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New Extended Beta Function Defined by Product of Two Wright Functions

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Abstract. This paper aims to introduce a new extension of extended(p_1, p_2) −Beta function by product of two wright functions. Various properties of this extended function are investigated such as integral representations, summation formulas and Mellin transform.

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Index to information contained in this paper

1. Introduction

Very recently, Ata, E. [1] introduced and investigated some properties of the generalized Beta and Gamma functions defined by

$$ {}^{\psi}\Gamma_{p_1}^{(\beta_1,\beta_2)}(\omega_1) = \int_0^\infty t^{\omega_1-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-t-\frac{p_1}{t}\right)dt, \tag{1}$$

$(Re(\omega_1) > 0, Re(\beta_1) > 0, Re(\beta_2) > 1, Re(p_1) > 0)$,

$$ {}^{\psi}B_{p_1}^{(\beta_1,\beta_2)}(\omega_1,\omega_2) = \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{t(1-t)}\right)dt, \tag{2}$$

$(Re(\omega_1) > 0, Re(\omega_2) > 0\ , Re(\beta_1) > 0, Re(\beta_2) > 1, Re(p_1) > 0)$,

where ${}_1\psi_1(.)$ denoted the Wright function defined by [11]

$$ {}_1\psi_1(\beta_1,\beta_2;\omega_1) = \sum_{\lambda=0}^{\infty}\frac{1}{\Gamma(\beta_1\lambda+\beta_2)}\frac{{\omega_1}^{\lambda}}{\lambda!}. \tag{3}$$

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For $\beta_1 = 0$ and $\beta_2 = 2$, equations (1) and (2) reduce to the extended Gamma and Beta type functions due to Chaudhry et al. [2] defined by

$$\Gamma_{p_1}(\omega_1) = \int_0^\infty t^{\omega_1 - 1} exp\left(-t - \frac{p_1}{t}\right) dt, \qquad (Re(p_1) > 0) \tag{4}$$

$$B(\omega_1, \omega_2; p_1) = \int_0^1 t^{\omega_1 - 1}(1-t)^{\omega_2 - 1} e^{\frac{-p_1}{t(1-t)}} dt, \tag{5}$$

$(Re(p_1) > 0, Re(\omega_1) > 0, Re(\omega_2) > 0)$,

which for $p_1 = 0$ give the classical Gamma and Beta functions defined by [11]

$$\Gamma(\omega_1) = \int_0^\infty t^{\omega_1 - 1} e^{-t} dt, \qquad (Re(\omega_1) > 0), \tag{6}$$

$$B(\omega_1, \omega_2) = \int_0^1 t^{\omega_1 - 1}(1-t)^{\omega_2 - 1} dt, \qquad (Re(\omega_1) > 0, Re(\omega_2) > 0), \tag{7}$$

Choi *et al.* [3] introduced the extended (p_1, p_2)-Beta function by the following form:

$$B(\omega_1, \omega_2; p_1, p_2) = \int_0^1 t^{\omega_1 - 1}(1-t)^{\omega_2 - 1} e^{\frac{-p_1}{t} - \frac{,p_2}{(1-t)}} dt, \tag{8}$$

$(Re(p_1) > 0, Re(p_2) > 0, Re(\omega_1) > 0, Re(\omega_2) > 0)$.

$$B_{p_1, p_2}(\omega_1, \omega_2) = B_{p_1}(\omega_1, \omega_2), \tag{9}$$

and for $p_1 = p_2 = 0$, we get

$$B_{p_1, p_2}(\omega_1, \omega_2) = B(\omega_1, \omega_2). \tag{10}$$

Recently, many authors have introduced certain extensions for Gamma and Beta functions (see[1,2,3,5,7,8,9,10,12,13]).

2. A new extension of Beta function

In this section, we define a new extension of (p_1, p_2) −Beta function and investigate some its properties such as Mellin transforms and integral representations.

Definition 2.1. Then we extended (p_1, p_2) −Beta function is defined as:

$$^{\psi}B_{p_1, p_2}^{(\beta_1, \beta_2)}(\omega_1, \omega_2)$$
$$= \int_0^1 t^{\omega_1 - 1}(1-t)^{\omega_2 - 1} \ {}_1\psi_1\left(\beta_1, \beta_2; -\frac{p_1}{t}\right) \ {}_1\psi_1\left(\beta_1, \beta_2; -\frac{p_1}{(1-t)}\right) dt, \tag{11}$$

where $Re(\omega_1) > 0, Re(\omega_2) > 0 , Re(\beta_1) > 0, Re(\beta_2) > 1,\ p_1, p_2 \geq 0$ and ${}_1\psi_1$ is the Wright function defined by equation (3).

Remark 2.1. Note that:

(i) If $\beta_1 = 0, \beta_2 = 2,$ then (11) reduces to the well-known extended Beta function given by equation (8).

(ii) If $\beta_1 = 0, \beta_2 = 2$ and $p_1 = p_2$, then (11) reduces to the extended Beta function given by equation (5).

(iii) If $\beta_1 = 0, \beta_2 = 2$ and $p_1 = p_2 = 0$, then (11) reduces to the classical Beta function given by equation (7).

Now, we introduce some properties of the new extended (p_1, p_2) −Beta function in the form of the following theorems:

Theorem 2.1. The new extended $(\mathrm{p}_1, \mathrm{p}_2)$ −Beta function has the following relation:

$$^{\psi}B_{p_1, p_2}^{(\beta_1, \beta_2)}(\omega_1 + 1, \omega_2) + {}^{\psi}B_{p_1, p_2}^{(\beta_1, \beta_2)}(\omega_1, \omega_2 + 1) = {}^{\psi}B_{p_1, p_2}^{(\beta_1, \beta_2)}(\omega_1, \omega_2). \tag{12}$$

Proof. Considering the left hand side of (12) and using definition (11), we have

$$^{\psi}B_{p_1, p_2}^{(\beta_1, \beta_2)}(\omega_1 + 1, \omega_2) + {}^{\psi}B_{p_1, p_2}^{(\beta_1, \beta_2)}(\omega_1, \omega_2 + 1)$$

$$= \int_0^1 \{t^{\omega_1}(1-t)^{\omega_2-1} + t^{\omega_1-1}(1-t)^{\omega_2}\}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{t}\right)\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{(1-t)}\right)dt,$$

$$= \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}\{t+(1-t)\}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{t}\right)\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{(1-t)}\right)dt,$$

$$= \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{t}\right)\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{(1-t)}\right)dt,$$

which proves the desired result.

Theorem 2.2. The new extended $(\mathrm{p}_1,\mathrm{p}_2)$ −Beta function has the following summation formula:

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1, 1-\omega_2) = \sum_{\lambda=0}^{\infty}\frac{(\omega_2)_\lambda}{\lambda!}\,{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1+\lambda, 1), \tag{13}$$

Proof. Considering the generalized binomial theorem [11]

$$(1-t)^{-\omega_2} = \sum_{\lambda=0}^{\infty}\frac{(\omega_2)_\lambda}{\lambda!}t^\lambda, \qquad (|t|<1). \tag{14}$$

Applying (14) to definition (11) of extended (p_1,p_2) −Beta function, we get

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1, 1-\omega_2)$$

$$= \int_0^1 \sum_{\lambda=0}^{\infty}\frac{(\omega_2)_\lambda}{\lambda!}t^{\omega_1+\lambda-1}(1-t)^{\omega_2-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{t}\right)\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_2}{(1-t)}\right)dt.$$

Now, interchanging the order of summation and integration in above equation and using (11), we get the desired result.

Theorem 2.3. The new extended $(\mathrm{p}_1,\mathrm{p}_2)$ −Beta function has the following summation formula:

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2) = \sum_{\lambda=0}^{\infty}{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1+\lambda,\omega_2+1), \tag{15}$$

Proof. Replacing the following series representation

$$(1-t)^{\omega_2-1} = (1-t)^{\omega_2}\sum_{\lambda=0}^{\infty}t^\lambda\ ,$$

in definition (11), we obtain

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2) =$$

$$\int_0^1 (1-t)^{\omega_2}\sum_{\lambda=0}^{\infty}t^{\omega_1+\lambda-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{t}\right)\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_2}{(1-t)}\right)dt,$$

interchanging the order of integration and summation in above equation and using (11), we get the desired result.

Theorem 2.4. For the new extended $(\mathrm{p}_1,\mathrm{p}_2)$ −Beta function, the following relation holds true:

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a-\lambda) = \sum_{\lambda=0}^{\infty}\binom{\lambda}{k}\,{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+k,-a-k), \quad (\lambda\in\mathbb{N}_0). \tag{16}$$

Proof. Taking $\omega_1 = a$and $\omega_2 = -a-\lambda$ in relation (2.2), we have

$$ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a-\lambda) = {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a-\lambda+1) + {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+1,-a-\lambda), $$

Starting with $\lambda = 1, 2, 3, \ldots$, we have

$$ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a-1) = {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a) + {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+1,-a-1), $$

$$ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a-2) $$
$$ = {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a) + 2\,{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+1,-a-1) + {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+2,-a-2), $$

$$ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a-3) $$
$$ = {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a,-a) + 3\,{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+1,-a-1) + 3\,{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+2,-a-2) + {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(a+3,-a-3) $$

and so on. The above series behaves like as finite binomials series does. Thus, we can finally obtain the desired result.
Note that, we can also prove the desired result (16) by applying induction on n.

Theorem 2.5. The new extended $(\mathrm{p}_1, \mathrm{p}_2)$ −Beta function has the following summation formula:

$$ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2) = \sum_{\lambda=0}^{\infty}\sum_{\kappa=0}^{\infty} \frac{(-p_1)^{\lambda}(-p_2)^{\kappa}}{\Gamma(\beta_1\lambda+\beta_2)\Gamma(\beta_1\kappa+\beta_2)\lambda!\,\kappa!} B(\omega_1-\lambda,\omega_2-\kappa). \tag{17} $$

Proof. Expanding the Wright functions in the right hand said of equation (11) by using definition (3) and then using relation (7) in the resultant equation, we get the desired result.

3. Integral formulas

In this section, we get some integral formulas for the new extended (p_1, p_2) − Beta function in form of the following theorems:

Theorem 3.1. The new extended $(\mathrm{p}_1, \mathrm{p}_2)$ − Beta function has the following Mellin transform relation:

$$ \mathcal{M}\left\{ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2); p_1 \longrightarrow r, p_2 \longrightarrow s \right\} = B(\omega_1+r,\omega_2+s)\, {}^{\psi}\Gamma_0^{(\beta_1,\beta_2)}(r)\, {}^{\psi}\Gamma_0^{(\beta_1,\beta_2)}(s) \tag{18} $$

$Re(\omega_1 + r) > 0, Re(\omega_2 + s) > 0, Re(\beta_1) > 0, Re(\beta_2) > 1, Re(p_1), Re(p_2) > 0, Re(s), Re(s) > 0.$

Proof. Applying the Mellin transform to definition (11), we have

$$ \mathcal{M}\left\{ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2); p_1 \longrightarrow r, p_2 \longrightarrow s \right\} = \int_0^{\infty} p^{r-1} \int_0^{\infty} q^{s-1} \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1} \times {}_1\psi_1\left(\beta_1,\beta_2; -\frac{p_1}{t}\right) {}_1\psi_1\left(\beta_1,\beta_2; -\frac{p_2}{(1-t)}\right) dt\,dp_1\,dp_2. \tag{19} $$

Interchanging the order of integrations, we get

$$ \mathcal{M}\left\{ {}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2); p_1 \longrightarrow r, p_2 \longrightarrow s \right\} = \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1} \left\{ \int_0^{\infty} p^{r-1}\, {}_1\psi_1\left(\beta_1,\beta_2; -\frac{p_1}{t}\right) dp_1 \right\} $$

$$\times\left\{\int_0^\infty {p_2}^{s-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_2}{(1-t)}\right)dp_2\right\}dt. \tag{20}$$

Substituting $u=\frac{p_1}{t}, v=\frac{p_2}{(1-t)}$ in (20), we get

$$\mathcal{M}\left\{{}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2);p_1\longrightarrow r,p_2\longrightarrow s\right\}$$
$$=\int_0^1 t^{\omega_1+r-1}(1-t)^{\omega_2+s-1}dt$$
$$\left\{\int_0^\infty u^{r-1}\ {}_1\psi_1(\beta_1,\beta_2;-u)du\right\}\left\{\int_0^\infty v^{s-1}\ {}_1\psi_1(\beta_1,\beta_2;-v)dv\right\}. \tag{21}$$

Using definition (1) (for$p_1=0$)in the right hand said of the above equation, we get the desired result.

Theorem 3.2. The new extended $(\mathrm{p}_1,\mathrm{p}_2)-$Beta function has the following Mellin transforms formula:

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2)$$
$$=\frac{1}{(2\pi i)^2}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\frac{\Gamma(\omega_1+r)\Gamma(\omega_2+s)\ {}^{\Psi}\Gamma^{(\beta_1,\beta_2)}(r)\ {}^{\Psi}\Gamma^{(\beta_1,\beta_2)}(s)}{\Gamma(\omega_1+\omega_2+r+s)}$$
$$\times {p_1}^{-r}{p_2}^{-s}drds, \tag{22}$$
$$(Re(\omega_1)>0,Re(\omega_2)>0\ ,Re(\beta_1)>0,Re(\beta_2)>1,p_1,p_2\geq 0,\gamma_1,\gamma_2>0).$$

Proof. Applying the inverse Mellin transform on both sides of (12), we get the desired result.

Theorem 3.3. For the extended $(\mathrm{p}_1,\mathrm{p}_2)-$ Beta function, the following integral representations holds true:

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2)$$
$$=2\int_0^{\frac{\pi}{2}} cos^{2\omega_1-1}\theta sin^{2\omega_2-1}\theta\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1}{\cos^2\theta}\right)\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_2}{\sin^2\theta}\right)d\theta, \tag{23}$$
$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2)$$
$$=\int_0^\infty\frac{u^{\omega_1-1}}{(1+u)^{\omega_1+\omega_2}}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1(1+u)}{u}\right)\ {}_1\psi_1\big(\beta_1,\beta_2;-p_2(1+u)\big)du, \tag{24}$$

$${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2)=(c-a)^{1-\omega_1-\omega_2}$$
$$\int_a^c (u-a)^{\omega_1-1}(c-u)^{\omega_1-1}\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_1(c-a)}{(u-a)}\right)$$
$$\times\ {}_1\psi_1\left(\beta_1,\beta_2;-\frac{p_2(c-a)}{(c-u)}\right)du, \tag{25}$$

Proof. Equations (23) - (25) can be easily obtained by taking the transformation $\mathrm{t}=\cos^2\theta,\ t=\frac{u}{1+u}$ and $t=\frac{u-a}{c-a}$ in (11), respectively.

4. Concluding Remarks

We know that the Tricomi functions $C_n(x)$ are defined by [11]

$$C_m(\omega_1)=\sum_{\lambda=0}^{\infty}\frac{(-\omega_1)^k}{\lambda!\,(n+\lambda)!}(m=0,1,2,\ldots), \tag{26}$$

Note that, from definitions (3) and (26) and using the relation $\Gamma(m+1)=m!$, we get

$${}_1\psi_1(1,m+1,\omega_3)=C_m(\omega_3). \tag{27}$$

Also, Dattoli *et al*. [4] have introduced the 2-index 3-variable Tricomi functions defined as:

$$C_m(\omega_1,\omega_2,\omega_3)=\sum_{\lambda=0}^{\infty}C_{m+\lambda}(\omega_1)C_{n+\lambda}(\omega_2)\frac{{\omega_3}^{\lambda}}{\lambda!}. \tag{28}$$

Here, we explore the possibility to define a further extension of extended (p_1,p_2) −Beta function in terms of integral its kernel contains product of two Tricomi functions. For this aim, we put $\beta_1=1$ and $\beta_2=m+1$ in definition (11) and using relation (27), we get

$${}^{\psi}B_{p_1,p_2}^{(1,m+1)}(\omega_1,\omega_2)={}^{C}B_{p_1,p_2}^{(m)}(\omega_1,\omega_2), \tag{29}$$

where we have defined ${}^{C}B_{p_1,p_2}^{(m)}(\omega_1,\omega_2)$ by

$${}^{C}B_{p_1,p_2}^{(m)}(\omega_1,\omega_2)=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}C_m\left(-\frac{p_1}{t}\right)C_m\left(-\frac{p_2}{(1-t)}\right)dt. \tag{30}$$

In view of the above definition, we conclude that all the properties of the extended (p,q) −Beta function ${}^{C}B_{p_1,p_2}^{(m)}(\omega_1,\omega_2)$ can be deduced from the corresponding ones for ${}^{\psi}B_{p_1,p_2}^{(\beta_1,\beta_2)}(\omega_1,\omega_2)$.

As further property, multiplying both side of equation (30) by $\frac{u^m}{m!}$ and summing up over m, we have

$$\sum_{m=0}^{\infty}{}^{C}B_{p_1,p_2}^{(m)}(\omega_1,\omega_2)\frac{u^m}{m!}=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}\sum_{m=0}^{\infty}C_m\left(-\frac{p_1}{t}\right)C_m\left(-\frac{p_2}{(1-t)}\right)\frac{u^m}{m!}dt, \tag{31}$$

which on using definition (28) (for $m=\lambda=0$) in the right hand said gives the following relation:

$$\sum_{m=0}^{\infty}{}^{C}B_{p_1,p_2}^{(m)}(\omega_1,\omega_2)\frac{u^m}{m!}=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}C_m\left(-\frac{p_1}{t},-\frac{p_2}{(1-t)},u\right)dt. \tag{32}$$

Finally, we conclude that if we letting $\beta_1=0$ and $\beta_2=2$ in all results in sections 2 and 3, then all results will be reduced to the work of Choi *et al*. [3].

Also, if we letting $\beta_1=0$, $\beta_2=2$ and $p_1=p_2$ in all results in sections 2and 3, then all results will be reduced to the work of Chaudhry *et al*. [2].

In a forthcoming investigation, the new extended (p_1,p_2) −Beta function defined in equation (11) will be used to introduce other extensions of the extended Gauss hypergeometric and theconfluent hypergeometric functions. For each of these new extensions we will obtain various properties.

5. Conclusion

In this study, the new extension of extended (p_1,p_2) −Beta function are defined by product of two Wright functions and various properties of this extended function are investigated. Also, further extension of extended (p_1,p_2) −Beta function in terms of integral its kernel contains product of two Tricomi functions are introduced .

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