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Permanence and Uniformly Asymptotic Stability of Almost Periodic Positive Solutions for a Dynamic Commensalism Model on Time Scales

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Abstract. In this paper, we study dynamic commensalism model with nonmonotic functional response, density dependent birth rates on time scales and derive sufficient conditions for the permanence. We also establish the existence and uniform asymptotic stability of unique almost periodic positive solution of the model by using Lyapunov functional method.

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1. Introduction

Ecology relates to the study of living beings in connection to their living styles. Research in the area of theoretical ecology was first studied by Volterra [29] and Lotka [23]. Later many ecologists and mathematicians contributed to the growth of this area of knowledge as reported in [3, 7, 12, 24, 25] and references therein.

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The ecological interactions can be broadly classified as prey-predator, competition, commensalism, ammensalism, and neutralism etc.

A two species Commensalisms is an ecological connection between two species where one species X gain benefits while those of the other species Y neither benfit nor harmed. Here, X may referred as the commensal species while Y the host. Some examples are Cattle Egret, Anemonetish and Barnacles etc. The host species Y supports the commensal species X which has a natural growth rate in spite of a support other than from X. The commensal species X, in spite of the limitation of its natural resources flourishes drawing strength from the host species Y. The model is characterized by a system of first order nonlinear differential equations. In the last decades, commensalism model studied many researchers [8, 9, 19, 20, 32].

Chen at el. [6] proposed the following two species commensal symbiosis models with nonmonotonic functional response,

$$\begin{aligned} u_1'(t) &= u_1(t)\left[a_{11} - b_{12}u_1(t) + \frac{cu_2(t)}{d+u_2^2(t)}\right], \\ u_2'(t) &= u_2(t)\left[a_{21} - b_{22}u_2(t)\right], \end{aligned}$$

where $a_{11}, a_{21}, b_{12}, b_{22}, c, d$ are all positive constants and showed that the system admits a unique globally asymptotically stable positive equilibrium.

Zhao et al. [35] proposed and analyzed a commensalism model with nonmonotonic functional response and density-dependent birth rates,

$$\left.\begin{aligned} u_1'(t) &= u_1(t)\left[\frac{a_{11}}{a_{12}+a_{13}u_1(t)} - a_{14} - b_1u_1(t) + \frac{cu_2(t)}{d+u_2^2(t)}\right], \\ u_2'(t) &= u_2(t)\left[\frac{a_{21}}{a_{22}+a_{23}u_2(t)} - a_{24} - b_2u_2(t)\right], \end{aligned}\right\} \tag{1}$$

where a_{ij} (i = 1, 2, j = 1, 2, 3, 4) and $b_1, c, d,$ and b_2 are all positive constants. Here $u_1(t)$ and $u_2(t)$ are the densities of the first and second species at time t, respectively. a_{11} and a_{21} stand for the total resources available per unit time for species u and v, respectively. By applying the differential inequality theory, they showed that each equilibrium can be globally attractive under suitable conditions.

Xie et al. [33] derived sufficient conditions for the existence of positive periodic solution of the following discrete Lotka-Volterra commensal symbiosis model

$$\begin{aligned} u(k+1) &= u(k)\exp\left\{a_1(k) - b_1(k)u(k) + c_1(k)v(k)\right\} \\ v(k+1) &= v(k)\exp\left\{a_2(k) - b_2(k)v(k)\right\} \end{aligned}$$

where $\{b_i(k)\}$, $i = 1, 2$, $\{c_i(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$, are ω-periodic sequences such that $\overline{a}_i = \frac{1}{\omega}\sum_{k=0}^{\omega-1} a_i(k) > 0$, $i = 1, 2$.

The differential, difference and dynamic equations on time scales are three equations play important role for modelling in the environment. Among them, the theory of dynamic equations on time scales is the most recent and was introduced by Stefan Hilger in his PhD thesis in 1988 with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [4, 5]. In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. In this paper we systematically unify the existence of almost periodic solutions of commensalism model with nonmonotic functional response and density dependent birth rates modelled by ordinary differential equations and their discrete analogues in the form of difference equations and to extend these results to more general time scales. The concept of almost periodic time scales was proposed by Li and Wang [13]. Based on this concept, some works have been done (see [14–18, 21, 22, 26, 28] and references therein).

Recently, Wang [30] established a criteria for global existence of multiple periodic solutions to the dynamic predator-prey model with delays,

$$\begin{aligned} u_1^{\Delta}(t) &= a(t) - b(t)\exp\{u_1(t)\} - \frac{c(t)\exp\{2u_2(t)\}}{m^2\exp\{2u_2(t)\} + \exp\{2u_1(t)\}} - h(t)\exp\{-u_1(t)\}, \\ u_2^{\Delta}(t) &= \frac{f(t)\exp\{u_1(t-\tau(t)) + u_2(t-\tau(t))\}}{m^2\exp\{2u_2(t-\tau(t))\} + \exp\{2u_1(t-\tau(t))\}} - d(t), \end{aligned}$$

by applying continuation theorem based on Gaines and Mawhin's coincidence degree theory, and the corresponding discrete system was studied by [11].

Wang et al. [31] considered the following competitive system on time scales,

$$\begin{aligned} u_1^{\Delta}(t) &= r_1(t) - a_1(t)\exp\{u_1(t)\} - \frac{b_1(t)\exp\{u_2(t)\}}{1 + \exp\{u_2(t)}, \\ u_2^{\Delta}(t) &= r_2(t) - a_2(t)\exp\{u_2(t)\} - \frac{b_2(t)\exp\{u_1(t)\}}{1 + \exp\{u_1(t)}. \end{aligned}$$

and established existence and uniformly asymptotic stability of unique positive almost periodic solutions by time scale calculus theory and Lyapunov functional method

Prasad et al. [27] studied the following 3-species predator-prey competition model on time scales,

$$\begin{aligned} u_1^{\Delta}(t) &= r_1(t) - \exp\{u_1(t)\} - \alpha\exp\{u_2(t)\} - \beta\exp\{u_3(t)\}, \\ u_2^{\Delta}(t) &= r_2(t) - \beta\exp\{u_1(t)\} - \exp\{u_2(t)\} - \alpha\exp\{u_3(t)\}, \\ u_3^{\Delta}(t) &= r_3(t) - \alpha\exp\{u_1(t)\} - \beta\exp\{u_2(t)\} - \exp\{u_3(t)\}, \end{aligned}$$

and established sufficient conditions for the existence and uniform asymptotic stability of unique positive almost periodic solution of system.

Motivated by the aforementioned reasons in this paper we study commensalism model with nonmonotic functional response and density dependent birth rates on time scales,

$$\left.\begin{aligned} \omega_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} \\ &+ \frac{c(t)\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}}, \\ \omega_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t)+a_{23}(t)\exp\{\omega_2(t)\}} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\}, \end{aligned}\right\} \quad (2)$$

where $\omega_i(t)$ are the densities of the i^{th}species at time $t \in \mathbb{T}^+ \big(\mathbb{T}^+$ is a nonempty closed subset of $\mathbb{R}^+ = [0,+\infty)\big)$ and $\omega_i(0) > 0$. ω_i^{Δ} express the delta derivative of the functions $\omega_i(t), i = 1,2$. $a_{ij}(t), i = 1,2, j = 1,2,3,4$ and $b_1(t), b_2(t), c(t), d(t)$ are bounded positive almost periodic functions. Clearly, if we set $u_i(t) = \exp\{\omega_i\}, i = 1,2$ and choose $\mathbb{T}^+ = \mathbb{R}^+$ the system (2) is reduced to the model (1) and $\mathbb{T}^+ = \mathbb{Z}^+$ ($\mathbb{Z}^+$ is the set of nonnegative integer numbers), then the system (2) is reduced to the following discrete system,

$$\begin{aligned} \omega_1(t+1) =& \omega_1(t)\exp\left[\frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)\omega_1(t)} - a_{14}(t) - b_1(t)\omega_1(t) + \frac{c(t)\omega_2(t)}{d(t)+\omega_2^2(t)}\right], \\ \omega_2(t+1) =& \omega_2(t)\exp\left[\frac{a_{21}(t)}{a_{22}(t)+a_{23}(t)\omega_2(t)} - a_{24}(t) - b_2(t)\omega_2(t)\right], \end{aligned}$$

The paper is organized in the following way. In Section 2, we provide some definitions and lemmas which are useful in establishing our main results. In Section 3, we derive sufficient conditions for the permanence of system (2). The sufficient conditions for the existence and uniform asymptotic stability of unique positive almost periodic solution of system (2) are derived in Section 4. In final section, the numeric simulations are given to illustrate the feasibility of the main results.

2. Preliminaries

In this section, we give some definitions and developed lemmas which are useful in the next sections.
As we assumed almost periodic functions on $\mathbb{T}^+$ are bounded, we use the notations

$$f^{\mathcal{L}} = \inf\Big\{f(t) : t \in \mathbb{T}^+\Big\},$$

and

$$f^{\mathcal{U}} = \sup\Big\{f(t) : t \in \mathbb{T}^+\Big\},$$

where $f(t)$ is an almost periodic function. We use the following notations in the paper:

$$\mathscr{A}_1 = \frac{a_{11}^{\mathcal{U}} a_{13}^{\mathcal{U}} e^{\kappa_1}}{\big(a_{12}^{\mathcal{L}} + a_{13}^{\mathcal{L}} e^{\ell_1}\big)^2}, \quad \mathscr{A}_2 = \frac{a_{11}^{\mathcal{L}} a_{13}^{\mathcal{L}} e^{\ell_1}}{\big(a_{12}^{\mathcal{U}} + a_{13}^{\mathcal{U}} e^{\kappa_1}\big)^2},$$

$$\mathscr{B}_1 = \frac{c^{\mathcal{U}}\left(d^{\mathcal{U}} - e^{3\ell_2}\right)}{\left(d^{\mathcal{L}} + e^{2\ell_2}\right)^2}, \quad \mathscr{B}_2 = \frac{c^{\mathcal{L}}\left(d^{\mathcal{L}} - e^{3\kappa_2}\right)}{\left(d^{\mathcal{U}} + e^{2\kappa_2}\right)^2},$$

$$\mathscr{C}_1 = \frac{a_{21}^{\mathcal{U}} a_{23}^{\mathcal{U}} e^{\kappa_2}}{\left(a_{22}^{\mathcal{L}} + a_{23}^{\mathcal{L}} e^{\ell_2}\right)^2}, \quad \mathscr{C}_2 = \frac{a_{21}^{\mathcal{L}} a_{23}^{\mathcal{L}} e^{\ell_2}}{\left(a_{22}^{\mathcal{U}} + a_{23}^{\mathcal{U}} e^{\kappa_2}\right)^2}.$$

Definition 2.1 [5] A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\},$$

$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\},$$

and

$$\mu(t) = \rho(t) - t,$$

respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$;otherwise $\mathbb{T}^k = \mathbb{T}$.
- A function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.2 [5] A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1+\mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Also, we denote the set

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}.$$

Lemma 2.3 *[10] If $a > 0, b > 0$ and $-b \in \mathcal{R}^+$. Then*

$$w^{\Delta}(t) \leq (\geq) a - bw(t), \ w(t) > 0, \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$w(t) \leq (\geq) \frac{a}{b}\Big[1 + \Big(\frac{bw(t_0)}{a} - 1\Big)e_{(-b)}(t, t_0)\Big], \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 2.4 [13] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\prod = \{\kappa \in \mathbb{R} : t + \kappa \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.5 [13] Let $\mathbb{T}$ be an almost periodic time scale. Then a function $w \in \mathcal{C}(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of w i.e.,

$$\mathcal{E}\{\varepsilon, w\} = \Big\{\kappa \in \prod : |w(t+\kappa) - w(t)| < \varepsilon, \forall t \in \mathbb{T}\Big\}$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε.

Definition 2.6 [13] Let $\mathbb{T}$ be a positive almost periodic time scale. Then a function $\phi \in \mathcal{C}(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $w \in \mathbb{D}$ if the ε-translation set of ϕ

$$\mathcal{E}\{\varepsilon, \phi, \mathbb{S}\} = \Big\{\kappa \in \prod : |\phi(t+\kappa) - \phi(t)| < \varepsilon, \forall (t, w) \in \mathbb{T} \times \mathbb{S}\Big\}$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε, and for each compact subset $\mathbb{S}$ of $\mathbb{D}$.

Next, consider the system

$$w^{\Delta}(t) = \psi(t, w), \tag{3}$$

and its associate product system

$$w^{\Delta}(t) = \psi(t, w), \quad z^{\Delta}(t) = \psi(t, z), \tag{4}$$

where $\psi : \mathbb{T}^+ \times \mathbb{S}_B \to \mathbb{R}^n$, $\mathbb{S}_B = \{w \in \mathbb{R}^n : \|w\| < B\}$, $\psi(t, w)$ is almost periodic in t uniformly for $w \in \mathbb{S}_B$ and is continuous in w.

Lemma 2.7 *[34] Let $\mathcal{V}(t, w, z)$ be Lyapunov function defined on $\mathbb{T}^+ \times \mathbb{S}_B^2$ and satisfies the following conditions*

(i) $\alpha\big(\|w - z\|\big) \le \mathcal{V}(t, w, z) \le \beta\big(\|w - z\|\big)$, *where* $\alpha, \beta \in \mathcal{P}$,

$$\mathcal{P} = \Big\{\gamma \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+) : \gamma(0) = 0 \quad \textit{and} \quad \gamma \quad \textit{is} \quad \textit{increasing}\Big\};$$

(ii) $|\mathcal{V}(t, w, z) - \mathcal{V}(t, w_1, z_1)| \le \mathcal{L}\big(\|w - w_1\| + \|z - z_1\|\big)$, *where* $\mathcal{L} > 0$ *is a constant,*

(iii) $\mathcal{D}^+\mathcal{V}^{\Delta}(t, w, z) \le -\lambda \mathcal{V}(t, w, z)$, *where* $\lambda > 0, -\lambda \in \mathcal{R}^+$.

Further, if there exists a solution $x(t) \in \mathbb{S}$ of system (3) for $t \in \mathbb{T}^+$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exist a unique almost periodic solution $f(t) \in \mathbb{S}$ of system (3), which is uniformly asymptotically stable.

Definition 2.8 System (2) is said to be permanent, if there exist positive constants ℓ, κ such that

$$\ell \le \liminf_{t \to +\infty} \omega_i(t) < \limsup_{t \to +\infty} \omega_i(t) \le \kappa, \; i = 1, 2,$$

for any solution $\big(\omega_1(t), \omega_2(t)\big)$ of (2).

3. Permanence

In this section, we derive the sufficient conditions for the system (2) to be permanent.

Lemma 3.1 *Suppose that*

$$\left.\begin{aligned} a_{11}^{\mathcal{U}} + c^{\mathcal{U}} a_{12}^{\mathcal{L}} &> \left[a_{14}^{\mathcal{L}} + b_1^{\mathcal{L}}\right] a_{12}^{\mathcal{L}} \\ a_{21}^{\mathcal{U}} &> \left[a_{24}^{\mathcal{L}} + b_2^{\mathcal{L}}\right] a_{22}^{\mathcal{L}}. \end{aligned}\right\} \tag{5}$$

Then any positive solution $\big(\omega_1(t), \omega_2(t)\big)$ *of the dynamic system (2) satisfies*

$$\limsup_{t\to+\infty} \omega_1(t) \le \kappa_1 := \frac{1}{b_1^{\mathcal{L}}}\left[\frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\right]$$

and

$$\limsup_{t\to+\infty} \omega_2(t) \le \kappa_2 := \frac{1}{b_2^{\mathcal{L}}}\left[\frac{a_{21}^{\mathcal{U}}}{a_{22}^{\mathcal{L}}} - a_{24}^{\mathcal{L}} - b_2^{\mathcal{L}}\right].$$

Proof It follows from the first equation of the system (2) that

$$\begin{aligned} \omega_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} \\ &\qquad + \frac{c(t)\exp\{\omega_2(t)\}}{d(t) + \exp\{2\omega_2(t)\}} \\ \le& \frac{a_{11}(t)}{a_{12}(t)} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} + c(t) \\ \le& \frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\exp\{\omega_1(t)\} \\ \le& \frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\big[\omega_1(t) + 1\big]. \end{aligned}$$

By using Lemma 2.3 we have

$$\limsup_{t\to+\infty} \omega_1(t) \le \kappa_1 := \frac{1}{b_1^{\mathcal{L}}}\left[\frac{a_{11}^{\mathcal{U}}}{a_{12}^{\mathcal{L}}} - a_{14}^{\mathcal{L}} + c^{\mathcal{U}} - b_1^{\mathcal{L}}\right].$$

Similarly from the second equation of the system (2) that

$$\begin{aligned} \omega_2^{\Delta}(t) &= \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{\omega_2(t)\}} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\} \\ &\le \frac{a_{21}(t)}{a_{22}(t)} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\} \\ &\le \frac{a_{21}^{\mathcal{U}}}{a_{22}^{\mathcal{L}}} - a_{24}^{\mathcal{L}} - b_2^{\mathcal{L}}\big[\omega_2(t) + 1\big]. \end{aligned}$$

From Lemma 2.3, we get

$$\limsup_{t\to+\infty} \omega_2(t) \le \kappa_2 := \frac{1}{b_2^{\mathcal{L}}}\left[\frac{a_{21}^{\mathcal{U}}}{a_{22}^{\mathcal{L}}} - a_{24}^{\mathcal{L}} - b_2^{\mathcal{L}}\right].$$

This completes the proof. ■

Lemma 3.2 *If the inequalities* (5) *and*

$$\left.\begin{aligned} a_{11}^{\mathcal{L}} &> a_{14}^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1\}\big) \\ a_{21}^{\mathcal{L}} &> a_{24}^{\mathcal{U}}\big(a_{22}^{\mathcal{U}} + \exp\{\kappa_2\}\big) \end{aligned}\right\} \tag{6}$$

hold, then any positive solution $\big(\omega_1(t), \omega_2(t)\big)$ *of system (2) satisfies*

$$\liminf_{t\to+\infty} \omega_1(t) \geq \ell_1 := \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1\}\big)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right],$$

$$\liminf_{t\to+\infty} \omega_2(t) \geq \ell_2 := \ln\left[\frac{a_{21}^{\mathcal{L}}}{b_2^{\mathcal{U}}\big(a_{22}^{\mathcal{U}} + \exp\{\kappa_2\}\big)} - \frac{a_{24}^{\mathcal{U}}}{b_2^{\mathcal{U}}}\right].$$

Proof From Lemma 3.1, we know that

$$\limsup_{t\to+\infty} \omega_1(t) \leq \kappa_1,$$

which means that for any $\varepsilon > 0$, there exists a $t_0 \in \mathbb{T}^+$ such that $\omega_1(t) \leq \kappa_1 + \varepsilon$ for all $t \geq t_0$. Then for $t \geq t_0$, it follows from the first equation of system (2) that

$$\begin{aligned} \omega_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} \\ &+ \frac{c(t)\exp\{\omega_2(t)\}}{d(t) + \exp\{2\omega_2(t)\}} \\ \geq& \frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}} - a_{14}^{\mathcal{U}} - b_1^{\mathcal{U}}\exp\{\omega_1(t)\}. \end{aligned}$$

Now we claim that for $t \geq t_0$,

$$\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}} - a_{14}^{\mathcal{U}} - b_1^{\mathcal{U}}\exp\{\omega_1(t)\} \leq 0. \tag{7}$$

By way of contradiction, assume that there exists a $\hat{t} \geq t_0$ such that

$$\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}} - a_{14}^{\mathcal{U}} - b_1^{\mathcal{U}}\exp\{\omega_1(t)\} > 0$$

and for any $t \in [t_0, \hat{t})_{\mathbb{T}^+}$,

$$\frac{a_{11}^{\mathcal{L}}}{a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}} - a_{14}^{\mathcal{U}} - b_1^{\mathcal{U}}\exp\{\omega_1(t)\} \leq 0.$$

Then

$$\omega_1(\hat{t}) < \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\big)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right]$$

and for any $t \in [t_0, \hat{t})_{\mathbb{T}^+}$,

$$\omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\big)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right],$$

which implies $\omega_1^{\Delta}(\hat{t}) < 0$. It is contradiction, and hence the inequality in (7) holds for all $t \geq t_0$, and

$$\omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\big)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right],$$

consequently

$$\liminf_{t\to+\infty} \omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1 + \varepsilon\}\big)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right].$$

Since ε is arbitrary small and from the first inequality in (6), we have

$$\liminf_{t\to+\infty} \omega_1(t) \geq \ln\left[\frac{a_{11}^{\mathcal{L}}}{b_1^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1\}\big)} - \frac{a_{14}^{\mathcal{U}}}{b_1^{\mathcal{U}}}\right].$$

Analogously, by the second inequality in (6), we obtain that

$$\liminf_{t\to+\infty} \omega_2(t) \geq \ln\left[\frac{a_{21}^{\mathcal{L}}}{b_2^{\mathcal{U}}\big(a_{22}^{\mathcal{U}} + \exp\{\kappa_2\}\big)} - \frac{a_{24}^{\mathcal{U}}}{b_2^{\mathcal{U}}}\right].$$

This completes the proof. ■

Theorem 3.3 *Under the assumptions* (5) *and* (6)*, the system* (2) *is permanent.*

Proof From Lemmas 3.1 and 3.2, the system (2) is permanent. ■

4. Positive almost periodic solution

In this section, we establish sufficient conditions for the existence, uniqueness and uniform asymptotic stability of positive almost periodic solution of system (2). Define

$$\Lambda = \Big\{\big(\omega_1(t), \omega_2(t)\big) : \big(\omega_1(t), \omega_2(t)\big) \text{ is a solution of (2)} \\ \text{and } 0 < \ell_i \leq \omega_i(t) \leq \kappa_i,\ i = 1, 2\Big\}.$$

It is clear that Λ is invariant set of system (2).

Theorem 4.1 *Suppose that* (5) *and* (6) *are satisfied, then* $\Lambda \neq \varnothing$.

Proof The almost periodicity of $a_{ij}(t)$, $i = 1, 2, 3, 4; j = 1, 2$ implies that there is a sequence $\{\theta_k\} \subseteq \mathbb{T}^+$ with $\theta_k \to +\infty$ such that

$$a_{ij}(t + \theta_k) \to a_{ij}(t),\ \text{as } k \to +\infty,\ i = 1, 2, 3, 4; j = 1, 2.$$

From Lemma 3.1 and 3.2, for each sufficiently small $\epsilon > 0$, there exists a $\tau \in \mathbb{T}^+$ such that

$$\ell_i - \epsilon \leq \omega_i(t) \leq \kappa_i + \epsilon,\ for\ all\ t \geq \tau,\ i = 1, 2.$$

Set $\omega_{ik}(t) = \omega_i(t + \theta_k)$ for $t \geq \tau - \theta_k$, $k = 1, 2, \cdots$. For any positive integer m, there exists a sequence $\{\omega_{ik}(t) : k \geq m\}$ such that the sequence $\{\omega_{ik}(t)\}$ has a subsequence, denoted by $\{\omega^*_{ik}(t)\}(\omega^*_{ik}(t) = \omega_i(t + \theta^*_k))$, converging on any finite interval of $\mathbb{T}^+$ as $k \to +\infty$. So we have a sequence $\{w_i(t)\}$ such that for $t \in \mathbb{T}^+$,

$$\omega^*_{ik}(t) \to w_k(t),\ as\ k \to +\infty,\ i = 1, 2. \tag{8}$$

It is easy to see that the above sequence $\{\theta^*_k\} \subseteq \mathbb{T}^+$ with $\theta^*_k \to +\infty$ for $k \to +\infty$ such that

$$a_{ij}(t + \theta^*_k) \to a_{ij}(t),\ as\ k \to +\infty,\ i = 1, 2, 3, 4; j = 1, 2.$$

Which, together with (8) and

$$\begin{aligned}
\omega_1^{*\Delta}(t) =& \frac{a_{11}(t + \theta^*_k)}{a_{12}(t + \theta^*_k) + a_{13}(t + \theta^*_k)\exp\{\omega_1(t)\}} - a_{14}(t + \theta^*_k) - b_1(t + \theta^*_k)\exp\{\omega_1(t)\} \\
&+ \frac{c(t + \theta^*_k)\exp\{\omega_2(t)\}}{d(t + \theta^*_k) + \exp\{2\omega_2(t)\}}, \\
\omega_2^{*\Delta}(t) =& \frac{a_{21}(t + \theta^*_k)}{a_{22}(t + \theta^*_k) + a_{23}(t + \theta^*_k)\exp\{\omega_2(t)\}} - a_{24}(t + \theta^*_k) - b_2(t + \theta^*_k)\exp\{\omega_2(t)\},
\end{aligned}$$

yields

$$\begin{aligned}
w_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{w_1(t)\}} - a_{14}(t) - b_1(t)\exp\{w_1(t)\} \\
&+ \frac{c(t)\exp\{w_2(t)\}}{d(t) + \exp\{2w_2(t)\}}, \\
w_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{w_2(t)\}} - a_{24}(t) - b_2(t)\exp\{w_2(t)\},
\end{aligned}$$

It is clear that $\big(w_1(t), w_2(t)\big)$ is a solution of system (2) and

$$\ell_i - \epsilon \leq w_i(t) \leq \kappa_i + \epsilon,\ for\ t \in \mathbb{T}^+,\ i = 1, 2.$$

Since ϵ was arbitrary, it follows that

$$\ell_i \leq w_i(t) \leq \kappa_i,\ for\ t \in \mathbb{T}^+,\ i = 1, 2.$$

This completes the proof. ■

Theorem 4.2 *Assume that* (5), (6), $\Gamma_1 > 0$ *and* $\Gamma_2 > 0$, *where*

$$\Gamma_1 = \Big[\big(2b_1^{\mathcal{L}} e^{\ell_1} + 2b_1^{\mathcal{L}} \mathscr{A}_2 e^{\ell_1} + \mu^{\mathcal{L}} b_1^{\mathcal{L}} e^{\ell_1} \mathscr{B}_2\big) - \big(2\mathscr{A}_1 + \mu^{\mathcal{U}} \left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}} \mathscr{A}_1^2 + \mu^{\mathcal{U}} \mathscr{A}_1 \mathscr{B}_1 + \mathscr{B}_1\big)\Big],$$

$$\Gamma_2 = \Big[\big(\mu^{\mathcal{L}} b_1^{\mathcal{L}} e^{\ell_1} \mathscr{B}_2 + 2b_2^{\mathcal{L}} e^{\ell_2} (1 + \mu^{\mathcal{L}} \mathscr{C}_2)\big) - \big(\mu^{\mathcal{U}} \mathscr{B}_1^2 + \mu^{\mathcal{U}} \mathscr{A}_1 \mathscr{B}_1 + \mathscr{B}_1 + 2\mathscr{C}_1 + \mu^{\mathcal{U}} \mathscr{C}_1^2 + \mu^{\mathcal{U}} (b_2^{\mathcal{U}})^2 e^{2\kappa_2}\big)\Big],$$

are satisfied. Then the dynamic system (2) *has a unique almost periodic solution* $\big(\omega_1(t), \omega_2(t)\big) \in \Lambda$ *and is uniformly asymptotically stable.*

Proof From Theorem 4.1 that there exists a solution $\big(\omega_1(t), \omega_2(t)\big)$ of system (2) such that

$$\ell_i \le \omega_i(t) \le \kappa_i,$$

for $t \in \mathbb{T}^+$, $i = 1, 2$.
Define

$$\|\big(\omega_1(t), \omega_2(t)\big)\| = |\omega_1(t)| + |\omega_2(t)|, \quad \big(\omega_1(t), \omega_2(t)\big) \in \mathbb{R}_+^2.$$

Assume that $\mathcal{W}_1(t) = \big(\omega_1(t), \omega_2(t)\big)$, $\mathcal{W}_2(t) = \big(w_1(t), w_2(t)\big)$ are any two positive solutions of system (2), then

$$\|\mathcal{W}_1\| \le \kappa_1 + \kappa_2$$

and

$$\|\mathcal{W}_2\| \le \kappa_1 + \kappa_2.$$

We consider the associate product system of system (2) as follows

$$\left.\begin{aligned}
\omega_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{\omega_1(t)\}} - a_{14}(t) - b_1(t)\exp\{\omega_1(t)\} \\
&+ \frac{c(t)\exp\{\omega_2(t)\}}{d(t) + \exp\{2\omega_2(t)\}}, \\
\omega_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{\omega_2(t)\}} - a_{24}(t) - b_2(t)\exp\{\omega_2(t)\}, \\
w_1^{\Delta}(t) =& \frac{a_{11}(t)}{a_{12}(t) + a_{13}(t)\exp\{w_1(t)\}} - a_{14}(t) - b_1(t)\exp\{w_1(t)\} \\
&+ \frac{c(t)\exp\{w_2(t)\}}{d(t) + \exp\{2w_2(t)\}}, \\
w_2^{\Delta}(t) =& \frac{a_{21}(t)}{a_{22}(t) + a_{23}(t)\exp\{w_2(t)\}} - a_{24}(t) - b_2(t)\exp\{w_2(t)\}.
\end{aligned}\right\} \quad (9)$$

Construct the following Lyapunov function $\mathcal{V}\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)$ on $\mathbb{T}^+\times\Omega\times\Omega$ by

$$\mathcal{V}\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)=\big(\omega_1(t)-w_1(t)\big)^2+\big(\omega_2(t)-w_2(t)\big)^2.$$

It is obvious that the norm

$$\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|=|\omega_1(t)-w_1(t)|+|\omega_2(t)-w_2(t)|$$

is equivalent to

$$\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|_*=\Big[\big(\omega_1(t)-w_1(t)\big)^2+\big(\omega_2(t)-w_2(t)\big)^2\Big]^{\frac{1}{2}},$$

in other words, there exist two constants $\delta_1>0$, $\delta_2>0$ such that

$$\delta_1\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|\le\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|_*\le\delta_2\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|,$$

and hence we have

$$\big(\delta_1\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|\big)^2\le\mathcal{V}\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)\le\big(\delta_2\|\mathcal{W}_1(t)-\mathcal{W}_2(t)\|\big)^2.$$

Let $\alpha,\beta\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$, $\alpha(\omega)=\delta_1^2\omega^2$, $\beta(\omega)=\delta_2^2\omega^2$, then the assumption (i) of Lemma 2.7 is satisfied. On the other hand, we have

$$\begin{aligned}
&\Big|\mathcal{V}\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)-\mathcal{V}\big(t,\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)\big)\Big|\\
&=\Big|\big(\omega_1(t)-w_1(t)\big)^2+\big(\omega_2(t)-w_2(t)\big)^2-\big(\omega_1^*(t)-w_1^*(t)\big)^2-\big(\omega_2^*(t)-w_2^*(t)\big)^2\Big|\\
&\le\Big|\big(\omega_1(t)-w_1(t)\big)-\big(\omega_1^*(t)-w_1^*(t)\big)\Big|\Big|\big(\omega_1(t)-w_1(t)\big)+\big(\omega_1^*(t)-w_1^*(t)\big)\Big|\\
&\quad\Big|\big(\omega_2(t)-w_2(t)\big)-\big(\omega_2^*(t)-w_2^*(t)\big)\Big|\Big|\big(\omega_2(t)-w_2(t)\big)+\big(\omega_2^*(t)-w_2^*(t)\big)\Big|\\
&\le\Big|\big(\omega_1(t)-w_1(t)\big)-\big(\omega_1^*(t)-w_1^*(t)\big)\Big|\Big(|\omega_1(t)|+|w_1(t)|+|\omega_1^*(t)|+|w_1^*(t)|\Big)\\
&\quad\Big|\big(\omega_2(t)-w_2(t)\big)-\big(\omega_2^*(t)-w_2^*(t)\big)\Big|\Big(|\omega_2(t)|+|w_2(t)|+|\omega_2^*(t)|+|w_2^*(t)|\Big)\\
&\le\mathcal{L}\Big(|\omega_1(t)-\omega_1^*(t)|+|\omega_2(t)-\omega_2^*(t)|+|w_1(t)-w_1^*(t)|+|w_2(t)-w_2^*(t)|\Big)\\
&=\mathcal{L}\Big(\|\mathcal{W}_1(t)-\mathcal{W}_1^*(t)\|+\|\mathcal{W}_2(t)-\mathcal{W}_2^*(t)\|\Big),
\end{aligned}$$

where $\mathcal{W}_1^*(t)=\big(\omega_1^*,\omega_2^*\big),\mathcal{W}_2^*(t)=\big(w_1^*,w_2^*\big)$, and $\mathcal{L}=4\max\{\kappa_i,\ i=1,2\}$. Hence, the assumption (ii) of Lemma 2.7 is satisfied.

Now, estimating the right derivative $\mathcal{D}^+\mathcal{V}^{\Delta}$ of $\mathcal{V}$ along with associate product system (9), we obtain

$$\begin{aligned}
&\mathcal{D}^+\mathcal{V}^\Delta\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)\\
&\quad= \big(\omega_1(t)-w_1(t)\big)^\Delta\big(\omega_1(t)-w_1(t)\big)+[\omega_1(\sigma(t))-w_1(\sigma(t))]\big(\omega_1(t)-w_1(t)\big)\\
&\quad+\big(\omega_2(t)-w_2(t)\big)^\Delta\big(\omega_2(t)-w_2(t)\big)+[\omega_2(\sigma(t))-w_2(\sigma(t))]\big(\omega_2(t)-w_2(t)\big)\\
&\quad= \big(\omega_1(t)-w_1(t)\big)^\Delta\big(\omega_1(t)-w_1(t)\big)+\big[\big(\mu(t)\omega_1^\Delta(t)+\omega_1(t)\big)\\
&\qquad\qquad-\big(\mu(t)w_1^\Delta(t)+w_1(t)\big)\big]\big(\omega_1(t)-w_1(t)\big)^\Delta\\
&\qquad+\big(\omega_2(t)-w_2(t)\big)^\Delta\big(\omega_2(t)-w_2(t)\big)+\big[\big(\mu(t)\omega_2^\Delta(t)+\omega_2(t)\big)\\
&\qquad\qquad-\big(\mu(t)w_2^\Delta(t)+w_2(t)\big)\big]\big(\omega_2(t)-w_2(t)\big)^\Delta\\
&\quad=\Big[2\big(\omega_1(t)-w_1(t)\big)+\mu(t)\big(\omega_1(t)-w_1(t)\big)^\Delta\Big]\big(\omega_1(t)-w_1(t)\big)^\Delta\\
&\qquad+\Big[2\big(\omega_2(t)-w_2(t)\big)+\mu(t)\big(\omega_2(t)-w_2(t)\big)^\Delta\Big]\big(\omega_2(t)-w_2(t)\big)^\Delta.
\end{aligned}$$

So,

$$\mathcal{D}^+\mathcal{V}^\Delta\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)=\mathcal{V}_1+\mathcal{V}_2, \tag{10}$$

where

$$\begin{aligned}
\mathcal{V}_1&=\Big[2\big(\omega_1(t)-w_1(t)\big)+\mu(t)\big(\omega_1(t)-w_1(t)\big)^\Delta\Big]\big(\omega_1(t)-w_1(t)\big)^\Delta,\\
\mathcal{V}_2&=\Big[2\big(\omega_2(t)-w_2(t)\big)+\mu(t)\big(\omega_2(t)-w_2(t)\big)^\Delta\Big]\big(\omega_2(t)-w_2(t)\big)^\Delta.
\end{aligned}$$

From the system (9), we have

$$\begin{aligned}
&\big(\omega_1(t)-w_1(t)\big)^\Delta=a_{11}(t)\left[\frac{1}{a_{12}(t)+a_{13}(t)\exp\{\omega_1(t)\}}-\frac{1}{a_{12}(t)+a_{13}(t)\exp\{w_1(t)\}}\right]\\
&-b_1(t)[\exp\{\omega_1(t)\}-\exp\{w_1(t)\}]+c(t)\left[\frac{\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}}-\frac{\exp\{w_2(t)\}}{d(t)+\exp\{2w_2(t)\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\big(\omega_2(t)-w_2(t)\big)^\Delta&=a_{21}(t)\left[\frac{1}{a_{22}(t)+a_{23}(t)\exp\{\omega_2(t)\}}-\frac{1}{a_{22}(t)+a_{23}(t)\exp\{w_2(t)\}}\right]\\
&\quad-b_2(t)[\exp\{\omega_2(t)\}-\exp\{w_2(t)\}].
\end{aligned}$$

By mean value theorem, there exit $\xi_i(t),\eta_i(t),\ i=1,2$ lie between $\omega_i(t)$ and $w_i(t)$, and $\xi(t)$ lie between $\omega_2(t)$ and $w_2(t)$ such that

$$\exp\{\omega_i(t)\}-\exp\{w_i(t)\}=\exp\{\xi_i(t)\}[\omega_i(t)-w_i(t)],$$

$$\frac{\exp\{\omega_2(t)\}}{d(t)+\exp\{2\omega_2(t)\}}-\frac{\exp\{w_2(t)\}}{d(t)+\exp\{2w_2(t)\}}=\left[\frac{d-\exp\{3\xi(t)\}}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)],$$

$$\frac{1}{a_{i2}(t)+a_{i3}(t)\exp\{\omega_i(t)\}}-\frac{1}{a_{i2}(t)+a_{i3}(t)\exp\{w_i(t)\}} = \left[\frac{a_{i3}(t)\exp\{\eta_i(t)\}}{(a_{i2}(t)+a_{i3}(t)\exp\{\eta_i(t)\})^2}\right][\omega_i(t)-w_i(t)].$$

Therefore,

$$\big(\omega_1(t)-w_1(t)\big)^{\Delta} = \left[\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right][\omega_1(t)-w_1(t)] - b_1(t)\exp\{\xi_1(t)\}[\omega_1(t)-w_1(t)] + \left[\frac{c(t)(d-\exp\{3\xi(t)\})}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)],$$

and

$$\big(\omega_2(t)-w_2(t)\big)^{\Delta} = \left[\frac{a_{21}(t)a_{23}(t)\exp\{\eta_2(t)\}}{(a_{22}(t)+a_{23}(t)\exp\{\eta_2(t)\})^2}\right][\omega_2(t)-w_2(t)] - b_2(t)\exp\{\xi_2(t)\}[\omega_2(t)-w_2(t)].$$

Now from (10), we have

$$\begin{aligned}
\mathcal{V}_1 =&\Bigg[2\big(\omega_1(t)-w_1(t)\big)+\mu(t)\Bigg(\left[\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right][\omega_1(t)-w_1(t)]\\
&- b_1(t)\exp\{\xi_1(t)\}[\omega_1(t)-w_1(t)] + \left[\frac{c(t)(d-\exp\{3\xi(t)\})}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)]\Bigg)\Bigg]\\
&\times\Bigg[\left[\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right][\omega_1(t)-w_1(t)]\\
&- b_1(t)\exp\{\xi_1(t)\}[\omega_1(t)-w_1(t)] + \left[\frac{c(t)(d-\exp\{3\xi(t)\})}{(d+\exp\{2\xi(t)\})^2}\right][\omega_2(t)-w_2(t)]\Bigg]\\
=&\Bigg[2\left(\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}-b_1(t)\exp\{\xi_1(t)\}\right)\\
&+\mu(t)(b_1(t))^2\exp\{2\xi_1(t)\}+\mu(t)\left(\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right)^2\\
&-\frac{2b_1(t)a_{11}(t)a_{13}(t)\exp\{\xi_1(t)+\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\Bigg][\omega_1(t)-w_1(t)]^2\\
&+\mu(t)\left[\frac{c(t)(d(t)-\exp\{3\xi(t)\})}{(d(t)+\exp\{2\xi(t)\})^2}\right]^2[\omega_2(t)-w_2(t)]^2\\
&+2\Bigg[\mu(t)\left(\frac{a_{11}(t)a_{13}(t)\exp\{\eta_1(t)\}}{(a_{12}(t)+a_{13}(t)\exp\{\eta_1(t)\})^2}\right)\left(\frac{c(t)(d(t)-\exp\{3\xi(t)\})}{(d(t)+\exp\{2\xi(t)\})^2}\right)\\
&+\big(1-\mu(t)b_1(t)\exp\{\xi_1(t)\}\big)\left(\frac{c(t)(d(t)-\exp\{3\xi(t)\})}{(d(t)+\exp\{2\xi(t)\})^2}\right)\Bigg]\\
&\times[\omega_1(t)-w_1(t)][\omega_2(t)-w_2(t)]
\end{aligned}$$

$$\begin{aligned}&\leq \Big[2\mathscr{A}_1 - 2b_1^{\mathcal{L}}e^{\ell_1} + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 - 2b_1^{\mathcal{L}}\mathscr{A}_2 e^{\ell_1}\Big][\omega_1(t) - w_1(t)]^2\\ &\quad + \mu^{\mathcal{U}}\mathscr{B}_1^2[\omega_2(t) - w_2(t)]^2\\ &\quad + 2[\mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1 - \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2][\omega_1(t) - w_1(t)][\omega_2(t) - w_2(t)]\end{aligned}$$

Since $2ab \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$, it follows that

$$\left.\begin{aligned}\mathcal{V}_1 \leq &-\bigg[\left(2b_1^{\mathcal{L}}e^{\ell_1} + 2b_1^{\mathcal{L}}\mathscr{A}_2 e^{\ell_1} + \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2\right)\\ &- \left(2\mathscr{A}_1 + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\right)\bigg][\omega_1(t) - w_1(t)]^2\\ &+ \Big[\mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2 - \left(\mu^{\mathcal{U}}\mathscr{B}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\right)\Big][\omega_2(t) - w_2(t)]^2.\end{aligned}\right\} \quad (11)$$

Similarly, we can find

$$\mathcal{V}_2 \leq -\Big[2b_2^{\mathcal{L}}e^{\ell_2}\left(1 + \mu^{\mathcal{L}}\mathscr{C}_2\right) - \left(2\mathscr{C}_1 + \mu^{\mathcal{U}}\mathscr{C}_1^2 + \mu^{\mathcal{U}}(b_2^{\mathcal{U}})^2 e^{2\kappa_2}\right)\Big][\omega_2(t) - w_2(t)]^2. \quad (12)$$

From (10), (11) and (12), we get

$$\begin{aligned}&D^+\mathcal{V}^{\Delta}\left(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\right) = \mathcal{V}_1 + \mathcal{V}_2\\ &= -\bigg[\left(2b_1^{\mathcal{L}}e^{\ell_1} + 2b_1^{\mathcal{L}}\mathscr{A}_2 e^{\ell_1} + \mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2\right)\\ &\qquad - \left(2\mathscr{A}_1 + \mu^{\mathcal{U}}\left(b_1^{\mathcal{U}}\right)^2 e^{2\kappa_1} + \mu^{\mathcal{U}}\mathscr{A}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1\right)\bigg][\omega_1(t) - w_1(t)]^2\\ &\quad - \bigg[\left(\mu^{\mathcal{L}}b_1^{\mathcal{L}}e^{\ell_1}\mathscr{B}_2 + 2b_2^{\mathcal{L}}e^{\ell_2}\left(1 + \mu^{\mathcal{L}}\mathscr{C}_2\right)\right)\\ &\quad - \left(\mu^{\mathcal{U}}\mathscr{B}_1^2 + \mu^{\mathcal{U}}\mathscr{A}_1\mathscr{B}_1 + \mathscr{B}_1 + 2\mathscr{C}_1 + \mu^{\mathcal{U}}\mathscr{C}_1^2 + \mu^{\mathcal{U}}(b_2^{\mathcal{U}})^2 e^{2\kappa_2}\right)\bigg][\omega_2(t) - w_2(t)]^2\\ &= -\Gamma_1[\omega_1(t) - w_1(t)]^2 - \Gamma_2[\omega_2(t) - w_2(t)]^2\\ &\leq -\lambda\mathcal{V}(t, \mathcal{W}_1(t), \mathcal{W}_2(t)).\end{aligned}$$

where $\lambda = \min\{\Gamma_i : i = 1, 2\} > 0$ and $-\lambda \in \mathcal{R}^+$. Thus, the assumption (iii) of Lemma 2.7 is satisfied and hence, it follows from Lemma 2.7 that there exists a unique uniformly asymptotically stable almost periodic solution $\big(\omega_1(t), \omega_2(t)\big)$ of dynamic system (2) and $\big(\omega_1(t), \omega_2(t)\big) \in \Lambda$. This completes the proof. ■

5. Numerical simulations

In this section we present an example to check the validity of our main results.

Example 5.1 Consider the following system for $\mathbb{T}^+ = \mathbb{R}^+$.

$$\left.\begin{aligned} u_1'(t) &= u_1(t)\left[\frac{a_{11}(t)}{a_{12}(t)+a_{13}(t)u_1(t)} - a_{14}(t) - b_1(t)u_1(t) + \frac{c(t)u_2(t)}{d(t)+u_2^2(t)}\right], \\ u_2'(t) &= u_2(t)\left[\frac{a_{21}(t)}{a_{22}(t)+a_{23}(t)u_2(t)} - a_{24}(t) - b_2(t)u_2(t)\right], \end{aligned}\right\} \tag{13}$$

where

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{bmatrix} = \begin{bmatrix} 50+0.1\sin(\sqrt{3}t) & 48+0.1\sin(\sqrt{5}t) \\ 15+0.2\sin(\sqrt{2}t) & 28+0.1\sin(\sqrt{3}t) \\ 0.2+0.1\sin(\sqrt{5}t) & 120+0.2\sin(\sqrt{2}t) \\ 0.03+0.01\sin(\sqrt{2}t) & 0.002+0.01\sin(\sqrt{3}t) \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1.4+0.1\cos(\sqrt{2}t) \\ 1.4-0.1\sin(\sqrt{5}t) \end{bmatrix}, \quad \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0.4+0.1\sin(\sqrt{2}t) \\ 3.2+0.1\sin(\sqrt{3}t) \end{bmatrix}.$$

By calculating, we get

$$57.5 = a_{11}^{\mathcal{U}} + c^{\mathcal{U}} a_{12}^{\mathcal{L}} > 19.536 = \left[a_{14}^{\mathcal{L}} + b_1^{\mathcal{L}}\right] a_{12}^{\mathcal{L}},$$
$$48.1 = a_{21}^{\mathcal{U}} > 36.0468 = \left[a_{24}^{\mathcal{L}} + b_2^{\mathcal{L}}\right] a_{22}^{\mathcal{L}},$$

which shows that (5) holds and $\kappa_1 = 1.973180873, \kappa_2 = 0.3323187208$. Now we check (6),

$$49.9 = a_{11}^{\mathcal{L}} > 0.8957408724 = a_{14}^{\mathcal{U}}\big(a_{12}^{\mathcal{U}} + \exp\{\kappa_1\}\big),$$
$$47.9 = a_{21}^{\mathcal{L}} > 0.3539303657 = a_{24}^{\mathcal{U}}\big(a_{22}^{\mathcal{U}} + \exp\{\kappa_2\}\big).$$

So, $\ell_1 = 0.3776703951, \ell_2 = 0.07204048280$. From these values we obtain,

$$\mathscr{A}_1 = 0.4840130676,\ \mathscr{A}_2 = 0.02416120093,\ \mathscr{B}_1 = 0.05685627445,$$
$$\mathscr{B}_2 = 0.04254742499,\ \mathscr{C}_1 = 0.3284875457,\ \mathscr{C}_2 = 0.1610553506.$$

By above values (note that for $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$), we get

$$\Gamma_1 = 2.859856503, \quad \Gamma_2 = 2.080385645.$$

$\lambda = \min\{\Gamma_i : i = 1,2\} > 0$ and $-\lambda \in \mathcal{R}^+$. From Fig. 1-3, it is easy to see that for system (13) there exists a positive almost periodic solution denoted by $\big(\omega_1^*(t), \omega_2^*(t)\big)$. Moreover, Fig. 4-5 shows that any positive solution $\big(\omega_1(t), \omega_2(t)\big)$ tends to the above almost periodic solution $\big(\omega_1^*(t), \omega_2^*(t)\big)$.

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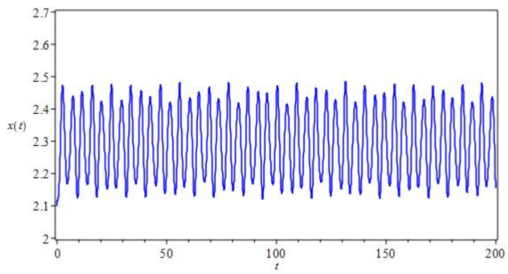

Figure 1. Positive almost periodic solution of system (13). Time series of $u_1^*(t)$ with initial value $u_1^*(0) = 2.1$ and t over $[0, 300]$.

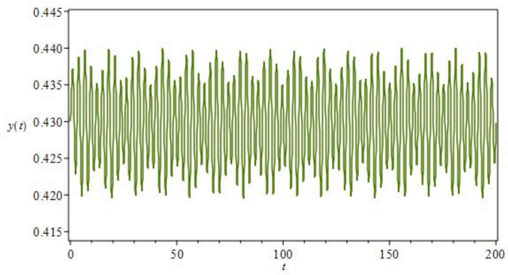

Figure 2. Positive almost periodic solution of system (13). Time series of $x_2^*(t)$ with initial value $x_1^*(0) = 0.43$ and t over $[0, 300]$.

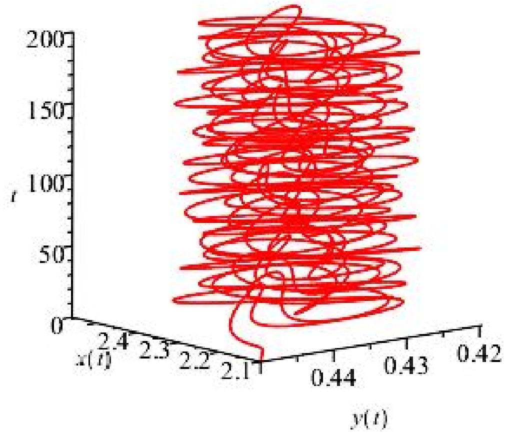

Figure 3. Positive almost periodic solution of system (13). 3-dimensional phase portrait of $u_1^*(t)$ and $u_2^*(t)$ with initial values $(2.1, 0.45)$ for $t \in [0, 200]$.

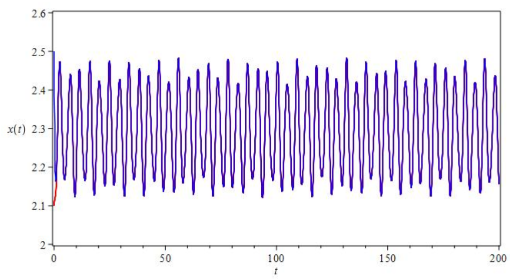

Figure 4. Uniformly asymptotic stability of system (13). Time series of $u_1(t)$ and $u_1^*(t)$ with initial values $u_1(0) = 2.1$, $u_1^*(0) = 2.5$ and t over $[0, 200]$.

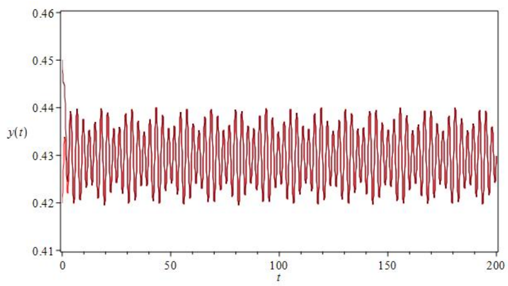

Figure 5. Uniformly asymptotic stability of system (13). Time series of $u_2(t)$ and $u_2^*(t)$ with initial values $u_2(0) = 0.42$, $u_2^*(0) = 0.45$ and t over $[0, 200]$.

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