Anumerical method based on RKHS for hyperbolic initial-boundary-value problem

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Abstract: The main aim of this article is to propose a computational method on the basis of the reproducing kernel Hilbert space method for solving a hyperbolic initial-boundary-value problem. The solution in reproducing kernel Hilbert space is constructed with series form, and the approximate solution v_m is given as an m-term summation. Furthermore, convergence of the proposed method is presented which provides the theoretical basis of the proposed method. Finally, some numerical experiments are considered to demonstrate the efficiency and applicability of proposed method.

Keywords: Hyperbolic differential-integral equations; Initial and boundary conditions; Approximate solution; Convergence analysis; Reproducing kernel Hilbert space.

AMS Subject Classification: 35L70; 35L10; 35L20.

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1. Introduction

In this paper, we consider a class of hyperbolic differential equations in the following form:

$$\varepsilon\frac{\partial^2 u}{\partial \overline{x}^2} = \frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial \overline{x}} + \beta(u^3 + \gamma_0 u^2 + \gamma u),\ (\overline{x}, t) \in (a, b) \times (0, T), \tag{1}$$

where $u = u(\overline{x}, t)$ is a sufficiently differentiable function, α is a real parameter, $\varepsilon > 0$ is a negligible positive parameter and $\beta \geqslant 0$, $\gamma \in (0, 1)$, $\gamma_0 = -(1 + \gamma)$. The initial and boundary conditions of these equations is given by:

$$u(\overline{x}, 0) = \tilde{u}_0(\overline{x}),\ a \leqslant \overline{x} \leqslant b, \tag{2}$$

$$u(a, t) = f_1(t),\ 0 \leqslant t \leqslant T, \tag{3}$$

and

$$u(b, t) = f_2(t),\ 0 \leqslant t \leqslant T. \tag{4}$$

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By changing of variable $x = \frac{\overline{x}-a}{b-a}$, we can rewritten the question's (1)-(4) in the following form:

$$\begin{cases} \frac{\varepsilon}{(b-a)^2}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \frac{\alpha}{b-a}u\frac{\partial u}{\partial x} + \beta(u^3+\gamma_0 u^2+\gamma u),\ (x,t)\in(0,1)\times(0,T), \\ u(x,0) = u_0(x), \\ u(0,t) = f_1(t),\ t\in[0,T], \\ u(1,t) = f_2(t)\ t\in[0,T]. \end{cases} \tag{5}$$

where $u_0(x) = \tilde{u}_0(a+(b-a)x)$. Using the following transformation:

$$\begin{cases} v(x,t) = u(x,t) - U(x,t) - u_0(x) + U_0(x), \\ U(x,t) = f_1(t)(1-x) + f_2(t)x,\ U_0(x) = U(x,0), \end{cases}$$

the equation (5) can be rewritten in the following form:

$$\begin{cases} \frac{\varepsilon}{(b-a)^2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - \frac{\alpha}{b-a}v\frac{\partial(U+u_0-U_0)}{\partial x} - \frac{\alpha}{b-a}(U+u_0-U_0)\frac{\partial v}{\partial x} - 3\beta(U+u_0-U_0)v \\ -2\beta\gamma_0(U+u_0-U_0)v - \beta\gamma v = F(x,t,v,\frac{\partial v}{\partial x}),\ (x,t)\in(0,1)\times(0,T), \\ v(x,0) = 0, \\ v(0,t) = 0,\ t\in[0,T], \\ v(1,t) = 0\ t\in[0,T], \end{cases} \tag{6}$$

where

$$\begin{aligned} F(x,t,v,\frac{\partial v}{\partial x}) &= \frac{\varepsilon}{(b-a)^2}\frac{\partial^2(U+u_0-U_0)}{\partial x^2} + \frac{\partial U}{\partial t} + \frac{\alpha}{b-a}(U+u_0-U_0)\frac{\partial(U+u_0-U_0)}{\partial x} \\ &\quad + \beta((U+u_0-U_0)^3 + \beta\gamma_0(U+u_0-U_0)^2 + \beta\gamma(U+u_0-U_0) \\ &\quad + \frac{\alpha}{b-a}v\frac{\partial v}{\partial x} + \beta v^3 + 3\beta(U+u_0-U_0)v^2 + \beta\gamma_0 v^2. \end{aligned}$$

The concept of reproducing kernel was first applied by Zaremba [1] to obtain the approximate solution of boundary value problems for harmonic functions. In 1909, Mercer [2] examined the functions which satisfy reproducing property in the theory of integral equations. He called these functions as positive definite kernels. The concept of reproducing kernels was systematized by Aronszajn [3] around 1948. From 1980, Cui and co-workers [4, 5] are pioneers in linear and nonlinear numerical analysis based on reproducing kernel theory. Recently, a lot of research works have been devoted to the application of reproducing kernel Hilbert space method to solve several linear and nonlinear problems such as variational problems depending on indefinite integrals [6], delay differential equations of fractional order [7], nonlocal initial-boundary value problems for hyperbolic and parabolic integro-differential equations [8], BlackScholes equation [9] and so on [10–14].
In this paper, we define the reproducing kernel space $\mathcal{H}_2^{(3,2)}(\Omega)$. In the following, by using two methods, we investigate the existence of the solution in the reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$.
The structure of this paper is as follows. In Section 2, we give our main results concerning to our numerical method. We present an analysis of the numerical method in this section. In Section 3, validations using typical cases with available numerical results in the literature are performed to demonstrate the accuracy and efficiency of the proposed method. Finally, some concluding remarks are presented.

2. Reproducing kernel Hillbert space

Definition 2.1 *[15] The reproducing kernel Hilbert space $\mathcal{H}_2^1[0,T]$ is defined by*

$$\mathcal{H}_2^1[0,T]=\left\{u:[0,T]\to\mathbf{R}|u\in\mathcal{AC}[0,T],\ u^{'}\in L^2[0,T]\right\}.$$

Also, the specific inner product in $\mathcal{H}_2^1[0,T]$ is of the following form

$$\langle u,v\rangle_{\ \mathcal{H}_2^1}=u(0)v(0)+\int_0^T u'(t)v'(t)dt,$$

and the norm is given by:

$$\|u\|_{\ \mathcal{H}_2^1}=\sqrt{\langle u,u\rangle_{\ \mathcal{H}_2^1}},$$

where $u,v\in\ \mathcal{H}_2^1$.

The Hilbert space $\mathcal{H}_2^1[0,T]$ admits the following reproducing kernel:

$$k_s^1(t)=\begin{cases}1+s, & s\leqslant t,\\ 1+t, & s>t.\end{cases}$$

Definition 2.2 *[16] The reproducing kernel Hilbert space ${}_c\mathcal{H}_2^3[0,1]$ is defined by*

$${}_c\mathcal{H}_2^3[0,1]=\left\{v:[0,1]\to\mathbf{R}|v,v',v''\in\mathcal{AC}[0,1],v(0)=v(1)=0\ v'''\in L^2[0,1]\right\}.$$

Also, the inner product in ${}_c\mathcal{H}_2^3[0,1]$ is of the following form

$$\langle v,u\rangle_{\ {}_c\mathcal{H}_2^3[0,1]}=v(0)u(0)+v'(0)u'(0)+v(1)u(1)+\int_0^1 v'''(x)u'''(x)dx,$$

and the norm is given by:

$$\|v\|_{\ {}_c\mathcal{H}_2^3[0,1]}=\sqrt{\langle v,v\rangle_{\ {}_c\mathcal{H}_2^3[0,1]}},$$

where $v,u\in\ {}_c\mathcal{H}_2^3$.

The Hilbert space ${}_c\mathcal{H}_2^3[0,1]$ admits the following reproducing kernel:

$$R_y^3(x)=\begin{cases}\frac{-1}{120}(x-1)y(yx^4-4yx^3+6yx^2\\ +(y^4-5y^3-120y+120)x+y^4), & x\geqslant y,\\ \frac{-1}{120}(y-1)x(xy^4-4xy^3+6xy^2\\ +(x^4-5x^3-120x+120)y+x^4), & y>x.\end{cases}$$

Definition 2.3 *[17] The reproducing kernel Hilbert space ${}_c\mathcal{H}_2^2[0,T]$ is defined by*

$${}_c\mathcal{H}_2^2[0,T]=\Big\{v:[0,T]\to\mathbf{R}|v,v^{'}\in\mathcal{AC}[0,T]\ ,\\ v(0)=0,\ v^{''}\in L^2[0,T]\Big\}.$$

Also, the specific inner product in ${}_c\mathcal{H}_2^2[0,T]$ *is of the following form*

$$\langle v, u\rangle_{\ {}_c\mathcal{H}_2^2[0,T]} = v(0)u(0) + v^{'}(0)u^{'}(0) + \int_0^T v^{''}(t)u^{''}(t)dt,$$

and the norm is given by:

$$\|v\|_{\ {}_c\mathcal{H}_2^2[0,T]} = \sqrt{\langle v, v\rangle_{\ {}_c\mathcal{H}_2^2[0,T]}}, \tag{7}$$

where $v, u \in {}_c\mathcal{H}_2^2$.

The Hilbert space ${}_c\mathcal{H}_2^2[0,T]$ admits the following reproducing kernel:

$$Q_s^2(t) = \begin{cases} ts + \frac{ts^2}{2} - \frac{s^3}{6}, \ t \geqslant s, \\ ts + \frac{st^2}{2} - \frac{t^3}{6}, \ s > t. \end{cases}$$

Definition 2.4 *[8] Let* $\Omega = [0,1]\times[0,T] \subseteq \mathbf{R}^2$*. The reproducing kernel Hilbert space* $\mathcal{H}_2^{(1,1)}(\Omega)$ *is defined by*

$$\mathcal{H}_2^{(1,1)}(\Omega) = \left\{u | u \in \mathcal{C}(\Omega),\ \partial_x\partial_t u \in L^2(\Omega)\right\},$$

Also, the specific inner product in $\mathcal{H}_2^{(1,1)}(\Omega)$ *is of the following form*

$$\begin{aligned} \langle u, v\rangle_{\mathcal{H}_2^{(1,1)}} &= u(0,0)v(0,0) + \int_0^1 \partial_x u(x,0)\partial_x v(x,0)dx \\ &+ \int_0^T \partial_t u(0,t)\partial_t v(0,t)dt + \int_0^T\int_0^1 \partial_x\partial_t u(x,t)\partial_x\partial_t v(x,t)dxdt, \end{aligned}$$

and the norm is given by:

$$\|u\|_{\mathcal{H}_2^{(1,1)}} = (\langle u, u\rangle_{\mathcal{H}_2^{(1,1)}})^{\frac{1}{2}},$$

where $u, v \in \mathcal{H}_2^{(1,1)}(\Omega)$.

Theorem 2.5 *[8] Let* $\Omega = [0,1]\times[0,T] \subseteq \mathbf{R}^2$*. Then the Hilbert space* $\mathcal{H}_2^{(1,1)}(\Omega)$ *admits the following reproducing kernel*

$$K_{(y,s)}^{1,1}(x,t) = k_y^1(x)k_s^1(t).$$

Definition 2.6 *[18] Let* $\Omega = [0,1]\times[0,T] \subseteq \mathbf{R}^2$*. The reproducing kernel Hilbert space* $\mathcal{H}_2^{(3,2)}(\Omega)$ *is defined by*

$$\begin{aligned} &\mathcal{H}_2^{(3,2)}(\Omega) = \{v | \partial_x^2\partial_t v(x,t) \in \mathcal{C}(\Omega),\ \partial_x^3\partial_t^2 v(x,t) \in L^2(\Omega), \\ &v(x,0) = v(0,t) = v(1,t) = 0\}. \end{aligned}$$

Also, the specific inner product in $\mathcal{H}_2^{(3,2)}(\Omega)$ is of the following form

$$\langle v,u\rangle_{\ {}^{c}\mathcal{H}_{(3,2)}(\Omega)} = \partial_x\partial_t v(0,0)\partial_x\partial_t u(0,0) + \int_0^1 \partial_x^3\partial_t v(x,0)\partial_x^3\partial_t u(x,0)dx$$
$$+\int_0^T \partial_x\partial_t^2 v(0,t)\partial_x\partial_t^2 u(0,t)dt + \int_0^T\int_0^1 \partial_x^3\partial_t^2 v(x,t)\partial_x^3\partial_t^2 u(x,t)dxdt,$$

and the norm is given by:

$$\|u\|_{\mathcal{H}_2^{(3,2)}} = (\langle u,u\rangle_{\mathcal{H}_2^{(3,2)}})^{\frac{1}{2}},$$

where $v,u\in\ \mathcal{H}_2^{(3,2)}(\Omega)$.

THEOREM 2.7 *[19] Let $\Omega=[0,1]\times[0,T]\subseteq\mathbf{R}^2$. Then the Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$ admits the following reproducing kernel*

$$K^{3,2}_{(y,s)}(x,t) = R^3_y(x)Q^2_s(t).$$

2.1 *Solution in the reproducing kernel Hilbert space*

2.1.1 First method to calculate the approximate solution

Suppose that the solution of the problem (6) belongs to reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$. Whenever a nonlinear operator $F(x,t,v,\frac{\partial v}{\partial x})$ belongs to reproducing kernel Hilbert space $\mathcal{H}_2^{(1,1)}(\Omega)$, then linear operator $\mathcal{L}:\mathcal{H}_2^{(3,2)}(\Omega)\to\mathcal{H}_2^{(1,1)}(\Omega)$ well-defined and as follows:

$$\mathcal{L}v(x,t) = \frac{\varepsilon}{(b-a)^2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - \frac{\alpha}{b-a}v\frac{\partial(U+u_0-U_0)}{\partial x} - \frac{\alpha}{b-a}(U+u_0-U_0)\frac{\partial v}{\partial x}$$
$$-3\beta(U+u_0-U_0)v - 2\beta\gamma_0(U+u_0-U_0)v - \beta\gamma v.$$

Hence, we can rewritten the problem (6) as follows:

$$\mathcal{L}v(x,t) = F(x,t,v,\frac{\partial v}{\partial x}), \tag{8}$$
$$v(x,0)=v(0,t)=v(1,t)=0.$$

THEOREM 2.8 *Let $\mathcal{L}:\mathcal{H}_2^{(3,2)}(\Omega)\to\mathcal{H}_2^{(1,1)}(\Omega)$. Then $\mathcal{L}$ is bounded linear operator.*

Proof It is sufficient to show that $\|\mathcal{L}v\|_{\mathcal{H}_2^{(1,1)}} \leqslant \|v\|_{\mathcal{H}_2^{(3,2)}}$. By using the norm of Hilbert space $\mathcal{H}_2^{(1,1)}$, we have

$$\|\mathcal{L}v\|_{\mathcal{H}_2^{(1,1)}} = \langle\mathcal{L}v,v\rangle_{\mathcal{H}_2^{(1,1)}} = \mathcal{L}v^2(0,0) + \int_0^1(\partial_x\mathcal{L}v(\varsigma,0))^2dx$$
$$+\int_0^T(\partial_t Lv(0,t))^2dt + \int_0^T\int_0^1(\partial_x\partial_t\mathcal{L}v(x,t))^2dxdt.$$

By applying the reproduction property, we obtain

$$\begin{aligned}
\langle v(.,.), \mathcal{L}K^{3,}_{(y,s)}(.,.)\rangle_{\mathcal{H}_2^{(3,2)}} &= \mathcal{L}v(y,s),\\
\langle v(.,.), \partial_y \mathcal{L}K^{3,2}_{(y,s)}(.,.)\rangle_{\mathcal{H}_2^{(3,2)}} &= \partial_y \mathcal{L}v(y,s),\\
\langle v(.,.), \partial_s \mathcal{L}K^{3,2}_{(y,s)}(.,.)\rangle_{\mathcal{H}_2^{(3,2)}} &= \partial_s \mathcal{L}v(y,s),\\
\langle v(.,.), \partial_\eta \partial_s \mathcal{L}K^{3,2}_{(y,s)}(.,.)\rangle_{\mathcal{H}_2^{(3,2)}} &= \partial_y \partial_s \mathcal{L}v(y,s),
\end{aligned}$$

We remind that

$$|\partial^{i+j}_{ys} v(y,s)| \leqslant M_{ij} \|v\|_{\mathcal{H}_2^{(3,2)}}, i=0,1,2, j=0,1, \tag{9}$$

where M_{ij} are positive real number.
Therefore, we get

$$|\mathcal{L}v(0,0)|^2 \leqslant c_1 \|v\|_{\mathcal{H}_2^{(3,2)}}, \tag{10}$$

$$\int_0^1 (\partial_y \mathcal{L}v(y,0))^2 dy \leqslant c_2 \|v\|_{\mathcal{H}_2^{(3,2)}}, \tag{11}$$

$$\int_0^T (\partial_s \mathcal{L}v(0,s))^2 ds \leqslant c_3 \|v\|_{\mathcal{H}_2^{(3,2)}}, \tag{12}$$

$$\int_0^T \int_0^1 (\partial_y \partial_s \mathcal{L}v(y,s))^2 dyds \leqslant c_4 \|v\|_{\mathcal{H}_2^{(3,2)}}, \tag{13}$$

where c_i, $i = 1,2,3,4$ are positive real number. By combining the inequalities (10)-(13), we obtain the inequality (9). Hence, the proof of the Theorem 2.8 is completed. ∎

Since $\mathcal{L}$ is a bounded linear operator, then we can define uniquely the adjoint operator $\mathcal{L}^* : \mathcal{H}_2^{(1,1)}(\Omega) \longrightarrow \mathcal{H}_2^{(3,2)}(\Omega)$. Suppose that $\{(x_i,t_i)\}_{i=1}^{\infty}$ be a subset countable dense in the domain Ω. By using the adjoint operator L^*, the functions $\vartheta_i(x,t)$ are defined by

$$\begin{aligned}
\psi_i(x,t) &= K^{1,1}_{(x_i,t_i)}(x,t), i=1,2,\cdots,\\
\vartheta_i(x,t) &= L^* \psi_i(x,t), i=1,2,\cdots
\end{aligned}$$

Theorem 2.9 *Suppose that $K^{3,2}_{(y,s)}(x,t)$ be a reproducing kernel of $\mathcal{H}_2^{(3,2)}(\Omega)$. Then we have*

$$\vartheta_i(x,t) = \mathcal{L}_{(y,s)} K^{3,2}_{(y,s)}(x,t)|_{(y,s)=(x_i,t_i)}, i=1,2,\cdots,$$

where the subscript (y,s) of the linear operator $\mathcal{L}$ indicates that $\mathcal{L}$ is a function of (y,s).

Proof By using the properties of reproducing kernel $K^{3,2}_{(y,s)}(x,t)$, we have

$$\begin{aligned}
\vartheta_i(x,t) = \mathcal{L}^*\psi_i(x,t) &= \langle \mathcal{L}^*\psi_i(y,s), K^{3,2}_{(y,s)}(x,t)\rangle_{\mathcal{H}_2^{(3,2)}(\Omega)}\\
&= \langle \psi_i(y,s), \mathcal{L}_{(y,s)} K^{3,2}_{(y,s)}(x,t)\rangle_{\mathcal{H}_2^{(1,1)}(\Omega)}\\
&= \mathcal{L}_{(y,s)} K^{3,2}_{(y,s)}(x,t)|_{(y,s)=(x_i,t_i)}.
\end{aligned}$$

where the subscript (y, s) of the linear operator $\mathcal{L}$ indicates that $\mathcal{L}$ is a function of (y, s). ■

THEOREM 2.10 *Suppose that the sequence* $\{(x_i, t_i)\}_{i=1}^{\infty}$ *be dense in* Ω. *Then* $\{\vartheta_i(x,t)\}_{i=1}^{\infty}$ *is a independent linear sequnce in the reproducing kernel space* $\mathcal{H}_2^{(3,2)}(\Omega)$.

THEOREM 2.11 *Suppose that the sequence* $\{(x_i, t_i)\}_{i=1}^{\infty}$ *be dense in* Ω. *Then the sequence* $\{\vartheta_i(x,t)\}_{i=1}^{\infty}$ *is complete in* $\mathcal{H}_2^{(3,2)}(\Omega)$.

Proof Let $v(x,t) \in \mathcal{H}_2^{(3,2)}(\Omega)$. Since $\langle v(x,t), \vartheta_i(x,t)\rangle_{\mathcal{H}_2^{(3,3)}} = 0$, we have

$$\langle v, \vartheta_i\rangle_{\mathcal{H}_2^{(3,2)}(\Omega)} = \langle v, \mathcal{L}^*\psi_i\rangle_{\mathcal{H}_2^{(3,2)}(\Omega)} = \langle \mathcal{L}v, \psi_i\rangle_{\mathcal{H}_2^{(1,1)}(\Omega)} = \mathcal{L}v(x_i, t_i) = 0,\ i \in N.$$

Since the subset $\{(x_i, t_i)\}_{i=1}^{\infty}$ be dense in Ω, we obtain that

$$\mathcal{L}v(x,t) = 0.$$

Since the solution of Equation (8) is unique, we have

$$v(x,t) = 0,\ \forall (x,t) \in \Omega.$$

Hence, the sequence $\{\vartheta_i(x,t)\}_{i=1}^{\infty}$ are completed in reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$. ■

The sequence $\{\vartheta_i(x,t)\}_{i=1}^{\infty}$ is convergent in completed reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$, so the solution $v(x,t)$ can be expressed as follows:

$$v(x,t) = \sum_{i=1}^{\infty} c_i \vartheta_i(x,t).$$

Now, with the choice m-sentence of (14), the approximate solution $P_m v(x,t)$ is presented by

$$P_m v(x,t) = v_m(x,t) = P_m v(x,t) = \sum_{i=1}^{m} c_i \vartheta_i(x,t).$$

where $P_m :\ \mathcal{H}_2^{(3,2)}(\Omega) \to \{\vartheta_i(x,t)\}_{i=1}^{m}$ be orthogonal projection.

Now, we approximate the coefficients c_i with a repeat process.

For the first approximation, we select the function $v_{1,m}(x,t) \in\ \mathcal{H}_2^{(3,2)}(\Omega)$ and assume that

$$v_{n,m} = \sum_{i=1}^{m} c_{i,n}\vartheta_i(x,t),\ n = 2, 3, \ldots,$$

where $c_{i,n},\ i = 1, ..., m,\ \ n = 2, 3, ...,$ with a duplicate process are calculated by using

$$\sum_{i=1}^{m} c_{i,n}\mathcal{L}\vartheta_i(x,t)|_{(x,t)=(x_j,t_j)}$$
$$= F(x, t, v_{n-1,m}, \frac{\partial v_{n-1,m}}{\partial x})|_{(x,t)=(x_j,t_j)}, j = 1, ..., m, n = 2, 3, ...$$

2.1.2 The second method to calculate the approximate solution

Let the subset $\{(x_i, t_i)\}_{i=1}^{\infty}$ be countable dense in Ω. We define the functions $\theta_i(x,t)$ as follows:

$$\theta_i(x,t) = K^{3,2}_{(y,s)}(x,t)|_{(y,s)=(x_i,t_i)}, i = 1, 2, \cdots$$

THEOREM 2.12 *Suppose that the sequence $\{(x_i, t_i)\}_{i=1}^{\infty}$ be dense in Ω, then the sequence $\{\vartheta_i(x,t)\}_{i=1}^{\infty}$ be linear independent in $\mathcal{H}_2^{(3,2)}(\Omega)$.*

Proof Assume that the relation $\sum_{i=1}^{m} a_i\theta_i(x,t) = 0$ is established for the sequence $\{a_i\}_{i=1}^{m}$.
Therefore, we chose $A_s(x,t) \in \mathcal{H}_2^{(3,2)}(\Omega)$ such that

$$A_s(x_s, t_s) = 1, \ A_s(x_l, t_l) = 0, \quad l = 1, 2, ..., m, \ l \neq s,$$

Hence, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{m} a_i\theta_i(x,t) = \langle A_s(x,t), \sum_{i=1}^{m} a_i\theta_i(x,t)\rangle_{\mathcal{H}_2^{(3,2)}(\Omega)} \\ &= \sum_{i=1}^{m} a_i A_s(x_i, t_i) = a_s, \ s = 1, 2, ..., m. \end{aligned}$$

Therefor the sequnce $\{\theta_i(x,t)\}_{i=1}^{\infty}$ be linear independent in reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$. ■

THEOREM 2.13 *Suppose that the sequence $\{(x_i, t_i)\}_{i=1}^{\infty}$ be dense in Ω, then the sequence $\{\theta_i(x,t)\}_{i=1}^{\infty}$ be complete in reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$.*

Proof Assume that $v(x,t) \in \mathcal{H}_2^{(3,2)}(\Omega)$. Since $\langle v(x,t), \theta_i(x,t)\rangle_{\mathcal{H}_2^{(3,2)}} = 0$, we have

$$v(x_i, t_i) = 0.$$

Since the subsequence $\{(x_i, t_i)\}_{i=1}^{\infty}$ be dense in Ω, we obtain that

$$v(x,t) = 0.$$

Therefore, by applying the Theorem 2.11 the sequence $\{\theta_i(x,t)\}_{i=1}^{\infty}$ be complete in reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$. This complete the proof of Theorem 2.13. ■

The sequence $\{\theta_i(x,t)\}_{i=1}^{\infty}$ be complete in reproducing kernel Hilbert space $\mathcal{H}_2^{(3,2)}(\Omega)$, so the solution $v(x,t)$ is given by:

$$v(x,t) = \sum_{i=1}^{\infty} C_i\theta_i(x,t).$$

Now, by chose m-sentence of the equation (14), the approximate solution $P_m v(x,t)$ is presented by

$$P_m v(x,t) = v_m(x,t) = P_m v(x,t) = \sum_{i=1}^{m} C_i\theta_i(x,t).$$

where $P_m:\ \mathcal{H}_2^{(3,2)}(\Omega) \to \{\theta_i(x,t)\}_{i=1}^m$ be orthogonal projection.
Now, we approximate the coefficients C_i with a repeat process.
For first approximation, we select the function $v_{1,m}(x,t) \in\ \mathcal{H}_2^{(3,2)}(\Omega)$ and assume that

$$v_{n,m} = \sum_{i=1}^{m} C_{i,n}\theta_i(x,t),\ n = 2,3,\dots,$$

where the coefficient $C_{i,n},\ i = 1,...,m,\ \ n = 2,3,...$ with a duplicate process are calculated by using

$$\begin{aligned}&\sum_{i=1}^{m} C_{i,n}\mathcal{L}\theta_i(x,t)|_{(x,t)=(x_j,t_j)}\\ &= F(x,t,v_{n-1,m},\frac{\partial v_{n-1,m}}{\partial x})|_{(x,t)=(x_j,t_j)}, j = 1,...,m, n = 2,...\end{aligned}$$

2.2 *Convergence analysis*

THEOREM 2.14 *Let γ is a real constant. Then*

$$B = \left\{v_{n,m}(x,t)|\ \|v_{n,m}\|_{\mathcal{H}_2^{(3,2)}} \leqslant \gamma\right\} \subset C(\Omega)$$

is a bounded set.

Proof We know that

$$\|v_{n,m}\|_\infty \leqslant \alpha\ \|v_{n,m}\|_{\mathcal{H}_2^{(3,2)}(\Omega)},$$

where α is a positive real constant.
Thus for all $(x,t) \in \Omega$ and $v_{n,m}(x) \in B$ there exist $\gamma < \infty$ such that $\|v_{n,m}\|_\infty \leqslant \gamma$.
This complete the proof of the Theorem 2.14. ■

THEOREM 2.15 *Let γ be real constant. Then $B = \left\{v_{n,m}(x,t)|\ \|v_{n,m}\|_{\mathcal{H}_2^{(3,2)}} \leqslant \gamma\right\} \subset C(\Omega)$ is eqicontinuous set.*

Proof By applying the Theorem (2.14), we obtain that

$$\begin{aligned}&|v_{n,m}(y^{'},s^{'}) - v_{n,m}(y^{''},s^{''})| = |\langle v_{n,m}(x,t), K^{3,2}_{(y^{'},s^{'})}(x,t) - K^{3,2}_{(y^{''},s^{''})}(x,t)\rangle_{\mathcal{H}_2^{(3,2)}}|\\ &\quad\leqslant \|v_{n,m}\|_{\mathcal{H}_2^{(3,2)}}\ \|K^{3,2}_{(y^{'},s^{'})} - K^{3,2}_{(y^{''},s^{''})}\|_{\mathcal{H}_2^{(3,2)}}\\ &\quad\leqslant \gamma\ \|\partial_y K^{3,2}_{(y,s)}(x,t)|_{y=(1-c)y^{'}+cy^{''}}(y^{'} - y^{''}) + \partial_s K^{3,2}_{(y,s)}(x,t)|_{s=(1-c)s^{'}+cs^{''}}(s^{'} - s^{''})\|_{\mathcal{H}_2^{(3,2)}}\\ &\quad\leqslant \omega(|y^{'} - y^{''}| + |s^{'} - s^{''}|),\end{aligned}$$

where ω is a real constant.
By chose $\delta = \frac{\epsilon}{\omega}$, we have

$$|y^{'} - y^{''}| + |s^{'} - s^{''}| < \delta \Rightarrow |v_{n,m}(y^{'},s^{'}) - v_{n,m}(y^{''},s^{''})| < \epsilon.$$

for all $(y^{'},s^{'}),(y^{''},s^{''}) \in \Omega$. This complete the proof of the Theorem 2.15. ■

Theorem 2.16 *Suppose that* $\|v_{n,m}\|_{\mathcal{H}_2^{(3,2)}}$ *be a bounded set. Then there exist subsequence* $\{v_{n_\kappa,m}\}_{\kappa=1}^{\infty} \subseteq B$ *and* $v(x,t) \in C(\Omega)$ *such that*

$$\lim_{\kappa\to\infty, m\to\infty} \|v_{n_\kappa,m} - v\|_\infty = 0.$$

Proof Assume that the set B be eqicontinuous and bounded. So, each sequence in B has a subsequence of convergence in $C(\Omega)$.
Thus there exist a subsequence $\{v_{n_\kappa,m}\}_{\kappa=1}^{\infty}$ in B such that

$$\lim_{\kappa\to\infty, m\to\infty} \|v_{n_\kappa,m} - v\|_\infty = 0,$$

Hence, we have

$$\mathcal{L}v_{n_\kappa,m}(x_j,t_j) = F(x,t,v_{n_\kappa-1,m},\frac{\partial v_{n_\kappa-1,m}}{\partial x})|_{(x,t)=(x_j,t_j)}, j = 1,...,m, \kappa = 1,2,...$$

Since $\mathcal{L}$ and F be a continuous functions of v, we obtain that

$$\mathcal{L}v(x,t) = F(x,t,v,\frac{\partial v}{\partial x}).$$

■

In the following theorem, we derive condition for existence and uniqueness of the solution $v_{n,m}$. Furthermore, we establish uniformly converges of the sequence $\{v_{n,m}\}_{n=1}^{\infty}$.

Theorem 2.17 *Assume that the condition of the theorem 2.16 is confirmed. If the solution of the equation* (8) *is exist and unique then*

$$\lim_{n\to\infty, m\to\infty} \|v_{n,m} - v\| \to 0.$$

Proof Assume that the sequence $\{v_{n,m}\}_{n\geqslant 1} \subset B$ is not convenes to v.
Thus, there exist a positive number ϵ_0 and subsequence $\{v_{n_\kappa,m}\}_{\kappa\geqslant 1} \subset B$ such that

$$\|v_{n_\kappa,m} - v\|_\infty \geqslant \epsilon_0, \quad \kappa = 1,2,... \tag{14}$$

Since $\{v_{n_\kappa,m}\}_{\kappa\geqslant 1} \subset B$ be a subset of bounded and continuous functions, then there exits a subsequence of $\{v_{n_\kappa,m}\}_{\kappa\geqslant 1}$ such that convergence to $\widehat{v}$. Without loss of generality, we may assume that the sequence $\{v_{n_\kappa,m}\}_{\kappa\geqslant 1}$ uniformly convergence to $\widehat{v}$.
Hence, we have

$$\lim_{\kappa\to\infty, m\to\infty} \|v_{n_\kappa,m} - \widehat{v}\|_\infty \to 0. \tag{15}$$

The existence and uniqueness of the solution of the equation(8) show that equation (15) contradicts with the equation (14). This completes the proof of the Theorem 2.17. ■

3. Numerical experiments

The methods presented in this paper are applied on two examples to illustrate the efficiency and the applicability of the proposed methods.

Example 3.1 The generalized Haxley-Burgers equation with initial and boundary and conditions are considered as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta(u^3 + \gamma_0 u^2 + \gamma u), \ (x,t) \in (0,1) \times (0,1), \\ u(x,0) = \frac{\gamma}{2}[1 + tanh(a_1 x)], \ 0 \leqslant x \leqslant 1, \\ u(0,t) = \frac{\gamma}{2}[1 + tanh(a_1(-a_2 t))], \ 0 \leqslant t \leqslant 1, \\ u(1,t) = \frac{\gamma}{2}[1 + tanh(a_1(1 - a_2 t))], \ 0 \leqslant t \leqslant 1. \end{cases}$$

The exact solution of this question is given by:

$$u(x,t) = \frac{\gamma}{2}[1 + tanh(a_1(x - a_2 t))].$$

where $a_1 = \frac{\gamma}{8}(-\alpha + \sqrt{\alpha^2 + 8\beta})$ and $a_2 = \frac{\alpha\gamma}{2} - \frac{(2-\gamma)(-\alpha+\sqrt{\alpha^2+8\beta})}{4}$.

x_i	t_i	Method 1 $m = 30$ $n = 10$	Method 2 $m = 30$ $n = 10$
0.1	0.05	$8.7412e-8$	$1.8756e-7$
0.1	0.10	$2.8756e-7$	$2.2321e-6$
0.1	1.00	$6.8135e-7$	$8.9134e-7$
0.5	0.05	$3.5820e-7$	$4.9612e-7$
0.5	0.10	$3.9124e-7$	$4.1258e-7$
0.5	1.00	$2.5621e-6$	$5.1263e-6$
0.9	0.05	$7.3785e-8$	$8.9125e-7$
0.9	0.10	$5.4236e-7$	$1.2031e-6$
0.9	1.00	$7.0219e-7$	$2.1206e-6$

Table 1. Absolute error for different values x and t ($\alpha = 0, \beta = 1$) (Example 3.1).

The absolute error values for proposed methods reported in tables 1 and 2. The results suggest that, the proposed methods are suitable for finding approximate solutions with high degree of accuracy.

Example 3.2 The generalized Haxley-Burgers equation with initial and boundary conditions are considered as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta(u^3 + \gamma_0 u^2 + \gamma u), \ (x,t) \in (0,1) \times (0,1), \\ u(x,0) = \frac{1}{2} - \frac{1}{2}[tanh(\frac{\beta x}{r-\alpha})], \ 0 \leqslant x \leqslant 1, \\ u(0,t) = \frac{1}{2} - \frac{1}{2}[tanh(\frac{-\beta q t}{r-\alpha})], \ 0 \leqslant t \leqslant 1, \\ u(1,t) = \frac{1}{2} - \frac{1}{2}[tanh(\frac{\beta(1-qt)}{r-\alpha})], \ 0 \leqslant t \leqslant 1. \end{cases}$$

The exact solution to this question is given by:

$$u(x,t) = \frac{1}{2} - \frac{1}{2}[tanh(\frac{\beta(x - qt)}{r - \alpha})].$$

where $r = \sqrt{\alpha^2 + 8\beta}$ and $q = \frac{(\alpha-r)(2\gamma-1)+2\alpha}{4}$.

The maximum error values for proposed methods are reported in tables 3 and 4. The results indicate that our methods are suitable for finding approximate solutions with high degree of accuracy.

x_i	t_i	Method 1 $m=30$ $n=10$	Method 2 $m=30$ $n=10$
0.1	0.05	$5.6534e-8$	$8.1743e-7$
0.1	0.10	$2.7430e-7$	$9.1496e-6$
0.1	1.00	$4.4321e-8$	$6.1298e-7$
0.5	0.05	$6.8921e-8$	$4.1075e-7$
0.5	0.10	$3.1209e-7$	$1.1690e-6$
0.5	1.00	$3.3407e-7$	$6.8356e-6$
0.9	0.05	$7.1763e-8$	$2.2567e-7$
0.9	0.10	$2.1093e-7$	$1.8754e-6$
0.9	1.00	$2.1856e-7$	$9.6114e-6$

Table 2. Absolute error for different values x and t ($\alpha=0, \beta=1$ $\gamma=0.001$)(Example 3.1).

$t=0.1$	$\alpha=5,\gamma=0.85$	$\alpha=5,\gamma=0.5$	$\alpha=3,\gamma=0.85$	$\alpha=3,\gamma=0.5$
$m=40,n=10$	$2.4521e-4$	$3.9814e-4$	$4.8927e-4$	$3.7310e-4$
$t=0.4$	$\alpha=5,\gamma=0.85$	$\alpha=5,\gamma=0.5$	$\alpha=3,\gamma=0.85$	$\alpha=3,\gamma=0.5$
$m=40,n=10$	$9.3745e-4$	$1.6721e-4$	$7.935e-4$	$9.5614e-4$
$t=0.9$	$\alpha=5,\gamma=0.85$	$\alpha=5,\gamma=0.5$	$\alpha=3,\gamma=0.85$	$\alpha=3,\gamma=0.5$
$m=40,n=10$	$3.8241e-3$	$1.4987e-3$	$4.5267e-3$	$5.1573e-3$

Table 3. Maximum error in the first method, $\beta=1$ for $t=0.1,0.4,0.9$ (Example 3.2).

$t=0.1$	$\alpha=5,\gamma=0.85$	$\alpha=5,\gamma=0.5$	$\alpha=3,\gamma=0.85$	$\alpha=3,\gamma=0.5$
$m=40,n=10$	$4.1512e-4$	$4.3212e-4$	$4.8927e-4$	$5.4192e-4$
$t=0.4$	$\alpha=5,\gamma=0.85$	$\alpha=5,\gamma=0.5$	$\alpha=3,\gamma=0.85$	$\alpha=3,\gamma=0.5$
$m=40,n=10$	$6.1096e-4$	$5.4012e-4$	$7.935e-4$	$1.8346e-3$
$t=0.9$	$\alpha=5,\gamma=0.85$	$\alpha=5,\gamma=0.5$	$\alpha=3,\gamma=0.85$	$\alpha=3,\gamma=0.5$
$m=40,n=10$	$4.2649e-3$	$4.2309e-3$	$2.1643e-3$	$6.2395e-3$

Table 4. Maximum error in the second method, $\beta=1$ for $t=0.1,0.4,0.9$ (Example 3.2)

4. Conclusion

In this paper, by applying the methods based on reproduction kernel Hilbert space, the approximate solution of the Generalized Huxley-Burgers equations are obtained. First, the appropriate reproducing kernel Hillbert space according to the initial and boundary conditions are defined. Afterwards, the building reproducing kernel are discussed. Since the Gram-Schmidt pronominalization process is unstable due to the rounding error of the process, this removal from the reproduction kernel method in solving this kind of nonlinear problems. In the following, we presented that the approximate solution be a uniformly convergence to the exact solution. In some of the future research, we can find error estimation and calculation the convergence rate of the reproduction kernel method for nonlinear partial differential equations.

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