HIGH-ORDER NOETHER'S TRANSFORMATION AND NEW DENSITIES FOR BBM EQUATION VIA CONTACT SYMMETRIES

Abstract. Noether's first theorem shows how symmetry groups of one-parameter transformations lead to the generation of conservation laws for the Euler-Lagrange equations. She states in her second theorem that there is a relationship between Euler's basic equation and Lagrange's basic equation. This one-to-one correspondence leads to a type of symmetry called generalized symmetry. According to these materials, in this paper, we want to obtain these types of symmetries for the Benjamin-Bona-Mahony (BBM) equation and show that each symmetry is connected to a specific conservation law. For this purpose, we obtain the symmetries of this equation using the Lie symmetry method, and then using the adjoint operator, we provide a classification on the group invariant solutions of this equation. Then, by applying Noether's method, we obtain a new conservation law for each symmetry.

1. Introduction

When we work with differential equations, the symmetry group can be very helpful in finding solutions. By integrating once and reducing the equation's order to one, we can solve simple equations using this method. However, for equations with partial derivatives, the general solution cannot be easily found using the symmetry group unless the system can be converted into a linear system. In this case, we can only find solutions that belong to specific subgroups of the symmetry group, which are called group invariant solutions. These solutions have fewer separate variables than the original system [6, 14].

One of Lie's important findings in the study of differential equations is that when a dynamic system is moved just a tiny bit, the complex rules that control the system become easier to understand. This is very important in the field of physics [18]. Another way we can use the symmetry group is to help us find conservation laws. In 1918, Emmy Noether explained two important ideas about symmetry groups. These ideas showed how the number of changes in a group relates to certain equations called Euler-Lagrange equations [14, 15]. Noether's first theorem explained how groups that have one parameter and involve changes in the way things are calculated can lead to the development of conservation laws for the Euler-Lagrange equations. This correspondence is a new set of symmetries called generalized symmetries [14]. The main distinction between this type of symmetry and geometric symmetries is that in addition to independent and dependent variables, these symmetries also include the derivative of dependent variables. Suppose

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the coefficients of the infinitesimal generators of the group of transformations are only functions of the independent and dependent variables of the base space. In that case, the transformations group is called the group of Lie point symmetries [2, 11]. If this restriction of coefficients is removed, then the coefficients of the infinitesimal generators of the group of transformations are also a function of dependent derivatives (like the coefficients of the generators of the extended group in the jet space), in this case, the obtained symmetries are higher (generalized) symmetries or Lie-Backlund symmetries. Geometrically, Lie-Backlund transformations are higher-order generalizations of contact transformations (first-order tangent). They can be considered as tangent transformations of infinite order. By generalizing infinitesimal generators of point and contact transformation groups, we approach the theory of Lie-Backlund transformation groups. This generalization is known as the Lie-Backlund operator. It's important to note that any one-parameter group could potentially be a variational problem, regardless of whether it comes from geometric or generalized symmetries [1]. This has the potential to create conservation laws, and conversely, any conservation law associated with these symmetries can be transformed into a variational problem. When it comes to studying a system, the conservation laws are essentially a divergence expression of the differential equations that becomes zero on all solutions. This means that if a system of differential equations has a conservation law, it's because there are certain numbers that can be used with specific operations to make the equations disappear.

In this work, we figure out the Lie symmetry and the generalized symmetry of the BBM equation. Then, we use this type of symmetry to obtain new conservation laws for this equation.

The BBM equation is a partial differential equation that describes wavelengths in some nonlinear systems. Consider the BBM or Regular-Long-Wave equation as follows:

$$\Delta := u_t + u_x + uu_x + u_{xxx} = 0, \tag{1.1}$$

which was foremost presented by Benjamin-Bona-Mahony to enhance the Kortewage-de-Vries (KdV) equation for short amplitude wavelengths with dimension 1+1 [4, 5, 8]. Ogawa [13] analyzes the reality of periodic waves and solitary waves of (1.1) in 1994 and shows the association between the wavelength and the amplitude. Also, in [12], from the scaling method, the conservation laws and the exact group invariant solutions of a type of equation (1.1) have been obtained. For the perturbed generalized BBM equation, approximate symmetries and approximate solutions have been obtained using the Ibragimov method [10]. In recent times, researchers have explored a new modified BBM local fractional equation that relies on the local fractional derivative. The study yielded practical and interesting outcomes that can be of significant value in academic and professional settings. The equation has potential applications in multiple domains and may provide a robust foundation for further research in the field [16]. If we terminate terms u_{2x} and u_{4x} in equation (1.1), the KdV equation is acquired, which is widely utilized in the analysis of water waves [9, 17]. Also, important and practical studies have been done regarding this equation and the conservation laws as well as its solutions [3, 7, 8]. The innovative research presented here sets itself apart from previous studies by utilizing direct methods and uncovering new conservation laws for this equation. The meticulous calculation technique employed leaves no doubt that these equations are non-equivalent and non-trivial. This groundbreaking research opens up new avenues for exploration and represents an important contribution to the field.

This article is organized as follows: In Section 2, we discuss necessary and useful concepts and obtain the 1-dimensional optimal system for the BBM equation. In the third Section, we find the conservation laws of the equation by the direct method and using multipliers. Finally, in Section, we get contact transformations and generalized symmetries or the Noether's symmetries of the BBM equation.

2. Group Invariant Solutions and Optimal System of BBM Equation

In this section, we will review the key definitions and theorems that are essential for the rest of the work. It is important to have a clear understanding of these concepts in order to properly apply them later on and then we obtain the group invariant solutions and optimal system of BBM equation.

Consider a system of p independent variables $x = (x^1, ..., x^p)$ and q dependent variables $u(x) = (u^1, ..., u^q)$ represented by partial differential equations of rank n as follows:

$$\Delta_\eta(x, u, u^{(n)}) \equiv 0, \qquad \eta = 1, 2, 3, .., k. \tag{2.2}$$

Such a system's solution is an equation of the type $u = f(x)$, where

$$u^\alpha = f^\alpha(x^1, x^2, x^3, ..., x^p), \qquad \alpha = 1, 2, 3, ..q,$$

are smooth functions of independent variables. Therefore, we assume that X represents a coordinate system on $\mathbb{R}^p$ and U represents a coordinate system on $\mathbb{R}^q$.

Definition 2.1. *A transformations local group like G that act on an open subset of $X \times U$ like $\mathcal{O}$ to change each solution of system $\Delta = 0$ into another solution is a symmetry group for a system of differential equations $\Delta = 0$.*

Definition 2.2. *Let $M \subset X \times U$ ba an open subset, the vector field on M is a vector like v and we can represent it as follows:*

$$v = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{2.3}$$

Also, the n-th prolongation of vector field v is defined as follows

$$Pr^{(n)}(v) = v + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha}. \tag{2.4}$$

Note that the coefficients in the above equation are defined as follows

$$\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha) + \sum_{i=1}^{p} \xi^i u_i^\alpha, \qquad \alpha = 1, 2, ..., q, \tag{2.5}$$

where

$$Q_\alpha(x,u^{(1)}) = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \qquad \alpha = 1,2,...,q, \tag{2.6}$$

and q-tuple $Q_\alpha(x,u^{(1)}) = (Q_1,...,Q_q)$ *called the characteristic of* v.

Theorem 2.3. *Let*

$$\Delta_k(x,u^{(n)}) \equiv 0, \qquad k = 1,2,...,l, \tag{2.7}$$

be a differential equations system of maximum rank that is defined on an open subset $\mathcal{O}$, *and* v *is the infinitesimal generator of* G *as a local group of transformations act on* $\mathcal{O}$. *Then* G *is the system's symmetry group if and only if*

$$Pr^{(n)}(\Delta_k(x,u^{(n)})) \equiv 0, k = 1,2,...,l, \Longleftrightarrow \Delta_k(x,u^{(n)}) \equiv 0. \tag{2.8}$$

Now, we consider the BBM equation

$$u_t + u_x + uu_x + u_{xxx} = 0$$

As we already understand, the foundation of the infinitesimal generating group method is in a way which reduces the independent variables in PDEs. it reduces the order in ODE equations. Now we put the infinitesimal group as below

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{2.9}$$

as well as the state of invariance (1.1). Invariance by applying (2.9) states that u remains a solution to (1.1) even after applying (2.9) (if u is solution then $\tilde{u}$ is solution too).
We have the following general form of vector field for the BBM equation

$$v = \xi(x,t,u)\partial x + \tau(x,t,u)\partial t + \phi(x,t,u)\partial u, \tag{2.10}$$

which must apply to relation $Pr^{(3)}(v(\Delta)) = 0$. To discover infinitesimal generators v, we must first calculate all feasible coefficient values of ξ, τ and ϕ. This necessitates calculating the third prolongation of v,

$$\begin{aligned} Pr^{(1)}(v) &= v + \phi^t \frac{\partial}{\partial u_t} + \phi^x \frac{\partial}{\partial u_x}, \\ Pr^{(2)}(v) &= v + Pr^{(1)}(v) + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{tx}\frac{\partial}{\partial u_{tx}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}, \\ Pr^{(3)}(v) &= Pr^{(2)}(v) + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxt}\frac{\partial}{\partial u_{xxt}} + \phi^{xtt}\frac{\partial}{\partial u_{xtt}} + \phi^{ttt}\frac{\partial}{\partial u_{ttt}}, \end{aligned} \tag{2.11}$$

with the coefficients

$$
\begin{aligned}
\phi^{x} &= D_x Q + \xi^1 u_{xx} + \xi^2 u_{xt},\\
\phi^{tx} &= D_x D_t Q + \xi^1 u_{xxt} + \xi^2 u_{xtt},\\
\phi^{xxx} &= D_x^3 Q + \xi^1 u_{xxxx} + \xi^2 u_{xxxt},\\
\phi^{xxt} &= D_x^2 D_t Q + \xi^1 u_{xxtt} + \xi^2 u_{xttt}.\\
&\vdots
\end{aligned} \tag{2.12}
$$

To find the infinitesimal generator for the BBM equation, we must to solve the following equation obtained by acting $Pr^{(3)}(v)$ to BBM equation.

$$
\begin{aligned}
&\phi_u^2 = 0, \quad \eta_x = \eta_u = \xi_u = 0, \quad \xi_x = -\frac{1}{2}\phi_u, \quad \eta_t = -\frac{3}{2}\phi_u,\\
&\xi_t = -\phi_u, \quad u - \phi_u + \phi.
\end{aligned} \tag{2.13}
$$

When we solve the system, we get these solutions for the vector's coefficients v:

$$
\xi^1 = \frac{1}{2}(-x-2t)c_1 + c_2 t + c_4, \quad \xi^2 = -\frac{3}{2}c_1 t + c_3, \quad \phi = c_1 u + c_2, \tag{2.14}
$$

such that c_1, c_2, and c_3 are arbitrary fixed numbers. So the BBM equation has four vector fields

$v_1 = (\frac{1}{2}x + t)\partial x + \frac{3}{2}t\partial t - u\partial u, \quad v_2 = \partial u + t\partial t, \quad v_3 = t\partial x + \partial u + \partial t, \quad v_4 = \partial t + \partial x,$
that create the Lie algebra $\mathfrak{g}$. Also their commutator table is:

Table 1. The table of commutators

$[,]$	v_1	v_2	v_3	v_4
v_1	0	v_2	$\frac{3}{2}v_2 - \frac{1}{2}v_3 - v_4$	$-\frac{3}{2}v_4$
v_2	$-v_2$	0	$-v_2 + v_3 - v_4$	$-v_2 + v_3 - v_4$
v_3	$-\frac{3}{2}v_2 + \frac{1}{2}v_3 + v_4$	$v_2 - v_3 + v_4$	0	$-v_2 + v_3 - v_4$
v_4	$\frac{3}{2}v_4$	$v_2 - v_3 + v_4$	$v_2 - v_3 + v_4$	0

Lemma 2.4. *The groups $W_i(\varepsilon)$ are made by vectors v_1, v_2, v_3 and v_4 as follows*

$$
\begin{aligned}
W_1(x,t,u) &= (e^{\frac{1}{2}\varepsilon}(-t+x) + te^{\frac{3}{2}\varepsilon}, te^{\frac{3}{2}\varepsilon}, ue^{-\varepsilon}),\\
W_2(x,t,u) &= (x + t\varepsilon, t, \varepsilon + u),\\
W_3(x,t,u) &= (\frac{1}{2}\varepsilon^2 + x + t\varepsilon, \varepsilon + t, u + \varepsilon),\\
W_4(x,t,u) &= (x+\varepsilon, t+\varepsilon, u).
\end{aligned} \tag{2.15}
$$

Because every part of a system has a different subgroup parameter in the full symmetry group, there are solutions called invariant solutions. Thus, the following theorem is proven.

Theorem 2.5. *Let $u = f(x,t)$ satisfies equation (1.1), then the following functions are solutions too:*

$$\begin{aligned} W_1(\varepsilon).f(x,t) &= e^{\varepsilon}(te^{\frac{3}{2}\varepsilon} + e^{\frac{1}{2}\varepsilon}(-t+x), te^{\frac{3}{2}\varepsilon}) \\ W_2(\varepsilon).f(x,t) &= (t\varepsilon + x, t) - \varepsilon, \\ W_3(\varepsilon).f(x,t) &= (\frac{1}{2}\varepsilon^2 + t\varepsilon + x, \varepsilon + t) - \varepsilon, \\ W_4(\varepsilon).f(x,t) &= (\varepsilon + x, \varepsilon + t), \end{aligned} \tag{2.16}$$

such that ε is a constant number.

We combine supplied vector fields as $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$ to find symmetries. and analyze the group.
Now we classify the group invariant solutions by using the classification of adjoint action orbits on the obtained subalgebras for the equation. Consider the group G, which is a Lie group. An optimum system is a list of groups that are similar to each other and have the same properties as the other groups in the list. It only belongs to one of the groups on the list. Correspondingly, an optimal system can be formed by a compilation of subalgebras. Subalgebra $\mathfrak{g}$, in particular, may be associated with a single member of the aforementioned list provided specific adjoint representation factors, namely $\tilde{h} = Ad\, g(h), g \in G$. The accurate solutions are obtained, and the symmetry of the differential equations is then established. The symmetry group affects qualities. We may find different solutions to the problem by recognizing these qualities.
Consider the adjoint action on the Lie series as follows

$$Ad(exp(\varepsilon v_i))v_j = \sum_{n=0}^{\infty}(\varepsilon^n/n!)(adv_i)^n(v_j),$$

so that $[v_i, v_j]$ represents the commutator for the Lie algebra, ε represents parameter and $i,j = 1,2,3,4$. Adjoint with (i,j) −th entry suggesting $Ad(exp(\varepsilon v_i))v_j$ in Table2

TABLE 2. The adjoint presentation table

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$e^{\varepsilon}v_2 + (-e^{\frac{1}{2}\varepsilon} + e^{-\varepsilon})v_3$	$e^{\frac{1}{2}\varepsilon}v_3$	$(e^{\frac{3}{2}\varepsilon} - e^{\frac{1}{2}\varepsilon})v_3 + e^{\frac{3}{2}\varepsilon}v_4$
v_2	v_1	$\varepsilon v_1 + v_2 + \varepsilon v_3 + \varepsilon v_4$	$(1-\varepsilon)v_3 - \varepsilon v_4$	$\varepsilon v_3 + (1+\varepsilon)v_4$
v_3	v_1	$(\frac{3}{2}\varepsilon - \frac{5}{4}\varepsilon^2)v_1$ $+(1-\varepsilon)v_2 + \varepsilon v_4$	$(-\frac{1}{2}\varepsilon + \frac{5}{4}\varepsilon^2)v_1$ $+\varepsilon v_2 + v_3 - \varepsilon v_4$	$(-\varepsilon - \frac{5}{4}\varepsilon^2)v_1$ $-\varepsilon v_2 + (1+\varepsilon)v_4$
v_4	v_1	$(1-\varepsilon)v_2 - \varepsilon v_3$	$\varepsilon v_2 + (1+\varepsilon)v_3$	$-\frac{3}{2}\varepsilon v_1 - \varepsilon v_2 - \varepsilon v_3 + v_4$

If $F_i^{\varepsilon} : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a linear map, by $v \mapsto Ad(exp(\varepsilon_i v_i).v)$, for $i = 1,2,3,4$, then the matrices M_i^{ε} for F_i^{ε} with relation to the base $\{v_i\}$, $i = 1,2,3,4$, are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\varepsilon} & -e^{\varepsilon^{\frac{1}{2}}} + e^{-\varepsilon} & 0 \\ 0 & 0 & e^{3\varepsilon} & e^{\frac{1}{2}\varepsilon} \\ 0 & 0 & e^{\frac{3}{2}\varepsilon} - e^{\frac{1}{2}\varepsilon} & e^{\frac{3}{2}\varepsilon} \end{pmatrix}, \qquad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 1 & \varepsilon & \varepsilon \\ 0 & 0 & 1-\varepsilon & -\varepsilon \\ 0 & 0 & \varepsilon & 1+\varepsilon \end{pmatrix},$$

$$M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{2}\varepsilon - \frac{5}{4}\varepsilon^2 & 1-\varepsilon & 0 & \varepsilon \\ -\frac{1}{2}\varepsilon + \frac{5}{4}\varepsilon^2 & \varepsilon & 1 & -\varepsilon \\ -\varepsilon - \frac{5}{4}\varepsilon^2 & -\varepsilon & 0 & 1+\varepsilon \end{pmatrix}, \qquad M_4^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{2}\varepsilon - \frac{5}{4}\varepsilon^2 & 1-\varepsilon & 0 & \varepsilon \\ -\frac{1}{2}\varepsilon + \frac{5}{4}\varepsilon^2 & \varepsilon & 1 & -\varepsilon \\ -\varepsilon - \frac{5}{4}\varepsilon^2 & -\varepsilon & 0 & 1+\varepsilon \end{pmatrix}.$$

Theorem 2.6. *An one-dimensional optimum system of Lie algebra $\mathfrak{g}$ is generated by*

$$Y_1 = v_1 + \alpha v_2, \qquad Y_2 = \beta v_2 + v_3, \qquad Y_3 = \gamma v_3, \tag{2.17}$$

where α, β and γ are arbitrary constants.

Proof:
If $V = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$ be a general element of $\mathfrak{g}$, first that suppose $a_1 \neq 0$, we can assume that $a_1 = 1$. According to Table (2), if we use such a V by $Ad(exp(-a_2/(1 + a_3 + a_4))$, we may eliminate the coefficient of v_2,

$$V' = v_1 + a_3' v_3 + a_4' v_4,$$

where certain values a_3', a_4' depending on the a_2, a_3, a_4 in the next step , we operate on V' with $Ad(exp(2a_4/3 + 2a_3))$ to destroy the coefficient of a_4 by leading to $V'' = v_1 + a_3' v_3$ and in the last step with $Ad(exp(a_3' v_3))$ by vanishing the residual coefficient, such that V is identical to v_1 in the adjoint expression. Any subalgebra in dimension one formed by a v with $a_1 \neq 0$ has the same meaning as the subgroup formed by v_1. Then we assume $a_1 = 0$ and $a_2 \neq 0$ and repeat the process.

3. BBM equation's local conservation law

Noether developed her renowned approach (known as Noether's theorem) for determining conservation laws for systems of equations that follow a specific pattern (known as a variational principle or action functional) in 1918. If a certain differential equations system has a principle variational, the ultimate solutions to its action differential equations will take the form of the Euler-Lagrange equations. In the current paper, Noether demonstrated that if an action functional has a symmetry, the fluxes of a local conservation law may be calculated using a formula that incorporates the infinitesimal values of the symmetry and the Lagrangian.

Let Δ be a system composed of p variables $x = (x^1, ..., x^p)$ and q dependent variables $u(x) = (u^1, ..., u^q)$, each of which is described by partial differential equations of order k,

$$\Delta_\eta(x, u; u^{(k)}) \equiv 0, \qquad \eta = 1, 2, ..., l. \tag{3.18}$$

Definition 3.1. *A local conservation law for (3.18) is the divergence expression as follows*

$$D_i\Phi^i\left[u\right] = D_1\Phi^1\left[u\right] + ... + D_n\Phi^n\left[u\right] = 0, \tag{3.19}$$

applicable to all obtained solutions for differential equation system (3.18). In (3.19):

$$D_i\Phi^i\left[u\right] = \Phi^i(x, u, \partial u, \partial^2 u, ..., \partial^r u) = 0, \quad i = 1, 2, 3, ..., n.$$

The operators that signify total derivative and the fluxes corresponding to the law of conservation are respectively present. The Noether's theorem is a well-known systematic approach to determine the conservation laws as related to variable symmetries in systems of Euler-Lagrange equations. Noether's explicit formula for local conservation laws is derived from a set of local multipliers that give rise to components of local symmetries in evolutionary form. As such, it can be observed that all local conservation laws that emerge from Noether's theorem are obtained through the direct method. The direct method is a versatile approach that can be applied to any differential equation (DE) system, regardless of whether its linearized system is self-adjoint. Notably, the method does not require the determination of any functional. Furthermore, any solution of an over-determined linear PDE system that satisfies the multipliers can represent a set of local conservation law multipliers. This over-determined linear PDE system is derived directly from the given DE system.

It is worth noting that if the linearized system is self-adjoint, the local symmetry-determining equations are a subset of this over-determined linear PDE system. This observation is significant because it highlights the potential for the direct method to identify local symmetries in DE systems even when the linearized system is not self-adjoint. By its versatility and directness, the direct method is a valuable tool in the study of DE systems.

3.1. **Finding the Conservation Law With Direct Method.** When it comes to establishing local conservation laws for a wide range of partial differential equation systems, the direct method is a systematic approach that provides a realistic mechanism. This involves searching for multipliers that can be used to combine different equations in a certain way, resulting in an expression that shows how much the equations spread out (divergence expression). Collecting the coefficients of the local conservation law is crucial in recreating the fluxes associated with it. Typically, this is done in the type (3.18) non-degenerate different equations system. By seeking scalar products, nontrivial local rules that combine linear combinations of the constitutive equations in the different equations system (3.18) backed by multipliers (i.e., factors that generate high divergence terms) may be identified. To obtain these formulations, we substitute different functions for the dependent variables and their derivatives, or multipliers, in the different equations system (3.18). If the coefficients are not unique, every solution to the different equations system (3.18) leads to the disappearance of these divergence terms.

Definition 3.2. *The Euler operator for U^μ define as follows*

$$E_{U^\mu} = \partial/\partial U^\mu - D_i \partial/\partial U_i^\mu + ... + (-1)^s D_{i_1} ... D_{i_s} \partial/\partial U^\mu_{i_1 ... i_s} + ...$$

Euler operator vanishes divergences. Now, we find local conservation law multipliers of the BBM equation. For this, we let $\Lambda = \Lambda(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$ for the BBM equation, then if Λ justifies the following condition, Λ can be considered as the local conservation law multiplier for the BBM equation

$$E_u \left(\Lambda(u_t + u_x + uu_x + u_{xxx})\right) \equiv 0. \tag{3.20}$$

Then, from equations (3.19) and (3.20) we have

$$\begin{aligned} &\Lambda = 1, \\ &\Lambda = u, \\ &\Lambda = t + tu - x, \\ &\Lambda = u_{xx} + \frac{1}{2}u^2, \\ &\Lambda = uu_{xx} + \frac{1}{6}u^3 + \frac{1}{2}u^2 + \frac{3}{5}u_{xxxx}. \end{aligned} \tag{3.21}$$

3.2. **Computation of Fluxes for the Conservation Laws.** In this subsection, we obtain the corresponding flux for each of the multipliers obtained in the previous subsection.

Case1: If the multiplier is u, then

$$u(u_t + u_x + uu_x + u_{xxx}) = D_x\Psi + D_t\Phi, \tag{3.22}$$

where D represents the total derivative operators. let $\Psi = \Psi(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$ and $\Phi = \Phi(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$. Now, we determine Φ and Ψ by extending the equation (3.22):

$$\Psi_x + \Psi_u u_x + \Psi_{u_x} u_{xx} + \Psi_{u_{xx}} u_{xxx} + ... + \Phi_t + \Phi_u u_t + \Phi_{u_x} u_{xt} + ... = u(u_t + u_x + uu_x + u_{xxx}).$$

By solving the equation, we obtain

$$\begin{aligned} &\Phi = \frac{1}{2}u^2, \\ &\Psi = uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{3}u^3 + \frac{1}{2}u^2. \end{aligned} \tag{3.23}$$

Case2: If the multiplier be $\Lambda = 1$, then

$$\begin{aligned} &\Phi = u, \\ &\Psi = u_{xx} + \frac{1}{2}u^2 + u. \end{aligned} \tag{3.24}$$

Case3: If the multiplier be $\Lambda = t + tu - x$, then

$$\begin{aligned}\Phi &= \frac{1}{2}tu^2 + \frac{1}{2}(2t - 2x)u,\\ \Psi &= \frac{1}{3}tu^3 + \frac{1}{6}(-3x + 6t)u^2 + \frac{1}{6}((6u_{xx} + 6)t - 6x)u\\ &+ \frac{1}{6}(-3u_x^2 + 6u_{xx})t + u_x - xu_{xx}.\end{aligned} \tag{3.25}$$

Case4: If the multiplier be $\Lambda = u_{xx} + \frac{1}{2}u^2$, then

$$\begin{aligned}\Phi &= \frac{1}{6}u^3 + uu_{xx},\\ \Psi &= \frac{1}{2}u_x^2 + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u^2u_{xx} + \frac{1}{8}u^4 + \frac{1}{6}u^3.\end{aligned} \tag{3.26}$$

Case5: If the multiplier be $\Lambda = uu_{xx} + \frac{1}{6}u^3 + \frac{1}{2}u^2 + \frac{3}{5}u_{xxxx}$, then

$$\begin{aligned}\Phi &= \frac{1}{24}u^4 + \frac{1}{2}u_{xx}u^2 + (\frac{1}{2}u_x^2 + \frac{3}{5}u_{xxxx})u,\\ \Psi &= \frac{1}{30}u^5 + \frac{1}{24}u^4 + \frac{1}{6}u_{xx}u^3 + \frac{1}{4}u^2u_x^2\\ &+ \frac{1}{120}(24u_{xx}^2 + 60u_x^2 + 72u_{xxx}u_x)u + \frac{3}{10}u_{xxx}^2 - \frac{1}{10}u_{xx}u_x^2 + \frac{3}{5}u_{xxx}u_x - \frac{3}{10}u_{xx}^2.\end{aligned} \tag{3.27}$$

4. Contact and Generalized Transformations

Suppose that we have n independent variables $x = (x_1, ..., x_n)$ and a single dependent variable $u(x)$, then we have the following.

Definition 4.1. *The definition of a contact transformation on E is as follows:*

$$(x, u, p) \mapsto (\tilde{x}, \tilde{u}, \tilde{p}) = (\varphi(x, u, p), \psi(x, u, p), \eta(x, u, p)), \tag{4.28}$$

where $p = (u_1, ..., u_p)$ and $u_i = \partial u/\partial x^i$ are the first-order derivatives of u with respect to the variables from which it is independent. It is defined in the k-th order of u in the jet space and is a one-to-one transformation of the whole space E on the first-order derivatives of u.

Theorem 4.2. [14] *The functions $\varphi = (\varphi^1, ..., \varphi^p)$ and $\chi = (\chi^1, ..., \chi^p)$ must all be satisfied in the following relations to be a contact transformations,*

$$\begin{aligned}\frac{\partial\psi}{\partial u^i} &= \chi^i\frac{\partial\varphi^i}{\partial u^i},\\ \frac{\partial\psi}{\partial x^i} + u_i\frac{\partial\psi}{\partial u} &= \chi^i(\frac{\partial\varphi^i}{\partial x^i} + u_i\frac{\partial\varphi^j}{\partial u}), \qquad i, j = 1, 2, ..., p.\end{aligned} \tag{4.29}$$

Similar to point transformations, contact transformations can be expressed as one-parameter. Let ε be the contact transformations group's parameter, then

$$\begin{aligned} \tilde{x}^i &= x^i + \varepsilon \xi^i(x,u,p) + O(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon \phi(x,u,p) + O(\varepsilon^2), \\ \tilde{u}_i &= u_i + \varepsilon \phi_i(x,u,p) + O(\varepsilon^2). \end{aligned} \tag{4.30}$$

In the above relations ξ^i and ϕ_i derived from the infinitesimal generator in the form

$$v = \sum_{i=1}^{p} (\xi^i(x,u,p)\partial x^i + \phi_i(x,u,p)\frac{\partial}{\partial u_i}) + \phi(x,u,p)\partial u_i, \tag{4.31}$$

such that

$$\phi_i = D_i\phi - \sum_{j=1}^{p} D_i\xi^j u_j = \frac{\partial \phi}{\partial u^i} - \sum_{j=1}^{p} \frac{\partial \xi^i}{\partial x^i}\frac{\partial u}{\partial u^j}. \tag{4.32}$$

Theorem 4.3. [14] *Equations (4.30) generate the infinitesimal generator if and only if the functions ξ^i and ϕ_i hold in the following relations,*

$$\frac{\partial \phi}{\partial u_i} - \sum_{j=1}^{p} u_j \frac{\partial \xi^j}{\partial u_i} = 0, \qquad i = 1,2,...,p. \tag{4.33}$$

Theorem 4.4. [14] *The contact transformation of one-parameter Lie groups with an infinitesimal generator is the same as any one-parameter local transformation with an infinitesimal generator*

$$v = Q(x,u,u_{(1)})\partial u,$$

where

$$v = \sum_{i=1}^{p} \xi^i(x,u^{(1)})\partial_{x^i} + \phi(x,u^{(1)})\partial_u + \sum_{i=1}^{p} \xi^i(x,u^{(1)})\phi^1(x,u^{(1)})\partial u_i. \tag{4.34}$$

Therefor, the coefficients of this generator are satisfy the following expressions

$$\begin{aligned} \xi^i(x,u^{(1)}) &= \frac{\partial Q}{\partial u_i}, \\ \phi(x,u^{(1)}) &= u_i\frac{\partial Q}{\partial u_i} - Q, \\ \phi_i^{(1)}(x,u^{(1)}) &= -\frac{\partial Q}{\partial u_i} - u_i\frac{\partial Q}{\partial u}, \qquad i = 1,...,p. \end{aligned} \tag{4.35}$$

Definition 4.5. *Consider the vector field v*

$$v = \sum_{i=1}^{p} \xi^i[u]\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha[u]\frac{\partial}{\partial u^\alpha}. \tag{4.36}$$

If ξ^i and ϕ_α are smooth differential functions, then v is a generalized vector field.

Definition 4.6. *If v is a generalized vector field, $u = f(x)$ and*

$$\Delta_\nu(u) \equiv \Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, .., l, \tag{4.37}$$

be a system of PDEs that satisfy the following relations

$$Pr(v)(\Delta_v) = 0, \qquad \nu = 1, 2, .., l, \tag{4.38}$$

then v is a generalized infinitesimal symmetry for this system.

Definition 4.7. *suppose you have a q-tuple of differential functions*

$$Q[u] = (Q_1[u], ..., Q_q[u]) \epsilon \mathcal{A}^q.$$

The generalized vector field v_Q is therefore referred to as an evolutionary vector field if and only if v_Q is defined as follows, and Q is referred to as its characteristic.

$$v_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}. \tag{4.39}$$

Lemma 4.8. [14] *Only when v_Q is the evolutionary representation of a system of differential equations, the vector field v considered to be symmetry.*

Now, consider the following Benjamin-Bona-Mahoney equation

$$\Delta := u_t + u_x + uu_x + u_{xxx} = 0.$$

Let $v_Q = Q[u]\partial u$ be the generalized symmetry in evolutionary form of the mentioned equation. We want to substitute any time-based changes in u in Q with expressions that only involve changes based on x and this won't change the equivalent category of v. For example, u_t is replaced by $-u_x - uu_x - u_{xxx}$ and u_{xt} by $-u_x^2 - u_{xx} - uu_{xx} - u_{xxxx}$,u_{xxt} by $-3u_{xx}u_x - uu_{xxx} - u_{x^5}$ and so on. Every symmetry corresponds to a characteristic $Q = Q(x, t, u, u_x, u_{xx}...)$. According to the definition of the infinitesimal condition, (4.6), for invariance we have

$$D_t Q = -(D_x Q + u_x Q + u D_x Q + D_{x^3}^3 Q), \tag{4.40}$$

which must be satisfied for all solutions. We have

$$Pr(v) = u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + ...,$$

and

$$Pr(v) = Q\partial u + D_x Q \partial u_x + D_x^2 Q \partial u_{xx} + ...$$

Since we only need to hold on the solutions, so we can use (1.1) and its prolongations to replace any t derivatives of u. After examining (4.40) closely, we can identify the coefficients of u's derivatives in descending order. Based on this data, we determine that the characteristic of every fourth order generalized symmetry for the BBM equation is a linear, constant-coefficient combination of five fundamental characteristics denoted by

$$
\begin{aligned}
&Q_1 = u_x tu - (1/3)u_x x + u_{xxx}t + 1/3 - (2/3)u,\\
&Q_2 = tu_1 - 1,\\
&Q_3 = uu_x + u_{xxx},\\
&Q_4 = u_x,\\
&Q_5 = u_x tu + (1/3) + u_{xx}u + (1/4)u^2 u_x - (1/3)u_x x + u_{xxx}t + (1/2)u_{xxx}u\\
&\ - (2/3)u + (3/10)u_{xxxxx}.
\end{aligned} \tag{4.41}
$$

Therefore Generalized symmetry group (Noether transformation) are obtained as follows

$$
\begin{aligned}
&v_1 = -(3u_x tu - u_x + 3u_{xxx}t + 1 - 2u)\partial u,\\
&v_2 = -(tu_x - 1)\partial u,\\
&v_3 = uu_x + u_{xxx}\partial u,\\
&v_4 = u_x\partial u,\\
&v_5 = (1/4u^2 u_x + u_{xx}u_x + 1/2u_{xxx}u)\partial u.
\end{aligned} \tag{4.42}
$$

Theorem 4.9. *As a result of theorems (4.4) and (4.2), the contact symmetries of the BBM equation are as follows:*

$$
\begin{aligned}
&v_2 = -(tu_x - 1)\partial u,\\
&v_4 = u_x\partial u.
\end{aligned} \tag{4.43}
$$

Remarks and Dissections. A local transformation in one parameter for a generator of type

$$
v = \eta(x, u, \partial u)\frac{\partial}{\partial u}, \tag{4.44}
$$

is exclusive identical to a producer of a Lie group in dimension one of contact generators. $\eta(x, u, \partial u)$, in particular, serves as a defining function in this situation.

It is demonstrated that any infinitesimal generator in the form of (4.44) is comparable to a producer of a Lie group, and that the transformation $\eta(x, u, \partial u)$ is linear concerning the dependency first derivative. We may also deduce this conclusion from the fact that each conservation law in the Euler-Lagrange equation has a corresponding variational symmetry, as demonstrated by Noether's theorem. As outlined in the introduction, numerous attempts have been made to identify exact solutions and conservation laws for the BBM equation. This study investigates new conservation laws. To ensure the credibility of the densities, it is necessary to demonstrate that they are nontrivial, independent, and accurate by verifying that no density is a total derivative concerning x. Additionally, the definition of a conservation law is only relevant when evaluated on the given PDEs' solutions. The densities must be functionally or linearly independent, meaning that they should not be a linear combination of other densities, a derivative of another density, or a combination of both. The conservation laws derived in this study satisfy the fundamental condition of a conservation law, as defined in equation (3.19). Regarding the construction method, it is evident that these conservation laws are non-equivalent and nontrivial, and we can therefore conclude that they are innovative.

Conclusion

In this article, using multipliers and the direct method, we obtained the conservation laws of the Benjamin-Bona-Mahony (BBM) equation. In this method, the fluxes were obtained by multipliers and solving the equations that are made for their coefficients. Also, the generalized symmetries and contact transformations of this equation, which are very important in examining the conservation laws, were obtained for this equation.

References

[1] R. Abedian, *The comparison of two high-order semi-discrete central schemes for solving hyperbolic conservation laws*, International journal of Mathematical Modelling & Computations, **7**(1)(2017), 39-54.

[2] Y. AryaNejad, M. Jafari and A. Khalili, *Examining (3+ 1)- Dimensional Extended Sakovich Equation Using Lie Group Methods*, International Journal of Mathematical Modelling & Computations, **13**(2)(2023).

[3] B. Azarnavid, M. Emamjomeh and M. Nabati, *An efficient kernel-based method for solving nonlinear generalized Benjamin-Bona-Mahony-Burgers equation in irregular domains,* Applied Numerical Mathematics, **181**(2022), 518-533.

[4] T. B. Benjamin, *Lectures on Nonlinear Wave Motion*, Lecture Notes in Applied Mathematics, **15**(1974), 3-47.

[5] T. B. Benjamin, J. L. Bona and J. J. Mahony, *Model Equations for long waves in Nonlinear Dispersive system*, Philos. Trans. Soc, London S. R., **272**(1972), 47-78.

[6] G. W. Bluman, A. F. Cheviakov and S. C. Anko, *Application of Symmetry Methods to Partial Differential Equations,* Springer, New York, Dordrecht Heudelberg London, (2010).

[7] J. L. Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, **278**(1287) (1975), 555-601.

[8] H. Fadhil, A. Manaa, A. Bewar and A. Y. Majeed, *Variational Homotopy perturbation Method for Solving Benjamin-Bona-Mahony equation*, Applied Mathematics, **6**(4)(2015), 55757.

[9] M. Izadi and M. E. Samei, *Time accurate solution to Benjamin-Bona-Mahony-Burgers equation via Taylor-Boubaker series scheme,* Boundary Value Problems, **17**(2022).

[10] M. Jafari and R. Darvazebanzade, *Analyzing of approximate symmetry and new conservation laws of perturbed generalized Benjamin-Bona-Mahony equation*, AUT Journal of Mathematics and Computing, **5**(1)(2024), 61-69.

[11] M. Jafar and S. S. Mahdion, *Investigating the New Conservation Laws of Hunter-Saxton Equation via Lie Symmetries*, International Journal of Mathematical Modelling & Computations, **13**(2)(2023).

[12] M. Jafari, A. Zaeim and M. Gandom, *On similarity reductions and conservation laws of the two non-linearity terms Benjamin-Bona-Mahoney equation*, Journal of Mathematical Extension, **17**(7)(2023), (1)1-22.

[13] T. Ogama, *Travelling wave solutions to a perturbed Korteweg-de Vries equation*, Hiroshima Math. J. **24**(1994), 401-422.

[14] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, **107**, Springer-Verlag, New York, (1993).

[15] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, (1982).

[16] K. J. Wang and F. Shi, *A new perspective on the exact solutions of the local fractional modified Benjamin-Bona-Mahony equation on Cantor sets*, Fractal and Fractional, **7**(1)(2023), 72.

[17] J. Wang, L. Zhang and J. Li, *New solitary wave solutions of a generalized BBM equation with distributed delays,* Nonlinear Dynamics, **111**(2023), 4631-4643.

[18] H. Yazdani, M. Nadjafikhah and M. Toomanian, *Apply New Optimized MRA* & *Invariant Solutions on the Generalized-FKPP Equation*, International Journal of Mathematical Modelling & Computations, **11**(4)(2021), 1-12.