta_9 & \eta_8 & \eta_7 & \eta_6 & \eta_5 & \eta_4 & \eta_3 & \eta_2 & \eta_1 \end{bmatrix}. \tag{15}$$

$$R^{(1)} = \begin{pmatrix} -65u_0, \\ -26u_0, \\ -u_0 - \lambda\alpha_1, \\ 0 \\ \vdots \\ 0 \\ -u_n - \lambda\alpha_1, \\ -26u_0, \\ -65u_0, \end{pmatrix}. \tag{16}$$

We assume that

$$A_0 \overline{U}^{(1)} + \lambda h^2 B \mathbf{f}^{(1)}(\overline{U}^{(1)}) = R^{(1)} + t^{(1)}, \tag{17}$$

where the vector $\overline{U}^{(1)} = u(x_i), (i = 1, 2, ..., n-1)$, is the exact solution and $t^{(1)} = [t_1, t_2, ..., t_{n-1}]^T$, is the vector of local truncation error.
By using (12) and (17) we get

$$AE^{(1)} = [A_0 + \lambda h^2 BF_k(U^{(1)})]E^{(1)} = t^{(1)}, \tag{18}$$

where

$$E^{(1)} = \overline{U}^{(1)} - U^{(1)},$$

$$\mathbf{f}^{(1)}(\overline{U}^{(1)}) - \mathbf{f}^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)}, \tag{19}$$

and $F_k(U^{(1)}) = \mathrm{diag}\{\frac{\partial f_i}{\partial u_i}\}, (i = 1, 2, ..., n-1)$, is a diagonal matrix of order $n-1$. To prove the existence of A^{-1}, since $A = A_0 + h^4 BF_k(U^{(1)})$, we have to show $A_0 = -P^3_{n-1}(1, 2, 1) + 30P^2_{n-1}(1, 2, 1) - 120P_{n-1}(1, 2, 1)$, is nonsingular.

By using Henrici [15] we have

$$\|(P_{n-1}(1, 2, 1))^{-1}\| \leqslant \frac{(b-a)^2}{8h^2}. \tag{20}$$

It is clear that the matrix A_0 is nonsingular and also $\|A_0^{-1}\| < \omega$ where ω is a positive number ($\|.\|$ is the L_∞ norm).

Theorem 3.1 *If* $Y < \frac{1}{\lambda h^2\|B\|\|A_0^{-1}\|}$, *then the matrix A given by (18) is monotone* ($Y = max|\frac{\partial f_i}{\partial u_i}|, i = 1, 2, ..., n-1$).

Proof From (18) we have

$$A = A_0 + \lambda h^2 BF_k(U^{(1)}),$$

hence $AA_0^{-1} = I + \lambda h^2 BF_k(U^{(1)})A_0^{-1}$, so that

$$A_0A^{-1} = (I + \lambda h^2 BF_k(U^{(1)})A_0^{-1})^{-1} =$$

$$= I - (\lambda h^2 BF_k(U^{(1)})A_0^{-1}) + (\lambda h^2 BF_k(U^{(1)})A_0^{-1})^2 - (\lambda h^2 BF_k(U^{(1)})A_0^{-1})^3 + ...$$

$$= [I - (\lambda h^2 BF_k(U^{(1)})A_0^{-1})][I + (\lambda h^2 BF_k(U^{(1)})A_0^{-1})^2 + (\lambda h^2 BF_k(U^{(1)})A_0^{-1})^4 + ...].$$

Also if $\rho(\lambda h^2 BF_k(U^{(1)})A_0^{-1}) < 1$ then, the two infinite series convergence.

Let $\|F_k(U^{(1)})\| \leqslant Y = \max|\frac{\partial f_i}{\partial u_i}|, i = 1, 2, ..., n-1$, then

$$A^{-1} =$$

$$[A_0^{-1} - A_0^{-1}\lambda h^2 BF_k(U^{(1)})A_0^{-1}][I + (\lambda h^2 BF_k(U^{(1)})A_0^{-1})^2 + (\lambda h^2 BF_k(U^{(1)})A_0^{-1})^4 + ...],$$

where the infinite series is nonnegative. Hence to show that A is monotone, it sufficient to show that $[A_0^{-1} - A_0^{-1}\lambda h^2 BF_k(U^{(1)})A_0^{-1}] > 0$. Here we have

$$I > A_0^{-1}\lambda h^2 BF_k(U^{(1)}),$$

$$\|\lambda h^2 A_0^{-1} BF_k(U^{(1)})\| \leqslant \lambda h^2 \|A_0^{-1}\|\|B\|\|F_k(U^{(1)})\| < 1. \tag{21}$$

Then

$$Y < \frac{1}{\lambda h^2 \|B\|\|A_0^{-1}\|}.$$

■

THEOREM 3.2 *Let $u(x)$ be the exact solution of the boundary value problem (4)-(5) and assume u_i, $i = 1, 2, ..., n-1$, be the numerical solution obtained by solving the system (12). Then we have*

$$\|E\| \equiv O(h^{10}),$$

provided $Y < \frac{17325}{279548636\lambda h^2 \omega}$, *where*

$$\alpha_1 = \frac{2867}{62559}, \alpha_2 = \frac{34180}{20853}, \alpha_3 = \frac{158339}{20853}, \alpha_4 = \frac{525032}{62559}, \beta_1 = \frac{6272}{993}, \beta_2 = -\frac{7265}{993}, \beta_3 = 0,$$

Proof We can write the error equation (18) in the following form

$$E^{(1)} = (A_0 + \lambda h^2 BF_k(U^{(1)}))^{-1} t^{(1)} = (I + \lambda h^2 A_0^{-1} BF_k(U^{(1)}))^{-1} A_0^{-1} t^{(1)},$$

$$\|E^{(1)}\| \leqslant \|(I + \lambda h^2 A_0^{-1} BF_k(U^{(1)}))^{-1}\|\|A_0^{-1}\|\|t^{(1)}\|\|,$$

it follows that

$$\|E^{(1)}\| \leqslant \frac{\|A_0^{-1}\|\|t^{(1)}\|}{1 - \lambda h^2 \|A_0^{-1}\|\|B\|\|F_k(U^{(1)})\|}, \tag{22}$$

provided that $\lambda h^2 \|A_0^{-1}\|\|B\|\|F_k(U^{(1)})\| < 1$. Following [22] we have

$$\|t^{(1)}\| \leqslant \frac{3319 h^{12} M_{12}}{35390520}, \tag{23}$$

where $M_{12} = \max|u^{(12)}(\xi)|, a \leqslant \xi \leqslant b$.
From inequalities (22), (23), $\|A_0^{-1}\| < \omega$, $\|F_k(U^{(1)})\| \leqslant Y$ ($Y = \max|\frac{\partial f_i}{\partial u_i}|, i = 1, 2, ..., n-1,$) and $\|B\| \leqslant \frac{279548636}{17325}$ we obtain

$$\|E\| \leqslant \frac{17325\omega h^{12}M_{12}}{252(1-279548636\lambda h^2\omega Y)} \equiv O(h^{10}), \tag{24}$$

provided that

$$Y < \frac{17325}{279548636\lambda h^2\omega}. \tag{25}$$

■

Corollary
In the same manner we can prove the convergence analysis of the other methods and we get:
(i) For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = \alpha_4 = 0, \beta_1 = 24, \beta_2 = 15$ and $\beta_3 = -80$, we have

$$\|E\| \equiv O(h^2).$$

(ii) For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = -450, \alpha_4 = 900, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we get

$$\|E\| \equiv O(h^4).$$

(iii) For $\alpha_1 = \frac{1}{42}, \alpha_2 = \frac{20}{7}, \alpha_3 = \frac{397}{14}, \alpha_4 = \frac{1208}{21}, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we obtain

$$\|E\| \equiv O(h^6).$$

(iv) For $\alpha_1 = \frac{337}{4927}, \alpha_2 = \frac{39712}{44343}, \alpha_3 = \frac{16285}{44343}, \alpha_4 = 0, \beta_1 = \frac{-33536}{14781}, \beta_2 = \frac{18755}{14781}$, and $\beta_3 = 0$, we have

$$\|E\| \equiv O(h^8).$$

(v) For $\alpha_1 = \frac{2867}{62559}, \alpha_2 = \frac{34180}{20853}, \alpha_3 = \frac{158339}{20853}, \alpha_4 = \frac{525032}{62559}, \beta_1 = \frac{6272}{993}, \beta_2 = -\frac{7265}{993}$, and $\beta_3 = 0$, we get

$$\|E\| \equiv O(h^{10}).$$

(vi) For $\alpha_1 = \frac{1857}{49483}, \alpha_2 = \frac{110322}{49483}, \alpha_3 = \frac{989739}{49483}, \alpha_4 = \frac{2175924}{49483}, \beta_1 = \frac{112266}{7069}, \beta_2 = \frac{112995}{7069}$, and $\beta_3 = -\frac{464660}{7069}$, we have

$$\|E\| \equiv O(h^{12}).$$

It follows $\|E\| \longrightarrow 0$ as $h \longrightarrow 0$. Therefore the convergence of the methods have been established.

4. Numerical Illustrations

In order to test the viability of the proposed methods based on non-polynomial spline and to demonstrate its convergence computationally, we consider three examples. Example 1 and 2 has been solved using our methods with different values of $n, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3$ in (8),

Table 1. Observed maximum absolute errors for Example 1.

n	Sixth-Order	Eight-Order	Tenth-Order	Twelve-Order
16	7.03×10^{-9}	1.98×10^{-8}	2.19×10^{-11}	2.31×10^{-11}
32	1.25×10^{-10}	7.13×10^{-11}	2.30×10^{-15}	2.50×10^{-15}
64	2.03×10^{-12}	2.72×10^{-13}	9.67×10^{-19}	2.05×10^{-19}
128	3.19×10^{-14}	1.05×10^{-15}	1.16×10^{-21}	1.45×10^{-23}
256	5.01×10^{-16}	4.09×10^{-18}	9.93×10^{-25}	9.56×10^{-28}
512	7.82×10^{-18}	1.60×10^{-20}	9.70×10^{-28}	6.06×10^{-32}
1024	1.22×10^{-19}	6.25×10^{-23}	9.48×10^{-31}	3.75×10^{-36}

(i) For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = \alpha_4 = 0, \beta_1 = 24, \beta_2 = 15$ and $\beta_3 = -80$, we have second-order method.
(ii) For $\alpha_1 = 50, \alpha_2 = 10, \alpha_3 = -450, \alpha_4 = 900, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we get fourth-order method.
(iii) For $\alpha_1 = \frac{1}{42}, \alpha_2 = \frac{20}{7}, \alpha_3 = \frac{397}{14}, \alpha_4 = \frac{1208}{21}, \beta_1 = 24, \beta_2 = 15$, and $\beta_3 = -80$, we obtain sixth-order method.
(iv) For $\alpha_1 = \frac{337}{4927}, \alpha_2 = \frac{39712}{44343}, \alpha_3 = \frac{16285}{44343}, \alpha_4 = 0, \beta_1 = \frac{-33536}{14781}, \beta_2 = \frac{18755}{14781}$, and $\beta_3 = 0$, we have eighth-order method.
(v) For $\alpha_1 = \frac{2867}{62559}, \alpha_2 = \frac{34180}{20853}, \alpha_3 = \frac{158339}{20853}, \alpha_4 = \frac{525032}{62559}, \beta_1 = \frac{6272}{993}, \beta_2 = -\frac{7265}{993}$, and $\beta_3 = 0$, we get tenth-order method.
(vi) For $\alpha_1 = \frac{1857}{49483}, \alpha_2 = \frac{110322}{49483}, \alpha_3 = \frac{989739}{49483}, \alpha_4 = \frac{2175924}{49483}, \beta_1 = \frac{112266}{7069}, \beta_2 = \frac{112995}{7069}$, and $\beta_3 = -\frac{464660}{7069}$, we have twelve-order method and also compared the obtained solution with the exact solution. The maximum absolute errors in solutions are tabulated in Tables 1 and 2 and the maximum absolute errors in solutions of example 1 are compared with methods in [23,32-34]. In example 3 which has no exact solution, the maximum absolute errors in solutions of example 3 are obtained with compared solutions in $n = 16$ and $n = 32$. The tables show that our results are more accurate. All calculations were implemented using Mathematica6.0 with Working Precision 50.

Example 1. We consider the following variational problem

$$minJ[u(x)] = \int_0^1 (u(x) + u'(x) - 4e^{3x})^2 dx, \tag{26}$$

with given boundary conditions

$$u(0) = 0, u(1) = e^3,$$

By using Euler-Lagrange (5) equation we get

$$u''(x) - u(x) - 8e^{3x} = 0, \tag{27}$$

with the same boundary conditions. The exact solution of this problem is $u(x) = e^{(3x)}$.

Example 2. Consider the following variational problem

$$minJ[u(x)] = \int_0^{\frac{\pi}{4}} ((u(x))^2 - (u'(x))^2) dx, \tag{28}$$

with the boundary conditions:

$$u(0) = 0, u(\frac{\pi}{4}) = \sqrt{2}.$$

Table 2. The maximum absolute errors for Example 1 in [32] and [34].

n	h	$\|E_u(h)\|_\infty$ in [32]	$\|E\|$ in [34]
4	0.2500000	3.52887×10^{-3}	-
8	0.1250000	3.96710×10^{-4}	6.9109×10^{-2}
16	0.0625000	2.85156×10^{-5}	1.7165×10^{-2}
32	0.0312500	1.85427×10^{-6}	4.2845×10^{-3}
64	0.0156250	1.17167×10^{-7}	1.0707×10^{-3}
128	0.0078125	7.34391×10^{-9}	2.6764×10^{-4}
256	0.00390625	-	6.6906×10^{-5}

Table 3. The maximum absolute errors for Example 1 in [33].

n	h	$\|E_u(h)\|_\infty$
5	0.200	3.10326×10^{-3}
10	0.100	1.96017×10^{-4}
20	0.050	1.22896×10^{-5}
30	0.033	2.43458×10^{-6}
40	0.025	7.70612×10^{-7}

Table 4. The maximum absolute errors for Example 1 in [23].

n	4	6	8	10	12	14
E_n	6.8×10^{-2}	6.5×10^{-4}	4.8×10^{-6}	3.3×10^{-8}	8.0×10^{-11}	2.1×10^{-13}

Table 5. Observed maximum absolute errors for Example 2.

n	Second-Order	Fourth-Order	Sixth-Order
16	8.71×10^{-4}	1.44×10^{-6}	4.51×10^{-14}
32	2.34×10^{-4}	9.66×10^{-8}	7.52×10^{-16}
64	5.97×10^{-5}	6.14×10^{-9}	1.20×10^{-17}
128	1.50×10^{-4}	3.85×10^{-10}	1.87×10^{-19}
256	3.75×10^{-6}	2.41×10^{-11}	2.93×10^{-21}
512	9.38×10^{-7}	1.51×10^{-12}	4.58×10^{-23}
1024	2.34×10^{-7}	9.41×10^{-14}	7.16×10^{-25}
n	Eight-Order	Tenth-Order	Twelve-Order
16	7.72×10^{-15}	6.15×10^{-20}	3.60×10^{-20}
32	2.96×10^{-17}	3.09×10^{-23}	2.24×10^{-24}
64	1.08×10^{-19}	2.87×10^{-26}	1.38×10^{-28}
128	4.41×10^{-22}	2.91×10^{-29}	8.44×10^{-33}
256	1.64×10^{-24}	2.73×10^{-32}	5.15×10^{-37}
512	6.42×10^{-27}	2.67×10^{-35}	3.15×10^{-41}
1024	2.51×10^{-29}	2.61×10^{-38}	1.92×10^{-45}

By using Euler-Lagrange equation we get

$$u''(x) + u(x) = 0, \tag{29}$$

with the same boundary conditions. The exact solution of this problem is $u(x) = sin(x) + cos(x)$.

Example 3. Consider the following variational problem

$$minJ[u(x)] = \int_0^1 \frac{1}{2}(u'(x) + e^{u(x)})dx, \tag{30}$$

with the boundary conditions:

$$u(0) = 0, u(1) = 1.$$

Table 6. Observed maximum absolute errors for Example 3.

x	Second-Order	Fourth-Order	Eight-Order	Twelve-Order
$\frac{1}{16}$	2.52×10^{-4}	1.13×10^{-5}	2.45×10^{-11}	2.04×10^{-12}
$\frac{4}{16}$	7.80×10^{-4}	2.27×10^{-5}	8.07×10^{-11}	3.50×10^{-12}
$\frac{8}{16}$	1.22×10^{-3}	3.85×10^{-5}	2.50×10^{-10}	5.65×10^{-12}
$\frac{12}{16}$	1.35×10^{-3}	3.57×10^{-5}	2.71×10^{-10}	8.09×10^{-12}
$\frac{15}{16}$	3.37×10^{-4}	9.62×10^{-6}	6.22×10^{-11}	1.06×10^{-12}

By using Euler-Lagrange (5) equation we get

$$u''(x) = \frac{1}{2}e^{u(x)}, \tag{31}$$

with the same boundary conditions. This example has no exact solution.

5. Conclusion

The approximate solutions of the extremum of a functional over the specified domain by using non-polynomial spline, shows that our methods are better in the sense of accuracy and applicability. These have been verified by the maximum absolute errors max$|e_i|$ given in tables. Some properties of band matrices are obtained, which are required in proving the convergence analysis of the finite difference and spline methods.

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