International Journal of
Mathematical Modelling & Computations
Vol. 11, No. 03, Summer 2021, 1- 11

Local Existence and Blow Up of Solutions for a System of Viscoelastic Wave Equations of Kirchhoff Type with Delay and Logarithmic Nonlinearity

K. Rajendra Prasad[a], Mahammad Khuddush[a,*] and B. Bharathi[a,b]

[a]*Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, 530003, India,*
[b]*College of Engineering for Women, Gayatri Vidya Parishad, Visakhapatnam, 530048, India.*

Abstract. In this paper, we consider a system of viscoelastic wave equations of Kirchhoff type with delay and logarithmic nonlinearity. We obtain the local existence of solution by using the Faedo-Galerkin approximation and under suitable conditions, we prove the blow up of solutions in finite time.

Received: 29 September 2020, Revised: 11 February 2021, Accepted: 04 March 2021.

Keywords: Wave equation; Local existence; Blow up; Logarithmic nonlinearity; Delay.

AMS Subject Classification: 35L05, 35L70, 35B44.

Index to information contained in this paper

1. Introduction

In this paper, we concerned with the following a system of logarithmic viscoelastic wave equations of Kirchhoff type with delay.

$$\left.\begin{aligned}&\partial_{tt}\theta - \mathcal{M}\big(\|\nabla\theta\|_2^2\big)\Delta\theta + \int_0^t \mathrm{g}(t-s)\Delta\theta(s)ds + \alpha|\partial_t\theta(x,t)|^{\ell-1}\partial_t\theta(x,t)\\ &\quad + \beta|\partial_t\theta(x,t-\tau)|^{\ell-1}\partial_t\theta(x,t-\tau) = \theta|\theta|^{\mathrm{p}-2}\ln|\theta|^{\mathrm{k}},\ x\in\Omega, t>0,\\ &\theta(x,t)=0,\ x\in\partial\Omega, t>0,\\ &\partial_t\theta(x,t-\tau) = \mathrm{f}_0(x,t-\tau),\ x\in\Omega, t\in(0,\tau),\\ &\theta(x,0)=\theta_0(x),\ \partial_t\theta(x,0)=\theta_1(x),\ x\in\Omega,\end{aligned}\right\} \tag{1}$$

*Corresponding author. Email: khuddush89@gmail.com

http://ijm2c.iauctb.ac.ir

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$. $\tau > 0$ is a time delay term and $\mathrm{k}, \alpha, \beta$ are positive real numbers and $\mathrm{p} > 2$. The initial datum $(\theta_0, \theta_1, \mathrm{f}_0)$ belong to a suitable space. The source term $\theta|\theta|^{\mathrm{p}-2}\ln|\theta|^k$ appears naturally in nuclear physics, optics, geophysics, supersymmetric and ination cosmology [1, 3]. The problems without delay (i.e., $\beta = 0$) has been considered by many authors during the past decades and many results have been obtained [2, 6, 10, 14] and the references therein. In [13], Wu and Tsai considered the following equation

$$\partial_{tt}\theta - \mathcal{M}\big(\|\nabla\theta\|_2^2\big)\Delta\theta + \int_0^t \mathrm{g}(t-s)\Delta\theta(s)ds - \Delta(\partial_t\theta) = \mathrm{f}(\theta).$$

with the same initial and boundary conditions as defined in (1) and established the global existence, asymptotic behavior and blow-up properties. Later, Wu [12], extended the result of [13] under a weaker condition on g. In [5], Mahdi, Ferhat and Hakem investigate the following initial boundary value problem for a system of viscoelastic wave equations of Kirchhoff type with a delay term in a bounded domain

$$\begin{aligned}\partial_{tt}\theta - \mathcal{M}\big(\|\nabla\theta\|_2^2\big)\Delta\theta + \int_0^t \mathrm{g}(t-s)\Delta\theta(s)ds + \alpha|\partial_t\theta(x,t)|^{\ell-1}\partial_t\theta(x,t)\\ + \beta|\partial_t\theta(x,t-\tau)|^{\ell-1}\partial_t\theta(x,t-\tau) = \theta|\theta|^{\mathrm{p}-1},\end{aligned}$$

with the same initial and boundary conditions as defined in (1) and established the energy decay rate and the blow up of solutions. Recently, Piskin and Yuksekkaya [11] consider a logarithmic nonlinear viscoelastic wave equation with a delay term in a bounded domain

$$\partial_{tt}\theta - \Delta\theta + \int_0^t \mathrm{g}(t-s)\Delta\theta(s)ds + \alpha\partial_t\theta(x,t) + \beta\partial_t\theta(x,t-\tau) = \theta|\theta|^{\mathrm{p}-2}\ln|\theta|^{\mathrm{k}},$$

with the same initial and boundary conditions as defined in (1) and they obtained the local existence of solution by using the Faedo-Galerkin approximation and under suitable conditions they proved the blow up of solutions in finite time. Motivated by the aforementioned works, in this paper by using the Faedo-Galerkin approximation we obtain local existence of solution and finite time blow up of solutions of the problem (1).

We assume the following conditions hold throughout the paper:

($\mathcal{A}_1$) $\mathcal{M}(t)$ is a nonnegative $\mathcal{C}^1$ function for $t \geq 0$ satisfying

$$\mathcal{M}(t) = a + bt^m, a > 0, b \geq 0 \text{ and } m \geq 0.$$

($\mathcal{A}_2$) the memory kernel $\mathrm{g}(t) : [0,\infty) \to [0,\infty)$ is a bounded $\mathcal{C}^1$ function satisfying

$$\mathrm{g}(t) \geq 0,\ a - \int_0^\infty \mathrm{g}(t)dt = \eta > 0 \text{ and } \mathrm{g}'(t) \leq 0.$$

($\mathcal{A}_3$) there exist positive constants d, γ, δ such that

$$\gamma\overline{\mathcal{M}}(s) - \left[\mathcal{M}(s) + \delta\int_0^s \mathrm{g}(y)dy\right] s \geq sd,\ \forall\ s \geq 0,$$

where $\overline{\mathcal{M}}(s) = \int_0^s \mathcal{M}(u)du$.

The organization of the remaining part of this paper is as follows. In the next section, we will give some preliminary lemmas which are useful in the main results of this paper. In section 3, we will prove local existence theorem and finite time blow up of solutions.

2. Preliminaries

In this section, we present some lemmas which are useful in our main results. As usual, $(\cdot,\cdot)$ and $\|\cdot\|_p$ indicates the inner product in the space $L^2(\Omega)$ and the norm of the space $L^p(\Omega)$, respectively.

Lemma 2.1 *([4], Lemma 3.2) There exists a positive constant* $C > 0$ *depending on* Ω *only such that*

$$\left[\int_\Omega |\theta|^p \ln|\theta|^k dx\right]^{s/p} \le C\left[\int_\Omega |\theta|^p \ln|\theta|^k dx + \|\nabla\theta\|_2^2\right]$$

for any $\theta \in L^{p+1}(\Omega)$ *and* $2 \le s \le p$, *provided that* $\int_\Omega |\theta|^p \ln|\theta|^k dx \ge 0$.

Lemma 2.2 *([4], Corollary 3.1) There exists a positive constant* $C > 0$ *depending on* Ω *only such that*

$$\|\theta\|_2^2 \le C\left[\left(\int_\Omega |\theta|^p \ln|\theta|^k dx\right)^{2/p} + \|\nabla\theta\|_2^{4/p}\right],$$

provided that $\int_\Omega |\theta|^p \ln|\theta|^k dx \ge 0$.

Lemma 2.3 *([4], Lemma 3.4) There exists a positive constant* $C > 0$ *depending on* Ω *only such that*

$$\|\theta\|_p^s \le C\left[\|\theta\|_p^p + \|\nabla\theta\|_2^2\right],$$

for any $\theta \in L^p(\Omega)$ *and* $2 \le s \le p$.

In order to prove the local existence result, we introduce the new variable z as in [9].

$$z(x,\sigma,t) = \partial_t\theta(x, t-\tau\sigma),\ \ x\in\Omega, \sigma\in(0,1),$$

which implies that

$$\tau\partial_t z(x,\sigma,t) + \partial_\sigma z(x,\sigma,t) = 0 \ \text{ in } \ \Omega\times(0,1)\times(0,1).$$

Therefore, problem (1) can be transformed as follows

$$\left.\begin{aligned}
&\partial_{tt}\theta - \mathcal{M}\big(\|\nabla\theta\|_2^2\big)\Delta\theta + \int_0^t g(t-s)\Delta\theta(s)ds + \alpha|\partial_t\theta(x,t)|^{\ell-1}\partial_t\theta(x,t)\\
&\quad + \beta|z(x,1,t)|^{\ell-1}z(x,1,t) = \theta|\theta|^{p-2}\ln|\theta|^{k},\ x\in\Omega, t>0,\\
&\tau\partial_t z(x,\sigma,t) + \partial_\sigma z(x,\sigma,t) = 0 \ \text{ in } \ \Omega\times(0,1)\times(0,1)\\
&z(x,1,t) = f_0(x,-\tau\sigma),\ x\in\Omega, t\in(0,\tau),\\
&\theta(x,0) = \theta_0(x),\ \partial_t\theta(x,0) = \theta_1(x),\ x\in\Omega,\\
&\theta(x,t) = 0,\ x\in\partial\Omega, t>0.
\end{aligned}\right\} \tag{2}$$

For any regular solution of (2), we define the energy as

$$\begin{aligned}
E(t) =&\frac{1}{2}\|\partial_t\theta\|_2^2 + \frac{1}{2}(g\circ\nabla\theta)(t) + \frac{1}{2}\left[a - \int_0^t g(s)ds\right]\|\nabla\theta\|_2^2 + \frac{b}{2(m+1)}(\|\nabla\theta\|_2^2)^{m+1}\\
&+ \frac{\varrho}{2}\int_\Omega\int_0^1 z^{\ell+1}(x,\sigma,s)d\sigma ds + \frac{k}{p^2}\|\theta\|_p^p - \frac{1}{p}\int_\Omega |\theta|^p\ln|\theta|^k dx,
\end{aligned}$$

where

$$\tau\frac{\ell\beta}{\ell+1} < \varrho < \tau\left[\frac{\alpha(\ell+1)-\beta}{\ell+1}\right], \tag{3}$$

$$(g\circ\nabla\theta)(t) = \int_0^t g(t-s)\|\nabla\theta(t) - \nabla\theta(s)\|^2 ds$$

and by simple calculations, we have

$$\begin{aligned}
\int_0^t g(t-s)\big(\nabla\theta(s),\nabla\theta_t(t)\big)ds =& -\frac{1}{2}g(t)\|\nabla\theta(t)\|^2 + \frac{1}{2}(g'\circ\nabla\theta)(t)\\
&- \frac{1}{2}\frac{d}{dt}\left[(g\circ\nabla\theta)(t) - \left(\int_0^t g(s)ds\right)\|\nabla\theta(t)\|^2\right].
\end{aligned}$$

Lemma 2.4 *Let* (θ, z) *be the solution of* (2)*. Then, for some* $C_0 \geq 0$*, the energy satisfies*

$$E'(t) = -C_0\left[\|\partial_t\theta(t)\|_{\ell+1}^{\ell+1} + \int_\Omega z^{\ell+1}(x,1,t)dx - (g'\circ\nabla\theta)(t) + g(t)\|\nabla\theta\|_2^2\right] \leq 0. \tag{4}$$

Proof Multiplying the first equation in (2) by $\partial_t\theta$, integrating over Ω and using integration by parts, we get

$$\frac{d}{dt}\Big[\frac{1}{2}\|\partial_t\theta(t)\|_2^2+\frac{1}{2}\left(a-\int_0^t g(s)ds\right)\|\nabla\theta(t)\|_2^2+\frac{1}{2}(g\circ\nabla\theta)(t)$$
$$+\frac{b}{2(m+1)}(\|\nabla\theta\|_2^2)^{m+1}+\frac{k}{p^2}\|\theta\|_p^p-\frac{1}{p}\int_\Omega|\theta|^p\ln|\theta|^k dx\Big]+\alpha\|\partial_t\theta(t)\|_{\ell+1}^{\ell+1}$$
$$+\beta\int_\Omega|z(x,1,t)dx|^{\ell-1}z(x,1,t)\partial_t\theta(x,t)dx+\frac{1}{2}g(t)\|\nabla\theta(t)\|_2^2-\frac{1}{2}(g'\circ\nabla\theta)(t)=0. \tag{5}$$

Integrating (5) over $(0,t)$, we get

$$\Big[\frac{1}{2}\|\partial_t\theta(t)\|_2^2+\frac{1}{2}\left(a-\int_0^t g(s)ds\right)\|\nabla\theta(t)\|_2^2+\frac{1}{2}(g\circ\nabla\theta)(t)$$
$$+\frac{b}{2(m+1)}(\|\nabla\theta\|_2^2)^{m+1}+\frac{k}{p^2}\|\theta\|_p^p-\frac{1}{p}\int_\Omega|\theta|^p\ln|\theta|^k dx\Big]+\int_0^t\alpha\|\partial_s\theta(s)\|_{\ell+1}^{\ell+1}ds$$
$$+\beta\int_0^t\int_\Omega|z(x,1,s)dx|^{\ell-1}z(x,1,s)\partial_s\theta(x,s)dsdx+\frac{1}{2}\int_0^t g(s)\|\nabla\theta(s)\|_2^2ds$$
$$-\frac{1}{2}\int_0^t(g'\circ\nabla\theta)(s)ds=\frac{1}{2}\|\theta_1\|_2^2+\frac{1}{2}\|\nabla\theta_0\|_2^2. \tag{6}$$

Multiplying the second equation in (2) by $\varrho|\theta|^{\ell-1}\theta$ and integrating over $\Omega\times(0,1)$, we get

$$\begin{aligned}\varrho\frac{d}{dt}\int_\Omega\int_0^1|z(x,\sigma,t)|^{\ell-1}z(x,\sigma,t)d\sigma dx\\
&=-\frac{\varrho}{\tau(\ell+1)}\int_\Omega\int_0^1\frac{\partial}{\partial\sigma}|z(x,\sigma,t)|^{\ell+1}d\sigma dx\\
&=-\frac{\varrho}{\tau}\int_\Omega\left[|z(x,1,t)|^{\ell+1}-|z(x,0,t)|^{\ell+1}\right]dx\\
&=-\frac{\varrho}{\tau}\int_\Omega|z(x,1,t)|^{\ell+1}dx+\frac{\varrho}{\tau}\|\partial_t\theta(t)\|_{\ell+1}^{\ell+1}.\end{aligned} \tag{7}$$

Now multiplying the equation (7) by $\frac{1}{2}$ and then adding to (6), we get

$$E(t)+\left[\alpha-\frac{\varrho}{\tau}\right]\int_0^t\|\partial_s\theta(s)\|_{\ell+1}^{\ell+1}ds+\frac{\varrho}{\tau}\int_0^t\int_\Omega|z(x,1,s)|^{\ell+1}dxds$$
$$-\beta\int_0^t\int_\Omega|z(x,1,s)|^{\ell-1}z(x,1,s)\partial_s\theta(x,s)dxds$$
$$+\frac{1}{2}\int_0^t g(s)\|\nabla\theta(s)\|_2^2ds-\frac{1}{2}\int_0^t(g'\circ\nabla\theta)(s)ds=E(0).$$

Differentiating with respect to t on both sides, we obtain

$$\mathrm{E}'(t) + \left[\alpha - \frac{\varrho}{\tau}\right] \|\partial_t\theta(t)\|_{\ell+1}^{\ell+1} + \frac{\varrho}{\tau}\int_\Omega |z(x,1,t)|^{\ell+1}dx$$
$$- \beta\int_\Omega |z(x,1,t)|^{\ell-1}z(x,1,t)\partial_s\theta(x,t)dx$$
$$+ \frac{1}{2}g(t)\|\nabla\theta(t)\|_2^2 - \frac{1}{2}(g'\circ\nabla\theta)(t) = 0.$$

From the Youngs inequality, we get

$$\mathrm{E}'(t) + \left[\alpha - \frac{\varrho}{\tau} - \frac{\beta}{\ell+1}\right] \|\partial_t\theta(t)\|_{\ell+1}^{\ell+1} + \left[\frac{\varrho}{\tau} - \frac{\beta\ell}{\ell+1}\right]\int_\Omega z^{\ell+1}(x,1,t)dx$$
$$+ \frac{1}{2}g(t)\|\nabla\theta(t)\|_2^2 - \frac{1}{2}(g'\circ\nabla\theta)(t) \leq 0.$$

By (3), for some $C_0 > 0$, we have

$$\mathrm{E}'(t) = -C_0\left[\|\partial_t\theta(t)\|_{\ell+1}^{\ell+1} + \int_\Omega z^{\ell+1}(x,1,t)dx - (g'\circ\nabla\theta)(t) + g(t)\|\nabla\theta\|_2^2\right] \leq 0.$$

■

3. Local existence and finite time blow up solutions

Theorem 3.1 *Let* $\theta_0 \in \mathcal{H}^2(\Omega)\cap\mathcal{W}$, $\omega_0 \in \mathcal{W}$ *and* $f_0 \in L^2(\Omega\times(0,1))$, *then there exists a unique solution* (θ, z) *of problem* (2) *defined on* $\Omega\times(0,\mathfrak{T})$ *for some constant* $\mathfrak{T} > 0$ *satisfying* $\theta \in L^\infty\big((0,\mathfrak{T}), \mathcal{H}^2(\Omega)\cap\mathcal{W}\big)$, $\partial_t\theta \in L^\infty((0,T),\mathcal{W})$, *where* $\mathcal{W} = \{\theta \in \mathcal{H}^2(\Omega) : \theta(0) = \theta_t(0) = 0\}$ *is the closed subspace of* $\mathcal{H}^2(\Omega)$ *endowed with the norm equivalent to the usual norm in* $\mathcal{H}^2(\Omega)$.

Proof We prove this theorem by Faedo-Galerkins method. In the next step, we obtain approximate solution of the problem (2).
Step 1: Approximate Solution: Let $\{\varphi_j\}_{j=1}^\infty$ be a complete orthogonal system of $\mathcal{W}$ and and $\mathcal{W}_r = span\{\varphi_1,\cdots,\varphi_r\}$, for each $r\in\mathbb{N}$ Moreover, we defne $\mathcal{V}_r = span\{\psi_1,\cdots,\psi_r\}$, $r\in\mathbb{N}$ and we can find a set of bases $\{\psi_i(x,\sigma)\}_{i=1}^r$, which is a subset of $L^2(\Omega\times(0,1))$, such that

$$\psi_i(x,0) = \phi_i(x),\ i = 1,2,\cdots,r.$$

Choosing $\{\theta_{0r}\}$ and $\{\omega_{0r}\}$ in $\mathcal{W}_r$ and $\{z_{0r}\}$ in $\mathcal{V}_r$ such that $\theta_{0r}\to\theta_0$ strongly in $\mathcal{W}$, $\omega_{0r}\to\omega_0$ strongly in $\mathcal{W}$, and $z_{0r}\to f_0$ strongly in $L^2(\Omega\times(0,1))$. We define approximates solution in the form

$$\theta_r(x,t) = \sum_{i=1}^r \phi_i(x)g_{ir}(t),$$

$$z_r(x,\sigma,t) = \sum_{i=1}^r \psi_i(x,\sigma)h_{ir}(t),$$

where $(\theta_r(t), z_r(t))$ are solutions of the following system:

$$\left.\begin{aligned}
&\int_\Omega (\partial_{tt}\theta_r)\phi_i dx - \int_\Omega \mathcal{M}\big(\|\nabla\theta_r\|_2^2\big)\Delta\theta_r\phi_i dx + \int_\Omega\int_0^t g(t-s)\Delta\theta_r(s)\phi_i ds dx \\
&+ \int_\Omega \alpha|\partial_t\theta_r(x,t)|^{\ell-1}\partial_{tt}\theta_r(x,t)\phi_i dx + \int_\Omega \beta|z(x,1,t)|^{\ell-1}z(x,1,t)\phi_i dx \\
&\qquad\qquad = \int_\Omega \theta_r|\theta_r|^{p-2}\ln|\theta_r|^k\phi_i dx \quad \text{in} \quad \Omega\times(0,\mathfrak{T}), \\
&\int_\Omega [\tau\partial_t z_r(x,\sigma,t) + \partial_\sigma z_r(x,\sigma,t)]\phi_i dx = 0 \quad \text{in} \quad \Omega\times(0,1)\times(0,\mathfrak{T}) \\
&z_r(x,1,t) = f_{0r}(x,-\tau\sigma), \quad \text{in} \quad \Omega\times(0,1), \\
&\theta_r(x,0) = \theta_{0r}(x),\ \partial_t\theta_r(x,0) = \theta_{1r}(x),\ x\in\Omega, \\
&\theta_r(x,t) = 0,\ x\in\partial\Omega, t\in(0,1).
\end{aligned}\right\}. \tag{8}$$

We obtain (8) has a unique solution $\big\{\big(g_{ir}(t), h_{ir}(t)\big)\big\}_{i=1}^r$ defined on $(0,\mathfrak{T})$ by using the theories of ordinary differential equation. Rest of the proof is very similar to Theorem 3.1. in [11, 15], so we omit details here. ■

Theorem 3.2 *Let* $\theta_0 \in \mathcal{H}_0^1(\Omega)\cap\mathcal{H}^2(\Omega)$, $\theta_1\in\mathcal{H}_0^1(\Omega)$ *and* $f_0 \in L^2\big(\Omega\times(0,1)\big)$ *be given. Assume that the assumptions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *are fulfilled. Let* $p > k$. *Then for any initial data satisfying* $E(0) < 0$, *the solution of* (2) *blows up in finite time.*

Proof Let $H(t) = -E(t)$ and using the Lemma 2.4, we obtain

$$\begin{aligned}
H'(t) \geq &\left[\alpha - \frac{\varrho}{\tau} - \frac{\beta}{\ell+1}\right]\|\partial_t\theta(t)\|_{\ell+1}^{\ell+1} + \left[\frac{\varrho}{\tau} - \frac{\beta\ell}{\ell+1}\right]\int_\Omega z^{\ell+1}(x,1,t)dx \\
&+ \frac{1}{2}g(t)\|\nabla\theta(t)\|_2^2 - \frac{1}{2}(g'\circ\nabla\theta)(t).
\end{aligned} \tag{9}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega |\theta|^p \ln|\theta|^k dx \leq \frac{k}{p}\|\theta\|_{p+1}^{p+1}. \tag{10}$$

Next, we let $N(t) = \|\theta\|_2^2$ and differentiating twice, we get

$$N'(t) = 2\int_\Omega (\partial_t\theta)\theta dx$$

and

$$N''(t) = 2\|\partial_t\theta\|_2^2 + 2\int_\Omega (\partial_{tt}\theta)\theta dx. \tag{11}$$

Using the first equation in (2), we get

$$\begin{aligned} \mathrm{N}''(t) =& 2\|\partial_t\theta\|_2^2 - 2\mathcal{M}\big(\|\nabla\theta\|_2^2\big)\|\nabla\theta\|_2^2 + 2\int_\Omega\int_0^t \mathrm{g}(t-s)\Delta\theta(s)\theta(t)dsdx \\ &- 2\alpha\int_\Omega |\partial_t\theta(x,t)|^{\ell-1}\theta(t)\partial_t\theta(x,t)dx \\ &- 2\beta\int_\Omega |\mathrm{z}(x,1,t)|^{\ell-1}\theta(t)\mathrm{z}(x,1,t)dx + 2\int_\Omega |\theta|^{\mathrm{p}}\ln|\theta|^{\mathrm{k}}dx. \end{aligned} \tag{12}$$

Using Hölder's and Youngs inequalities to get

$$\begin{aligned} &\int_\Omega\int_0^t \mathrm{g}(t-s)\nabla\theta(s)\nabla\theta(t)dsdx \\ &= \int_0^t \mathrm{g}(t-s)\|\nabla\theta(t)\|_2^2 ds + \int_\Omega\int_0^t \mathrm{g}(t-s)\nabla\theta(t)\big(\nabla\theta(s) - \nabla\theta(t)\big)dsdx \\ &\geq -(\mathrm{g}\circ\nabla\theta)(t) + \frac{3}{4}\|\nabla\theta(t)\|_2^2\int_0^t \mathrm{g}(s)ds \end{aligned} \tag{13}$$

Using (13) in (12), we get

$$\begin{aligned} \mathrm{N}''(t) \geq& \mathrm{p}\|\partial_t\theta\|_2^2 + 2\mathrm{p}\mathrm{H}(t) + (\mathrm{p}-2)(\mathrm{g}\circ\nabla\theta)(t) + \left[\frac{3-2\mathrm{p}}{2}\right]\|\nabla\theta(t)\|_2^2\int_0^t \mathrm{g}(s)ds \\ &- 2a\|\nabla\theta(t)\|_2^2 - 2b\big(\|\nabla\theta(t)\|_2^2\big)^{m+1} + a(\mathrm{p}+1)\|\nabla\theta(t)\|_2^2 \\ &+ \frac{b\mathrm{p}}{m+1}\big(\|\nabla\theta(t)\|_2^2\big)^{m+1} + d\|\nabla\theta\|_2^2 - 2\alpha\int_\Omega |\partial_t\theta(x,t)|^{\ell-1}\theta(t)\theta'(x,t)dx \\ &- 2\beta\int_\Omega |\mathrm{z}(x,1,t)|^{\ell-1}\theta(t)\mathrm{z}(x,1,t)dx + \varrho\mathrm{p}\int_\Omega\int_0^1 \mathrm{z}^{\ell+1}(x,\sigma,t)d\sigma dx. \end{aligned} \tag{14}$$

Using $(\mathcal{A}_3)$ to get

$$\begin{aligned} \mathrm{N}''(t) \geq& \mathrm{p}\|\partial_t\theta\|_2^2 + 2\mathrm{p}\mathrm{H}(t) + (\mathrm{p}-2)(\mathrm{g}\circ\nabla\theta)(t) + d\|\nabla\theta\|_2^2 \\ &- 2\alpha\int_\Omega |\partial_t\theta(x,t)|^{\ell-1}\theta(t)\theta'(x,t)dx - 2\beta\int_\Omega |\mathrm{z}(x,1,t)|^{\ell-1}\theta(t)\mathrm{z}(x,1,t)dx \\ &+ \varrho\mathrm{p}\int_\Omega\int_0^1 \mathrm{z}^{\ell+1}(x,\sigma,t)d\sigma dx. \end{aligned} \tag{15}$$

Now we define the functional

$$\mathcal{G}(t) = \mathrm{H}(t)^{1-\sigma} + 2\varepsilon\mathrm{N}'(t).$$

Differentiating on both sides and using (15), we obtain

$$\begin{aligned}\mathcal{G}'(t) &= (1-\sigma)\mathrm{H}(t)^{-\sigma}\mathrm{H}'(t) + 2\varepsilon \mathrm{N}''(t)\\ &\geq (1-\sigma)\mathrm{H}(t)^{-\sigma}\mathrm{H}'(t) + 2\varepsilon \mathrm{p}\|\partial_t\theta\|_2^2 + 2\mathrm{p}\varepsilon \mathrm{H}(t) + \varepsilon(\mathrm{p}-2)(\mathrm{g}\circ\nabla\theta)(t)\\ &\quad - 2\varepsilon\alpha\int_\Omega |\partial_t\theta(x,t)|^{\ell-1}\theta(t)\theta'(x,t)dx\\ &\quad - 2\varepsilon\beta\int_\Omega |\mathrm{z}(x,1,t)|^{\ell-1}\theta(t)\mathrm{z}(x,1,t)dx\\ &\quad + \varepsilon\varrho\mathrm{p}\int_\Omega\int_0^1 \mathrm{z}^{\ell+1}(x,\sigma,t)d\sigma dx + \varepsilon d\|\nabla\theta\|_2^2.\end{aligned} \tag{16}$$

By Hölder's inequality and from (10), we have

$$\begin{aligned}\left|\int_\Omega \theta|\partial_t\theta|^{\ell-1}(\partial_t\theta)dx\right| &\leq \|\partial_t\theta\|_{\ell+1}^{\ell}\|\theta\|_{\ell} \leq c_1\|\theta\|_{\ell+1}^{\frac{\ell+1}{\mathrm{p}+1}}\|\theta\|_{\ell+1}^{1-\frac{\ell+1}{\mathrm{p}+1}}\|\partial_t\theta\|_{\ell+1}^{\ell}\\ &\leq c_2\|\theta\|_{\ell+1}^{\frac{\ell+1}{\mathrm{p}+1}}\mathrm{H}(t)^{\frac{1}{\mathrm{p}+1}-\frac{\ell+1}{(\mathrm{p}+1)^2}}\|\partial_t\theta\|_{\ell+1}^{\ell}\end{aligned} \tag{17}$$

By Youngs inequality and (9), we get

$$\left|\int_\Omega \theta|\partial_t\theta|^{\ell-1}(\partial_t\theta)dx\right| \leq c_3\left[\xi^{\frac{1}{\mathrm{p}+1}}\|\theta\|_{\ell+1}^{\ell+1}\mathrm{H}(0)^{-l'} - \xi^{-\ell'}\mathrm{H}(0)^{l-l'}\mathrm{H}'(t)\mathrm{H}(t)^{-l}\right], \tag{18}$$

where $\sigma = \frac{1}{\mathrm{p}+1} - \frac{\ell+1}{(\mathrm{p}+1)^2} >, \xi > 0, \sigma' = \frac{\ell+1}{\ell}$. Letting $0 < \sigma < \sigma'$. Similarly, we can have

$$\begin{aligned}\left|\int_\Omega |\mathrm{z}(x,1,t)|^{\ell-1}\theta(t)\mathrm{z}(x,1,t)dx\right| \leq c_3\Big[&\xi^{\frac{1}{\mathrm{p}+1}}\|\theta\|_{\ell+1}^{\ell+1}\mathrm{H}(0)^{-\sigma'}\\ &- \xi^{-\ell'}\mathrm{H}(0)^{\sigma-\sigma'}\mathrm{H}'(t)\mathrm{H}(t)^{-\sigma}\Big]\end{aligned} \tag{19}$$

Applying (19) and (18) in (16), we have

$$\begin{aligned}\mathcal{G}'(t) \geq &\left[(1-\sigma) - 2\varepsilon(\alpha+\beta)\mathrm{H}(0)^{\sigma-\sigma'}\xi^{-\ell'}\right]\mathrm{H}(t)^{-\sigma}\mathrm{H}'(t)\\ &- 2\varepsilon(\alpha+\beta)\xi^{\frac{1}{\mathrm{p}+1}}\|\theta\|_{\ell+1}^{\ell+1}\mathrm{H}(0)^{-\sigma'}\\ &+ 2\varepsilon\mathrm{p}\|\partial_t\theta\|_2^2 + 2\mathrm{p}\varepsilon\mathrm{H}(t) + \varepsilon(\mathrm{p}-2)(\mathrm{g}\circ\nabla\theta)(t)\\ &+ \varepsilon\varrho\mathrm{p}\int_\Omega\int_0^1 \mathrm{z}^{\ell+1}(x,\sigma,t)d\sigma dx + \varepsilon d\|\nabla\theta\|_2^2.\end{aligned} \tag{20}$$

But, for sufficiently small ε, we have

$$\left[(1-\sigma) - 2\varepsilon(\alpha+\beta)\mathrm{H}(0)^{\sigma-\sigma'}\xi^{-\ell'}\right] \geq 0.$$

So, put $s = \ell + 1 \leq \mathrm{p} + 1$ such that $\|\theta\|_{\ell+1}^{s} \leq c_1\big(\|\nabla\theta\|_2^2 + \|\theta\|_{\mathrm{p}+1}^{\mathrm{p}+1}\big)$, where

$c = 2(\alpha+\beta)\xi^{\frac{1}{p+1}}\mathrm{H}(0)^{-\sigma'}c_1$ and taking $d > c$. Then

$$\begin{aligned}\mathcal{G}'(t) \geq\, &\varepsilon(d-c)\|\nabla\theta\|_2^2 - \varepsilon c\|\theta\|_{p+1}^{p+1} + 2\varepsilon p\|\partial_t\theta\|_2^2 + 2p\varepsilon\mathrm{H}(t)\\ &+\varepsilon(p-2)(g\circ\nabla\theta)(t) + \varepsilon\varrho p\int_\Omega\int_0^1 z^{\ell+1}(x,\sigma,t)d\sigma dx + \varepsilon d\|\nabla\theta\|_2^2 \\ \geq\, &\varepsilon\mu_1\|\nabla\theta\|_2^2 - \varepsilon\mu_2\|\theta\|_{p+1}^{p+1} + \varepsilon\mu_3\mathrm{H}(t) + \varepsilon\mu_4\|\partial_t\theta\|_2^2 + \varepsilon\mu_5(g\circ\nabla\theta)(t),\end{aligned}\tag{21}$$

where $\mu_1 = d-c, \mu_2 = c, \mu_3 = 2p, \mu_4 = 2p$ and $\mu_5 = p-2$. Next, using the technique of Messaoudi [8], we suppose that $p = 2\mu_6 + (p-2\mu_6)$, where $\mu_6 < \min\{\mu_1,\mu_2,\mu_3,\mu_5\}$, then (21) takes the form

$$\begin{aligned}\mathcal{G}'(t) \geq\, &\varepsilon(\mu_1-\mu_6)\|\nabla\theta\|_2^2 - \varepsilon(\mu_6-\mu_2)\|\theta\|_{p+1}^{p+1} + \varepsilon(\mu_3-\mu_6)\mathrm{H}(t)\\ &+\varepsilon(\mu_4-\mu_6)\|\partial_t\theta\|_2^2 + \varepsilon(\mu_5-\mu_6)(g\circ\nabla\theta)(t)\\ \geq\, &\varepsilon\chi\left[\|\nabla\theta\|_2^2 + \|\theta\|_{p+1}^{p+1} + \mathrm{H}(t) + \|\partial_t\theta\|_2^2 + (g\circ\nabla\theta)(t)\right]\\ \geq\, &\varepsilon\chi\left[\|\theta\|_{p+1}^{p+1} + \mathrm{H}(t) + \|\partial_t\theta\|_2^2\right],\end{aligned}\tag{22}$$

where $\chi > 0$ is the minimum of the coefficients of $\|\theta\|_{p+1}^{p+1}$, $\mathrm{H}(t)$ and $\|\partial_t\theta\|_2^2$. Now choose an ε so that

$$\mathcal{G}(0) = \mathrm{H}(0)^{1-\sigma} + 2\varepsilon\int_\Omega \theta^0\theta^1 dx > 0.$$

Thus, we have

$$\mathcal{G}(t) \geq \mathcal{G}(0) > 0,\ \forall\, t \geq 0.$$

Similarly as in [7], we also obtain

$$\mathcal{G}(t)^{1/(1-\sigma)} \leq C\left[\|\theta\|_{p+1}^{p+1} + \mathrm{H}(t) + \|\partial_t\theta\|_2^2\right],\ \forall\ t\geq 0,\tag{23}$$

for some number $C > 0$. Combining (22) and (23), we arrive at

$$\mathcal{G}'(t) \geq \frac{\varepsilon\chi}{C}\mathcal{G}(t)^{1/(1-\sigma)},\ \forall\ t \geq 0.$$

A simple integration over $(0,t)$ yields

$$\mathcal{G}(t)^{\sigma/(1-\sigma)} \geq \frac{1}{\mathcal{G}(0)^{-\sigma/(1-\sigma)} - \varepsilon\chi t\sigma/[C(1-\sigma)]},\ \forall\ t\geq 0.$$

This shows that $\mathcal{G}(t)$ blows up in finite time

$$\mathrm{T}^* \leq \frac{C(1-\sigma)}{\varepsilon\chi\sigma\mathcal{G}(0)^{\sigma/(1-\sigma)}}.$$

This completes the proof of theorem. ■

Acknowledgments

The authors would like to thank the referees for their valuable suggestions and comments for the improvement of the paper.

References

[1] K. Bartkowski and P. Gorka, One dimensional Klein-Gordon equation with logarithmic nonlinearities, Journal of Physics A: Mathematical and Theoretical, **41 (35)** (2008) 355201, doi:10.1088/1751-8113/41/35/355201.

[2] M. Gobbino, Quasilinear degenerate parabolic equations of Kirchhof type, Mathematical Methods in Applied Sciences, **22** (1999) 375–388.

[3] P. Gorka, Logarithmic Klein-Gordon equation, Acta Physica Polonica, **B40** (2009) 59–66.

[4] M. Kafini and S. A. Messaoudi, Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay, Applicable Analysis, **99 (3)** (2020) 530–547.

[5] F. Z. Mahdi, M. Ferhat and A. Hakem, Blow up and asymptotic behavior for a system of viscoelastic wave equations of Kirchhoff type with a delay term, Advances in the Theory of Nonlinear Analysis and its Application, **2 (3)** (2018) 146–167.

[6] T. Matsuyama and R. Ikehata, On global solutions and energy decay for the wave equation of Kirchhoff type with nonlinear damping terms, Journal of Mathematical Analysis and Applications, **204 (3)** (1996) 729–753.

[7] S. A. Messaoudi, Blow up and global existence in a nonlinear viscoelastic wave equation, Mathematische Nachrichten, **260** (2003) 58–66.

[8] S. A. Messaoudi, Blow up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation, Journal of Mathematical Analysis and Applications, **320** (2006) 902–915.

[9] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Differential Integral Equations, **21**(2008) 935–958.

[10] K. Ono, Global existence, decay and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings, Journal of Differential Equations, **137 (2)** (1997) 273–301.

[11] E. Pişkin and H. Yuksekkaya, Local existence an blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay, Computational Methods for Differential Equations, doi:10.22034/cmde.2020.35546.1608.

[12] S. T. Wu, Exponential energy decay of solutions for an integro-differential equation with strong damping, Journal of Mathematical Analysis and Applications, **364 (2)** (2010) 609–617.

[13] S. T. Wu and L. Y. Tsai, On global existence and blow-up of solutions for an integro-differential equation with strong damping, Taiwanese Journal of Mathematics, **10 (4)** (2006) 979–1014.

[14] R. Zeng, C. L. Mu and S. M. Zhou, A blow-up result for Kirchhoff-type equations with high energy, Mathematical Methods in the Applied Sciences, **34 (4)** (2011) 479–486.

[15] Y. Zhang and X. Liu, Blowup of solution for beam equation with delay and dynamic boundary conditions, Mathematical Methods in the Applied Sciences, **43 (8)** (2020) 5430–5440.