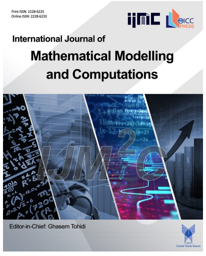

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Convergence and Stability Analysis of Particle Swarm Optimization Using The Fixed Point Method

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Abstract:
This paper presents a fixed point analytical approach to one of the most commonly used optimization techniques known as particle swarm optimization (PSO) and established that the solution space of PSO is a Banach space. With the help of well constructed fixed point theorem, the iterative intelligence algorithm of the PSO was shown to converge to the unique global fixed point. The PSO iterative algorithm was further proven to be T-stable. An example was provided and used to demonstrate the applicability of the stability result. Our results complement other methods of obtaining solutions for PSO in the literature.

Keywords: Fixed point, Banach space, Particle swarm optimization, Iteration algorithm.

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1. Introduction and Preliminary Definitions

Particle swarm optimization (PSO) is one of the classical optimization techniques. It is a heuristic global optimization technique based on bird flocking or fish schooling. It is one of the Swarm intelligence algorithms. The iteration emanated from Kennedy and Eberhart in 1995. It has enormous applications in diverse subject areas including Biology (artificial immune systems), Chemistry, Communication Systems, Engineering, Mathematics (Topology) and Medicine. The techniques of PSO have gained in-depth insight and a lot of theoretical analyses have been done on the algorithm. Many authors have given their perspectives on the convergence and stability of PSO, chief among them are: [1], [2]-[3], [4]- [8], [9], [10] and [11]. Many of these papers concentrated on the behaviour of a single particle in PSO and analyzing the particle's trajectory or its stability using a deterministic approach. In addition, a huge amount of work was done through empirical simulations in advancing the original version of PSO. In [12], Clerc considered an alternative version of PSO by including a parameter called constriction factor which should replace the restriction on velocities. In order to provide the most general result possible, rather than focusing on just the original or the canonical PSO (CPSO), a large class of PSO variants are considered. In the general case, all positional memory, such as the personal and neighborhood best positions, are modeled as sequences of random variables. This generality implies that variants such as the fully informed PSO [8] and the unified PSO [10]. Concerning PSO's success, a considerable amount of theoretical work has been performed on the stochastic search algorithm to try and predict and understand its underlying behaviour (see [2] and [4]). Almost all the theoretical research performed has to some extent relied upon the stagnation assumption, whereby the personal and neighborhood best positions of a particle are assumed to be fixed, which is not a true reaction of the behavior of PSO algorithms. It is important to state that the non-stagnant distribution assumption is a weaker assumption than all previously made assumptions placed on the particles' personal and neighborhood best positions, as each of the mentioned assumptions can be constructed as a specialization of the non-stagnant distribution assumption. Also, it is known that the non-stagnant distribution assumption is in fact a necessary assumption for order-1 and order-2 stabilities. In [9], Minglun et al. optimized PSO algorithm based on the simplicial algorithm of fixed point theory.

Nature - inspired optimization algorithms make use of iterative procedure in order to solve real-world optimization problems and provide near- optimal solutions corresponding to such problems [1]. Since near-optimal solution is obtained, it leads to generation of error in subsequent iterations. Thus, parameter selection based on stability analysis plays a vital role in making algorithm efficient.

Fixed point approach is another very useful way of proving the convergence and stability analysis of PSO. Fixed point theorems for contraction mappings, have always been a vibrant field of research with the celebrated Banach contraction mapping principle since 1922. Several other fixed point theorems of various iterative schemes for contraction mappings have been proved to date, chief among them are: [13], [14] , [15], [16], [17] and [18].

The study of stability of iterative schemes plays a crucial role in numerical mathematics due to chaotic behaviour of functions and discretization of computations in computer programming. Several authors have contributed to the study of stability of iterative schemes, for a comprehensive survey of the literature, see [15], [3], [16], [19], [20] and [21] for details. The first predominant result on T-stable contractive mapping was obtained by Ostrowski [19] for Picard iteration.

The long-established result on stability due to Ostrowski has been extended to multi-valued mappings by Singh and Chadha [20] and further broaden by Singh and Bhatnagar [22] and Singh et al. [21]. Supplementary, stability of iterative schemes has an exceptional importance in fractal graphics while generating fractals. Its functionality lies in the fact that in fractal graphics, fractal objects are generated by an infinite recursive process of successive approximations. An iterative scheme produces a sequence of results and tends towards one final object called a set attractor (fractal), which is independent of the initial choice. This stable disposition of set attractor is due to the stability of iterative scheme, otherwise, the system of underlying successive approximations would show chaotic behaviour and will never settle into a stationary state. Nevertheless, fractals themselves have a variety of applications in digital imaging, mobile computing, architecture and construction, different branches of engineering and applied sciences. For a survey of study on potential applications of fractal geometry in related fields see: [23], [24] and [25]. The link between the round-off stability with the concept of limit shadowing for a fixed point problem involving multi-valued maps can be found in [26]. The complete normed linear space is well suitable for the investigation of physical events, and also has important applications in nonlinear analysis. Thus, in this article, we shall propose a fixed point approach for obtaining convergence and stability of PSO. Firstly, we shall prove that the solution space of PSO is a complete normed linear space. Also, we shall employ convergence theorem to obtain fixed point of the PSO algorithm in the space and finally, establish that a PSO algorithm is stable with respect to the mapping T.

Definition 1.1 *[6] Let PSO represent particle swarm optimization.*

(i) Particle: A particle i, in a PSO is defined as the basic component of solution in the solution space. The dimension of the solution is N. The number of iteration is t. The position of particle, i, is given by $x_i(t) = x_{i1}(t), x_{i2}(t), x_{i3}(t), ..., x_{iN}(t)$.

(ii) Particle Swarm: This is made up of N particles. It stands for the N candidate solution. The population after iteration t is given by $POP(t) = x_1(t), x_2(t), x_3(t), ..., x_N(t)$.

(iii) The velocity of a particle: It is the variation of one iteration. It represents the displacement of the solution in the space C. It is defined as: $v_i(t) = v_{i1}(t), v_{i2}(t), v_{i3}(t), ..., v_{iN}(t)$.

Definition 1.2 *[6] Let C be the solution space or the search space of a particle swarm optimization (PSO). The iterative scheme defined on C, is given by:*

$$v_i(t+1) = wv_i(t)+c_1r_1[p_i(t)-x_i(t)]+c_2r_2[p_g(t)-x_i(t)]. \tag{1.1}$$

Equation (1.1) is called PSO velocity update equation.

$$x_i(t+1) = x_i(t) + v_i(t+1),\ 1 \leq i \leq N. \tag{1.2}$$

Where $x_i(t+1)$ is the position of particle, i, at iteration $t+1$;
$x_i(t)$ is the position of particle, i, at iteration t;
$v_i(t+1)$ is the velocity of particle, i, at iteration $t+1$;
$v_i(t)$ is the velocity of particle, i, at iteration t;
c_1, c_2 are acceleration factors;
r_1, r_2 are unoformly distributed random variables in $[0,1]$;
p_i is the $PBest$ (the best position of particle I);
p_g is the $GBest$ (the best position of population I);
w is inertial factor.

Let $\alpha_1 = c_1r_1$, and $\alpha_2 = c_2r_2$ and let $\alpha = \alpha_1 + \alpha_2$.
Using $\alpha_1, \alpha_2, \alpha$ in (1.1), substituting in (1.2) and simplifying, we have

$$x_i(t+1) = (1-\alpha)x_i(t)+\alpha_1p_i(t)+\alpha_2p_g(t)+wv_i(t). \tag{1.3}$$

Where $w < 1$, $0 < \alpha_1+\alpha_2 < 2w+2$. (1.3) is called the PSO algorithm or the PSO iterative scheme.

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Definition 1.3 *[23] Let E be a linear space over a field K. A norm on E is a real valued function $\|.\| : E \longrightarrow [0,\infty)$ which satisfies the following conditions:*
$N1 : \|x\| \geq 0, \forall x \in E$;
$N2 : \|x\| = 0$, *if and only if* $x = 0$;
$N3 : \|\alpha x\| = |\alpha|\ \|x\|,\ \forall \alpha \in K,\ x \in E$;
$N4 : \|x + y\| \leq \|x\| + \|y\|,\ \forall\, x, y \in E$.
Therefore, $(E, \|.\|)$ is a normed linear space. A complete normed linear space E is called a Banach space. A normed linear space E is said to be complete if every Cauchy sequence in E converges to a point or an element of E.

Definition 1.4 *[27] Let E be a normed linear space and $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in E. $\{x_n\}_{n=0}^{\infty}$ converges to a point $x \in E$ if for any $\epsilon > 0$, there exists n_0 such that*

$$\|x_n - x\| < \epsilon,\ \forall\, n \geq n_0. \tag{1.4}$$

Definition 1.5 *[27] Let E be a normed linear space and $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in E. $\{x_n\}_{n=0}^{\infty}$ is called a Cauchy sequence if for any $\epsilon > 0$, there exists n_0 such that*

$$\|x_n - x_m\| < \epsilon,\ \forall\, n,\ m \geq n_0. \tag{1.5}$$

Definition 1.6 *[27] Let E be a normed linear space and $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in E. $\{x_n\}_{n=0}^{\infty}$ is called a global Cauchy sequence if for any $\epsilon > 0$, there exists n_0 such that*

$$\|x_n - x_\epsilon\| < \epsilon,\ \forall\, n \geq n_0. \tag{1.6}$$

Definition 1.7 *[13] Let E be a Banach space, C a nonempty closed convex subset of E and $T : C \longrightarrow C$ be a selfmap of C. A point $x^* \in C$ is called a fixed point of the selfmap T if $Tx^* = x^*$. The set of all fixed point of T is written as: $F_T = \{x^* \in E : Tx^* = x^*\}$.*

Definition 1.8 *Let X be a metric space, and $T : X \longrightarrow X$ be a selfmap of X. A mapping T is called a contraction if there exists a number δ, satisfying $\delta \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \delta d(x, y),\ \forall\, x, y \in X. \tag{1.7}$$

Theorem 1.9 (Banach Contraction Principle). Let X be a complete metric space, and $T : X \longrightarrow X$ be a selfmap of X. Then
(i) T has a unique fixed point x^* in X;
(ii) $T^n x \longrightarrow x^*,\ \forall x \in X$; and
(iii) $d(T^n x, x^*) \leq \frac{\delta^n}{1-\delta} d(x, Tx)$.

Definition 1.10 *[14] Let (X, d) be a metric space and $T : X \longrightarrow X$ be a selfmap of X. Assume that $F_T = p \in X : T_p = p$ is the set of fixed points of T. For $x_0 \in X$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by*

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, 3, \ldots, \tag{1.8}$$

is called the Picard iterative scheme.

Definition 1.11 *[17] Let E be a Banach space and $T : E \longrightarrow E$ a self map of E. For $x_0 \in E$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n = 0, 1, 2, 3, \ldots, \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in (0,1), such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Mann iterative scheme.

Note that, if $\alpha_n = 1$ in (1.9), we have the Picard iterative scheme (1.8).

Berinde [15], gave a well illustrative explanation on how to derive the stability of iterative schemes as follows: Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an iterative scheme involving the mapping T

$$x_{n+1} = f(T, x_n),\ n = 0, 1, 2, 3, \ldots, \tag{1.10}$$

where $x_0 \in X$ is the initial approximation and f is some function. For example, the Picard iterative scheme (1.8) is obtained from (1.10) for $f(T, x_n) = Tx_n$, while the Mann iterative scheme (1.9) is obtained for $f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n$, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in (0,1) and E a Banach space. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of T. When calculating $\{x_n\}_{n=0}^{\infty}$, then we cover the following steps:
1. We choose the initial approximation $x_0 \in X$.
2. Then we compute $x_1 = f(T, x_0)$, but due to various errors (rounding errors, numerical approximations of functions, derivatives or integrals), we do not get the exact value of x_1, but a different one u_1, which is very close to x_1
3. Consequently, when computing $x_2 = f(T, x_1)$ we shall have actually $x_2 = f(T, u_1)$ and instead of the theoretical value x_2, we shall obtain a closed value and so on. In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iterative method, we get an approximant sequence $\{u_n\}_{n=0}^{\infty}$. We say the iteration method is stable if and only if for u_n closed enough to x_n, $\{u_n\}_{n=0}^{\infty}$ still converges to the fixed point x^* of T.

The following definition and lemma will be employed in proving the main result.

Definition 1.12 *[16] Let (X, d) be a metric space and $T : X \longrightarrow X$ a self map, $x_0 \in X$ and the iterative scheme defined by (1.8) such that the generated sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of T. Let $\{u_n\}_{n=0}^{\infty}$ be arbitrary sequence in X, and set $\epsilon_n = d(u_{n+1}, f(T, u_n))$, for $n = 0, 1, 2, \ldots$. We say the iterative scheme (1.8) is $T-$stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} u_n = x^*$.*

Lemma 1.13 [15] Let δ be a real number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, n=0,1,2,..., we have $\lim_{n\to\infty} u_n = 0$.

In the main result, we shall make use of fixed point method, using the contraction mapping to obtain convergence and stability results for particle swarm optimization (PSO) iterative scheme (1.3).

2. Main Result

2.1 Strong Convergence and Stability Results for PSO in Banach Spaces

In this section, the results will be obtained in three folds:
1. We will show that the space of a particle swarm

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optimization (PSO) is a Banach space.
2. We will also, prove that the PSO iterative scheme converges strongly to the unique fixed point of the contraction map T.
3. Finally, we will prove that the PSO iterative scheme is $T-$ stable.

We start by showing that the space of the PSO is a Banach space.

Proposition 2.1 Let E be a vector space and $x_i(t)$ be the position of particle, i, on E defined by: $x_i(t) = x_{i1}(t), x_{i2}(t), x_{i3}(t), \ldots, x_{iN}(t)$, where t is the number of iterations. Let the modulus of $x_i(t)$ be given by:

$$\|x_i(t)\| = \sqrt{\sum_{i=1}^{N} x_i^2(t)}, \tag{2.1}$$

then, $(E, \|.\|)$ is a Banach space.

Proof. Firstly, we prove that (2.1) satisfies the properties of a normed linear(vector) space as follows:
$N1 : \|x\| \geq 0, \ \forall x \in E.$
$x_i^2(t) \geq 0, \ \forall x_i \in E$, for each i.
Thus, $\|x\| = \|x_i(t)\| = \sqrt{\sum_{i=1}^{N} x_i^2(t)}, \ \forall x \in E.$
Hence, $N1$ is satisfied.

$N2 : \|x\| = 0$, if and only if $x = 0$.
If $\|x\| = \|x_i(t)\| = \sqrt{\sum_{i=1}^{N} x_i^2(t)} = 0$, then $x_i^2(t) = 0$, for each $i \Rightarrow x_i(t) = 0$, for each i. $\Rightarrow x = 0$.
Conversely, if $x = 0$, then, $x_i(t) = 0$, for each i. $\Rightarrow x_i^2(t) = 0$, for each i
Thus, $\|x\| = \|x_i(t)\| = \sqrt{\sum_{i=1}^{N} x_i^2(t)} = 0.$
Therefore, $\|x\| = 0$, if and only if $x = 0$.

$N3 : \|\alpha x\| = |\alpha| \ \|x\|, \ \forall \alpha \in K, \ x \in E;$

$$\begin{aligned} \|\alpha x\| &= \|\alpha x_i(t)\| = \sqrt{\sum_{i=1}^{N} \alpha^2 x_i^2(t)} = |\alpha|\sqrt{\sum_{i=1}^{N} x_i^2(t)} \\ &= |\alpha|\|x_i(t)\| = |\alpha|\|x\|. \end{aligned}$$

$N4 : \|x + y\| \leq \|x\| + \|y\|, \ \forall \ x, y \in E.$
Let $\|x + y\| = \|x_i(t) + y_i(t)\| = \sqrt{\sum_{i=1}^{N} (x_i(t) + y_i(t))^2}$

Let $N = 2$, then,

$$\begin{aligned} \|x + y\|^2 &= \|x_i(t) + y_i(t)\|^2 = \sum_{i=1}^{2} (x_i(t) + y_i(t))^2 \\ &= \sum_{i=1}^{2} (x_i^2(t) + 2x_i(t)y_i(t) + y_i^2(t)) \\ &= \sum_{i=1}^{2} x_i^2(t) + 2\sum_{i=1}^{2} |x_i(t)y_i(t)| + \sum_{i=1}^{2} y_i^2(t) \\ &\leq \sum_{i=1}^{2} x_i^2(t) + 2(\sum_{i=1}^{2} x_i^2(t))^{\frac{1}{2}} (\sum_{i=1}^{2} y_i^2(t))^{\frac{1}{2}} + \sum_{i=1}^{2} y_i^2(t) \end{aligned}$$

(By Cauchy Schwartz's inequality).
$= K_1^2 + 2K_1K_2 + K_2^2$, where $K_1 = (\sum_{i=1}^{2} x_i^2(t))^{\frac{1}{2}}$ and $K_2 = (\sum_{i=1}^{2} y_i^2(t))^{\frac{1}{2}}$.
$= (K_1 + K_2)^2 = (\|x\| + \|y\|)^2$.
Thus, $\|x + y\| \leq \|x\| + \|y\|, \ \forall \ x, y \in E.$
Since the modulus of the position of a particle, i, satisfies all the properties, therefore, $(E, \|.\|)$ is a normed linear (vector) space. This means that the solution space is a normed linear (vector) space.

Next, we show completeness.

Let $x_i(t + 1) = T(x_i(t)) = Tx_i(t)$.
By the definition of fixed point, there is a point $x \in E$ such that $x = Tx$.
Let x^* be the limit of the sequence $\{x_i(t)\}_{t=1}^{\infty}$, which is the unique fixed point. That is, $\lim_{t\to\infty} x_i(t) = x^*$.
Then, $Tx^* = T(\lim_{t\to\infty} x_i(t)) = \lim_{t\to\infty} T(x_i(t))$.
$= \lim_{t\to\infty} x_i(t + 1) = \lim_{t\to\infty} x_i(t) = x^*$.

Next, using the contraction mapping, we show that there is a Cauchy sequence in E that converges to the unique fixed point in E.
From definition 1.8, if $X = E$, then contraction condition (1.7) is written as:

$$\|Tx - Ty\| \leq \delta\|x - y\|, \ \forall \ x, y \in E, \ \delta \in [0, 1). \tag{2.2}$$

Notice that

$$\|x - y\| \leq \|x - Tx\| + \|Tx - Ty\| + \|Ty - y\|, \ \forall \ x, y \in E. \tag{2.3}$$

Applying (2.2) in (2.3), we have

$$\|x-y\| \leq \|x-Tx\|+\delta\|x-y\|+\|Ty-y\|, \ \forall \ x, y \in E. \tag{2.4}$$

Thus,

$$\|x - y\| \leq \frac{(\|x - Tx\| + \|y - Ty\|)}{1 - \delta}, \ \forall \ x, y \in E. \tag{2.5}$$

Notice also that using (2.2), we have

$$\|x_i(t) - Tx_i(t)\| \leq \delta^t \|x_i(1) - Tx_i(1)\|, \ \delta^t \in [0, 1). \tag{2.6}$$

$$\|y_i(s) - Ty_i(s)\| \leq \delta^s \|y_i(1) - Ty_i(1)\|, \ \delta^s \in [0, 1). \tag{2.7}$$

Thus,

$$\|x_i(t) - x_i(s)\| \leq \frac{(\delta^t + \delta^s)}{1 - \delta} \|x_i(1) - Tx_i(1)\|, \ \delta \in [0, 1). \tag{2.8}$$

Hence, $\{x_i(t)\}_{t=1}^{\infty}$ is a Cauchy sequence in E and it converges to $x^* \in E$ since

$$\|x_i(t) - x_i(s)\| \leq \|x_i(t) - x^*\| + \|x^* - x_i(s)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.9}$$

Therefore, since the properties of normed linear space E are satisfied and $\{x_i(t)\}_{t=1}^{\infty}$ is a Cauchy sequence in E and it converges to $x^* \in E$, then E is a Banach space. This ends the proof.

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Theorem 2.2 Let E be a Banach space, C (the solution or search space) be a subset of E and $T : C \to C$ be the contraction mapping satisfying the condition

$$\|Tx - Ty\| \leq \delta\|x - y\|, \tag{2.10}$$

for each $x, y \in C$, $0 \leq \delta < 1$. Let x^* be the unique fixed point of T. For $x_i(0) \in E$, let $\{x_i(t)\}_{t=0}^{\infty}$ be the PSO algorithm (iterative scheme) (1.3) defined by $x_i(t+1) = (1-\alpha)x_i(t) + \alpha_1 p_i(t) + \alpha_2 p_g(t) + wv_i(t)$, for $i = 1, 2, 3, ..., n$. Then, the PSO iterative scheme (1.3) converges strongly to x^* of T.

Proof:

$$\begin{aligned} \|x_i(t+1) - x^*\| &= \|[(1-\alpha)x_i(t) + \alpha_1 p_i(t) \\ &\quad +\alpha_2 p_g(t) + wv_i(t)] - x^*\| \\ &\leq (1-\alpha)\|x_i(t) - x^*\| \\ &\quad +\alpha_1\|p_i(t) - x^*\| + \alpha_2\|p_g(t) - x^*\| \\ &\quad +w\|v_i(t) - x^*\|. \end{aligned} \tag{2.11}$$

Applying contraction condition (2.10) in (2.11), we obtain (2.12), (2.13), (2.14a) and (2.14b) as follows

$$\begin{aligned} \|x_i(t) - x^*\| &= \|Tx_i(t-1) - x^*\| \\ &\leq \delta\|x_i(t-1) - x^*\| \\ &\vdots \\ &\leq \delta^{t-1}\|x_i(1) - x^*\|. \end{aligned} \tag{2.12}$$

$$\begin{aligned} \|v_i(t) - x^*\| &= \|Tv_i(t-1) - x^*\| \\ &\leq \delta\|v_i(t-1) - x^*\| \\ &\vdots \\ &\leq \delta^{t-1}\|v_i(1) - x^*\|. \end{aligned} \tag{2.13}$$

$$\begin{aligned} \|p_i(t) - x^*\| &= \|Tp_i(t-1) - x^*\| \\ &\leq \delta\|p_i(t-1) - x^*\| \\ &\vdots \\ &\leq \delta^{t-1}\|p_i(1) - x^*\|. \end{aligned} \tag{2.14a}$$

$$\begin{aligned} \|p_g(t) - x^*\| &= \|Tp_g(t-1) - x^*\| \\ &\leq \delta\|p_g(t-1) - x^*\| \\ &\vdots \\ &\leq \delta^{t-1}\|p_g(1) - x^*\|. \end{aligned} \tag{2.14b}$$

Substituting (2.12) - (2.14b) in (2.11), we obtain

$$\begin{aligned} \|x_i(t+1) - x^*\| &\leq \delta^{t-1}[(1-\alpha)\|x_i(1) - x^*\| \\ &\quad +\alpha_1\|p_i(1) - x^*\| \\ &\quad +\alpha_2\|p_g(1) - x^*\| \\ &\quad +w\|v_i(1)] - x^*\|]. \end{aligned} \tag{2.15}$$

Since $\delta^{t-1} \in [0, 1)$ then $\|x_i(t+1) - x^*\| \to 0$ as $t \to \infty$. That is, $\lim_{t\to\infty} x_i(t) = x^*$. This ends the proof.

Theorem 2.3 Let E be a Banach space, C (the solution or search space) be a subset of E and $T : C \to C$ be the contraction mapping with a unique fixed point x^* satisfying the contraction condition

$$\|x^* - Ty\| \leq \delta\|x^* - y\|, \tag{2.16}$$

for each $x, y \in C$, $0 \leq \delta < 1$. For $x_i(0) \in E$, let $\{x_i(t)\}_{t=0}^{\infty}$ be the PSO iterative scheme (1.3), $i = 1, 2, 3, ..., n$. Then, the PSO iterative scheme (1.3) is $T-$stable.

Proof: Assume that $\lim_{t\to\infty} x_i(t) = x^*$ or $x_i(t) \to x^*$ as $t \to \infty$.
Define $\{z_i(t)\}_{t=1}^{\infty}$ by $z_i(t+1) = (1-\alpha)z_i(t) + \alpha_1 q_i(t) + \alpha_2 q_g(t) + wu_i(t)$.
Then,

$$\begin{aligned} \|z_i(t+1) - x^*\| &= \|z_i(t+1) - [((1-\alpha)z_i(t) + \alpha_1 q_i(t) \\ &\quad +\alpha_2 q_g(t) + wu_i(t)) + ((1-\alpha)z_i(t) \\ &\quad +\alpha_1 q_i(t) + \alpha_2 q_g(t) + wu_i(t))] - x^*\| \\ &\leq \|z_i(t+1) - ((1-\alpha)z_i(t) + \alpha_1 q_i(t) \\ &\quad +\alpha_2 q_g(t) + wu_i(t))\| + \|((1-\alpha)z_i(t) \\ &\quad +\alpha_1 q_i(t) + \alpha_2 q_g(t) + wu_i(t)) - x^*\| \\ &\leq \epsilon_i(t) + \delta^{t-1}[(1-\alpha)\|z_i(1) - x^*\| \\ &\quad +\alpha_1\|q_i(1) - x^*\| + \alpha_2\|q_g(1) - x^*\| \\ &\quad +w\|u_i(1) - x^*\|]. \end{aligned} \tag{2.17}$$

Using lemma 1.13 on (2.17), it follows that $\lim_{t\to\infty}\|z_i(t+1) - x^*\| = 0$. That is, $\lim_{t\to\infty} z_i(t) = x^*$.

Conversely, let $\lim_{t\to\infty} z_i(t) = x^*$, we prove that $\lim_{t\to\infty}\epsilon_i(t) = 0$ as follows:

$$\begin{aligned} \epsilon_i(t) &= \|z_i(t+1) - [((1-\alpha)z_i(t) + \alpha_1 q_i(t) \\ &\quad +\alpha_2 q_g(t) + wu_i(t))]\| \\ &\leq \|z_i(t+1) - x^*\| + \|x^* - ((1-\alpha)z_i(t) \\ &\quad +\alpha_1 q_i(t) + \alpha_2 q_g(t) + wu_i(t))\| \\ &\leq \|z_i(t+1) - x^*\| + \delta^{t-1}[(1-\alpha)\|z_i(1) - x^*\| \\ &\quad +\alpha_1\|q_i(1) - x^*\| + \alpha_2\|q_g(1) - x^*\| \\ &\quad +w\|u_i(1) - x^*\|]. \end{aligned} \tag{2.18}$$

Since $\lim_{t\to\infty}\|z_i(t) - x^*\| = 0$ by assumption, then $\lim_{t\to\infty}\epsilon_i(t) = 0$.
Therefore, the PSO iterative scheme defined by (1.3) is $T-$stable. This ends the proof.

Example 2.4 *Let $E = [0, 1]$ and $T : [0, 1] \to [0, 1]$. Let $Tx = \frac{x}{2}$ and $z_i(t) = \frac{1}{t}$, $u_i(t) = \frac{1}{4t}$, $q_i(t) = \frac{1}{8t}$, $q_g(t) = \frac{1}{16t}$. Let $F_T = [0, 1]$, where $[0, 1]$ has the usual metric. Then T satisfies the inequality $\|Tx(t) - x^*\| \leq \delta\|x(t) - x^*\|$.*

We shall prove that the PSO iterative scheme (1.3) is $T-$stable.
Let $x^* = 0$, take $\alpha = \frac{1}{2}$, $w = \frac{1}{2}$, $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{4}$,
then, $\lim_{t\to\infty} z_i(t) = \lim_{t\to\infty}\frac{1}{t} = 0$. Also,

$$\begin{aligned} \lim_{t\to\infty}\epsilon_i(t) &= \lim_{t\to\infty}|z_i(t+1) - [((1-\alpha)z_i(t) \\ &\quad +\alpha_1 q_i(t) + \alpha_2 q_g(t) + wu_i(t))]| \\ &\leq \lim_{t\to\infty}|\frac{1}{t+1} - (1-\frac{1}{2})(\frac{1}{t}) - \frac{1}{4}(\frac{1}{8t}) \\ &\quad -\frac{1}{4}(\frac{1}{16t}) - \frac{1}{2}(\frac{1}{4t})| = 0. \end{aligned}$$

Hence, the PSO iterative scheme (1.3) is $T-$stable.

3. Conclusion

In this research, the solution space of a particle swarm optimization (PSO) was proved to be a Banach space. With the help of Banach contraction mapping, it was shown that the PSO algorithm converges strongly to the unique fixed point of the self map T and also that the PSO algorithm is $T-$ stable in a Banach space. The PSO iterative algorithms employed in this study have good potentials for further applications.

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The authors declare that they have no competing interest (s).

Authors' contributions

All the authors contributed equally and significantly to this research work, read and approved the final manuscript.

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Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.
Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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The author states that there is no conflict of interest.

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