

## A Three-Point Iterative Method for Solving Nonlinear Equations with High Efficiency Index

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**Abstract.** In this paper, we proposed a three-point iterative method for finding the simple roots of non-linear equations via mid-point and interpolation approach. The method requires one evaluation of the derivative and three(3) functions evaluation with efficiency index of  $8^{1/4} \approx 1.682$ . Numerical results reported here, between the proposed method with some other existing methods shows that our method is promising.

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Received: 03 April 2019, Revised: 10 June 2019, Accepted: 15 August 2019.

**Keywords:** Eight-order convergence; Non-linear equations; Mid-point; Efficiency index.

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## 1. Introduction

Consider the problem of finding the root of nonlinear equation

$$f(x) = 0, \tag{1}$$

where  $f : R \rightarrow R$  which is defined on an interval  $D$ , which assumed to satisfied the following assumption:

A1.  $f$  is continuously differentiable in an open interval  $D \subset R$ .

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A2. There exist a solution  $\alpha$  of (1) in  $D$  such that  $f(\alpha) = 0$ .

A3. The derivative  $f'(\alpha) \neq 0$ .

The famous method for finding the root of (1) is the Newton's method [17,18,16]. Newton's method for solving nonlinear equation is a natural extension of Newton's method for single equation and it is the source of numerous variants methods [9]. This method generates an iterative sequence  $\{x_k\}$  from any initial point  $x_0$  in the neighborhood of the solution  $\alpha$ , via

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad (2)$$

where  $k \doteq 0, 1, 2, \dots$  [18]. Newton method is probably the most widely used iterative method and it is an example one-point iteration which converges quadratically [10].

One of the attractive area of numerical analysis is solving nonlinear equations and mostly iterative method are used to find the solution of nonlinear equation. Throughout this paper, we consider iterative method to find a simple root  $\alpha$  of (1). In the recent years, many new iterative methods have been invented to increase the order of the convergence and efficiency index of the classical iterative methods [9]. There are numerous variants iterative methods for solving (1). For example, in 1974, Kung and Traub in the fundamental paper [7] provide the following derivative-free method for solving nonlinear equation by using the inverse interpolation given by

$$\begin{cases} y_n = x_n + \beta f(x_n), \\ z_n = y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)} \left[ \frac{1}{f[y_n-x_n]} - \frac{1}{f[z_n-y_n]} \right]. \end{cases} \quad (3)$$

In 2007, Jain [6] Proposed a steffence type methods for solving nonlinear equation which is a derivative free method of order three, in which we have three functions evaluation given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = x_n - \frac{f^3(x_n)}{[f(x_n+f(x_n))-f(x_n)][f(x_n)-f(y_n)]}. \end{cases} \quad (4)$$

In 2009 Bi et al. [2] Constructed a two(2) difference iterative methods of three step with eight-order convergence for solving nonlinear equations which is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{1-t_n/2}{1-5t_n/2}, \\ x_{n+1} = z_n - \frac{f(z_n)w(u_n)}{f[z_n,y_n]+(z_n-y_n)f[z_n,x_n,x_n]}, \end{cases} \quad (5)$$

with weight function  $w(u_n) = \frac{1}{(1-\alpha u_n)^{2/\alpha}}$  where  $\alpha \in R$ ,  $t_n = \frac{f(y_n)}{f(x_n)}$ ,  $u_n = \frac{f(z_n)}{f(x_n)}$ , and

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{[2f(x_n)-f(y_n)]f(y_n)}{[2f(x_n)-5f(y_n)]f'(x_n)}, \\ x_{n+1} = z_n - \frac{H(u_n)f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{cases} \quad (6)$$

where  $u_n = \frac{f(z_n)}{f(x_n)}$  and  $H(t)$  represents real valued function with  $H(0) = 1$ ,  $H'(0) = 2$  and  $|H''(0)| < \infty$ .

Also in the same year 2009, Bi et al. [3] constructed another family of eight-order iterative method using the function difference defined by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - h(u_n) \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n) + (\gamma + 2)f(z_n)}{f(x_n) + \gamma f(z_n)} \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n)f[z_n, x_n, x_n]}, \end{cases} \tag{7}$$

where  $\gamma \in R$  is constant,  $u_n = \frac{f(y_n)}{f(x_n)}$ , and  $h(t)$  represents a real valued function with  $h(0) = 1$ ,  $h'(0) = 2$ ,  $h''(0) = 10$ , and  $h'''(0) < \infty$ .

In 2010, Wang and Liu [13], constructed a new method for solving non linear equation with eight-order convergence and efficiency index. The method has four functions evaluation which is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{1}{1 - 2t_n}, \\ x_{n+1} = z_n - \frac{f(z_n)}{2(f[x_n, z_n] - f[x_n, y_n]) + f[y_n, z_n] + \frac{y_n - z_n}{y_n - x_n} (f[x_n, y_n] - f'(x_n))}, \end{cases} \tag{8}$$

where  $t_n = \frac{f(y_n)}{f(x_n)}$ .

In 2010, Wang and Liu [14] proposed a robust optimal eight order method by using weight function given by:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left[ \frac{1}{2} + u \left( \frac{1}{2} + \frac{f(z_n)}{f(y_n)} \right) \right], \end{cases} \tag{9}$$

where  $u = \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)}$ .

Also in 2010, Thukral and Petkovic [12] proposed a family of three point methods of optimal order for solving nonlinear equations as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + bf(y_n)}{f(x_n) + (b-2)f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} [Q(\frac{f(y_n)}{f(x_n)}) + v(x_n, y_n, z_n)], \end{cases} \tag{10}$$

where  $Q(\frac{f(y_n)}{f(x_n)}) = \frac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) - f(y_n)^2}$ ,  $v(x_n, y_n, z_n) = \frac{f(z_n)}{f(y_n) - 9f(z_n)} + 4\frac{f(z_n)}{f(x_n)}$ , and  $Q(0) = 1$ ,  $Q'(0) = 2$ ,  $Q''(0) = 10 - 4b$ ,  $Q'''(0) = 12b^2 - 72b + 72$ . And also develop another method using weight function given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{1+\beta t_n}{1+(\beta-2)t_n}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} [w(t_n) + \frac{s_n}{1+\alpha s_n} + 4u_n]. \end{cases} \quad (11)$$

with weight function

$$w(t_n) = \frac{5 - 2\beta - (2 - 8\beta + 2\beta^2)t_n + (1 + 4\beta)t_n^2}{5 - 2\beta - (12 - 12\beta + 2\beta^2)t_n},$$

where  $\alpha, \beta \in R$ ,  $t_n = \frac{f(y_n)}{f(x_n)}$  and  $u_n = \frac{f(z_n)}{f(x_n)}$ .

In 2010, another third-order iterative algorithm had been develop by Dehghan and Hajarian [5]

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = x_n - \frac{f(x_n)[f(y_n)+f(x_n)]-f(x_n)}{f(x_n+f(x_n))-f(x_n)}. \end{cases} \quad (12)$$

As we can see, this algorithm also includes three evaluation of the function per iteration and therefore is not optimal with high efficiency index.

In 2010, an accurate optimal fourth-order method [8] was proposed by Liu et al. as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = y_n - \frac{f[x_n, y_n] - f[y_n, z_n] + f[x_n, z_n]}{f[x_n, y_n]^2} f(y_n), \end{cases} \quad (13)$$

where  $z_n = x_n + f(x_n)$ . This method consists of three evaluations of the function per iteration in order to obtain fourth-order convergence. In this method  $f[x_n, y_n]$ ,  $f[y_n, z_n]$ ,  $f[x_n, z_n]$  are divide difference of  $f(x)$ . We recall that they can defined recursively via

$$f[x_i] = f(x_i); f[x_i, x_j] = \frac{f[x_i] - f[x_j]}{x_i - x_j}, x_i \neq x_j.$$

In 2011, Zheng et al. in [19] extended the approach given by Liu et al. [8] to provide a three parameter family of iterations with optimal convergence rate four.

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = y_n - \frac{f[x_n, y_n] + (p-1)f[y_n, z_n] - (p-1)f[x_n, z_n] - \beta(y_n - x_n)(y_n - z_n)}{f[x_n, y_n] + pf[y_n, z_n] - pf[x_n, z_n] + a(y_n - x_n)(y_n - z_n)} \frac{f(y_n)}{f[x_n - y_n]}, \end{cases} \quad (14)$$

where  $z_n = x_n + f(x_n)$  and  $\beta, a, p$  are real valued parameters.

In 2013, Sharma and Kaur [11], obtained a two-point iterative methods for solving

nonlinear equations which is derivative free method

$$\begin{cases} y_k = x_k - \frac{\gamma f(x_k)^2}{f(x_k + \gamma f(x_k)) - f(x_k)}, \\ x_{k+1} = y_k - \frac{f(y_k)f(x_k + \gamma f(x_k))}{(f(x_k + \gamma f(x_k)) - f(y_k))f[x_k, y_k]}. \end{cases} \tag{15}$$

The convergence order of these presented method is four and it is more efficient compared to some class of two points methods using numerical example.

In 2016, Ababneh [1] constructed the two point iterative method for solving nonlinear which has the convergence order four with two function evaluation given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - 2 \frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)(f(x_n) + (\beta - 2)f(y_n))}{f'(x_n)(f(x_n) + \beta f(y_n))} - \frac{f'(x_n)f(y_n)}{f(x_n)(f(x_n) + \beta f(y_n))} \left(\frac{f(y_n)}{f'(x_n)}\right)^2, \end{cases} \tag{16}$$

where  $\beta \in R$  is a constant and  $n = 0, 1, 2, \dots$ .

## 2. Method and convergence analysis

In this section, we present a new three point iterative method for solving nonlinear equation via mid-point and Newton interpolation approach.

Let  $f : R \rightarrow R$  is eight times continuously differentiable on an interval  $D \in R$  and has a simple zero  $\alpha \in D$ . Consider the two point iterative method that was constructed by Ababneh [1].

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - 2 \frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)(f(x_n) + (\beta - 2)f(y_n))}{f'(x_n)(f(x_n) + \beta f(y_n))} - \frac{f'(x_n)f(y_n)}{f(x_n)(f(x_n) + \beta f(y_n))} \left(\frac{f(y_n)}{f'(x_n)}\right)^2, \end{cases} \tag{17}$$

where  $\beta \in R$  is a constant and  $n = 0, 1, 2, \dots$ . In order to increase the convergence (17), we added one Newton step and our method is given:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - 2 \frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)(f(x_n) + (\beta - 2)f(y_n))}{f'(x_n)(f(x_n) + \beta f(y_n))} - \frac{f'(x_n)f(y_n)}{f(x_n)(f(x_n) + \beta f(y_n))} \left(\frac{f(y_n)}{f'(x_n)}\right)^2, \\ x_{n+1} = z_n - \frac{f(z_n)q_n}{2(f[x_n, z_n] - f[x_n, y_n]) + f[y_n, z_n] + \frac{y_n - z_n}{y_n - x_n}(f[x_n, y_n] - f'(x_n))}, \end{cases} \tag{18}$$

where  $q_n$  is the mid-point between the first two points.

The convergence analysis of the proposed method is presented in the following theorem.

**Theorem 2.1** Assume that the function  $f : D \subset R \rightarrow R$  is sufficiently differentiable and  $f$  has a simple zero  $\alpha \in D$ . If the initial point  $x_0$  is sufficiently close to  $\alpha$ , then the method defined in (18) is eight-order convergence and satisfy the error

equation

$$e_{n+1} = (59c_3^2c_2^2 - 10\beta^2c_2^4c_3 + 96c_2^6 + \beta^3c_2^6 - \frac{3}{2}c_5c_2^2 + \frac{55}{2}c_2^3c_4 - 2c_3^3 - 7c_3c_4c_2 - 166c_2^4c_3 + 12\beta c_2^3c_4 + 24\beta c_3^2c_2^2 + \frac{33}{2}\beta^2c_2^6 - 110\beta c_2^4c_3 + \frac{173}{2}\beta c_2^6)e_n^8 + (e_n^9). \quad (19)$$

*Proof* Consider the Taylor expansion of the function  $f(x_n)$  around  $\alpha$  which is given by

$$f(x_n) = f(\alpha) + \frac{1}{1!}f'(\alpha)(x_n - \alpha) + \frac{1}{2!}f''(\alpha)(x_n - \alpha)^2 + \frac{1}{3!}f'''(\alpha)(x_n - \alpha)^3 + \dots + \frac{1}{8!}f^{(8)}(\alpha)(x_n - \alpha)^8 + o(x_n - \alpha)^9. \quad (20)$$

Let  $e_n = x_n - \alpha$  be the error in  $n^{th}$  iteration with the assumption that  $f(\alpha) = 0$  and  $f'(\alpha) \neq 0$ , then we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + o(e_n^9)]. \quad (21)$$

Furthermore, we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + o(e_n^9)], \quad (22)$$

where  $c_n = \frac{f^{(n)}(\alpha)}{n!f'(\alpha)}$  for  $n = 2, 3, 4, \dots$  and  $e_n = x_n - \alpha$ . Then we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + 2(3c_3^2 - 10c_2^2c_3 + 5c_2c_4 + 4c_2^4 - 2c_5)e_n^5 + (-16c_2^5 - 28c_2^2c_4 + 17c_3c_4 + 52c_2^3c_3 - c_2(33c_3^2 - 13c_5) - 5c_6)e_n^6 + o(e_n^7).$$

Let  $e_{n,y} = y_n - \alpha$  be the error in  $y_n$  iteration where  $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$  and  $e_n = x_n - \alpha$ . Then we have

$$e_{n,y} = c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 - 2(3c_3^2 - 10c_2^2c_3 + 5c_2c_4 + 4c_2^4 - 2c_5)e_n^5 - (-16c_2^5 - 28c_2^2c_4 + 17c_3c_4 + 52c_2^3c_3 - c_2(33c_3^2 - 13c_5) - 5c_6)e_n^6 + o(e_n^7).$$

Also, finding the Taylor expansion of  $f(y_n)$  and simplifying it, we have

$$f(y_n) = f'(\alpha)[e_{n,y} + c_2e_{n,y}^2 + c_3e_{n,y}^3 + c_4e_{n,y}^4 + c_5e_{n,y}^5 + c_6e_{n,y}^6 + c_7e_{n,y}^7 + c_8e_{n,y}^8 + o(e_{n,y}^9)]. \quad (23)$$

Substituting equations (20), (21) and (22) in  $z_n$  above at (18) we obtained

$$e_{n,z} = (4c_2^3 - c_2c_3)e_n^4 + (12\beta c_2^2c_3 - 26c_2^4 - 19\beta c_2^4 - 2c_2c_4 - 2\beta^2c_2^4 + 26c_2^2c_3 - 2c_3^2)e_n^5 + o(e_n^6),$$

but  $e_{n,z} = z_n - \alpha$  which is the error in the second point  $z_n$ , and

$$z_n = y_n - 2 \frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)(f(x_n) + (\beta - 2)f(y_n))}{f'(x_n)(f(x_n) + \beta f(y_n))} - \frac{f'(x_n)f(y_n)}{f(x_n)(f(x_n) + \beta f(y_n))} \left(\frac{f(y_n)}{f'(x_n)}\right)^2,$$

therefore, we have the error equation in  $z_n$  as

$$e_{n,z} = [2(2 + 2\beta)c_2^3 - c_2c_3]e_n^4 + [-(26 + \beta(19 + 2\beta))c_2^4 + 2(13 + 6\beta)c_2^2c_3 - 2c_3^2 - 2c_2c_4]e_n^5 + o(e_n^6). \tag{24}$$

Similarly, the Taylor expansion of  $f(z_n)$  we have

$$f(z_n) = f'(\alpha)[e_{n,z} + c_2e_{n,z}^2 + c_3e_{n,z}^3 + c_4e_{n,z}^4 + c_5e_{n,z}^5 + c_6e_{n,z}^6 + c_7e_{n,z}^7 + c_8e_{n,z}^8 + o(e_{n,z}^9)]. \tag{25}$$

Simplifying (24) we have

$$f(z_n) = (4c_2^3 + 2\beta c_2^3 - c_2c_3)e_n^4 + (12\beta c_2^2c_3 - 19\beta c_2^4 - 2c_2c_4 - 2c_3^2 - 26c_2^4 + 26c_2^2c_3 - 2\beta^2c_2^4)e_n^5 + (-8c_2^5c_3 + 4\beta^2c_2^7 - 4\beta c_2^5c_3 + 16c_2^7 + 16\beta c_2^7 + c_2^3c_3^2)e_n^8 + o(e_n^9), \tag{26}$$

$$q_n = \frac{1}{2}c_2e_n^2 - (c_2^2 - c_3)e_n^3 + (4c_2^3 - 4c_2c_3 + \frac{3}{2}c_4 + \beta c_2^3)e_n^4 + (6\beta c_2^3c_3 + 23c_2^2c_3 - 17c_2^4 - 4c_2^3 - 6c_2c_4 - \beta^2c_2^4 + 2c_5 - \frac{19}{2}\beta c_2^4)e_n^5 + a(e_n^6) + b(e_n^7) + c(e_n^8) + o(e_n^9), \tag{27}$$

where  $a$ ,  $b$  and  $c$  are some terms in equation (26). Multiplying equations (25) by (26) we have the numerator of our proposed method as follows

$$f(z_n)q_n = (2c_2^4 - \frac{1}{2}c_2^3c_3 + \beta c_2^4)e_n^6 + (-c_2^2c_4 - 2c_2^3c_2 - \beta^2c_2^5 - \frac{23}{2}\beta c_2^5 + 18c_2^3c_3 + 8\beta c_2^3c_3 - 17c_2^5)e_n^7 + (31\beta c_2^6 - 72c_2^4c_3 + 3\beta c_2^3c_4 - 2c_3^3 + 12\beta c_2^3c_2^2 + 4\beta^2c_2^6 - 40\beta c_2^4c_3 + 8c_2^3c_4 + 42c_2^6 + 32c_2^3c_2^2 - \frac{7}{2}c_3c_4c_2 - 2\beta^2c_2^4c_3)e_n^8 + o(e_n^9), \tag{28}$$

but the denominator of our proposed method was found to be

$$\begin{aligned} Den = & 1 + (11c_2^4 - 2c_2c_4 + 6\beta c_2^4)e_n^4 + (-c_2^2c_4 - 57\beta c_2^5 - 4c_3c_4 + 77c_2^3c_3 - 2c_2c_5 \\ & + 40\beta c_2^3c_3 - 76c_2^5 - 6\beta^2 c_2^5)e_n^5 + (-36\beta^2 c_2^4c_3 + 196c_2^6 - 6c_4^2 + 43c_2^3c_4 - 368c_2^4c_3 \\ & - 7c_2c_6 + 191c_2^2c_3^2 - 2c_2(33c_3^2 - 13c_5)c_2 + 4\beta^3 c_2^6 + 15c_2c_3c_4 - 4c_3^3 + 222\beta c_2^6 \\ & + 40\beta c_2^3c_4 - 13c_2^2c_5 + 50\beta^2 c_2^6 - 4c_3c_5 + 72\beta c_3^2c_2^2 - 318\beta c_2^4c_3)e_n^6 + a(e_n^7) \\ & + b(e_n^8) + o(e_n^9). \end{aligned} \quad (29)$$

Dividing equation (27) by (28) we have the error equation as

$$\begin{aligned} e_{n+1} = & (59c_3^2c_2^2 - 10\beta^2 c_2^4c_3 + 96c_2^6 + \beta^3 c_2^6 - \frac{3}{2}c_5c_2^2 + \frac{55}{2}c_2^3c_4 - 2c_3^3 - 7c_3c_4c_2 \\ & - 166c_2^4c_3 + 12\beta c_2^3c_4 + 24\beta c_3^2c_2^2 + \frac{33}{2}\beta^2 c_2^6 - 110\beta c_2^4c_3 + \frac{173}{2}\beta c_2^6)e_n^8 + (e_n^9). \end{aligned} \quad (30)$$

Thus, the convergence order of our proposed method is eight for any real value of the parameter  $\beta$  which complete the proof of the theorem. ■

### 3. Numerical results

In this section, we test the numerical results of our new proposed method (18) named KBD and compare with existing eight order method which develop by:

(i) Thukral and Petkovic (TP) [12]:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{1+\beta t_n}{1+(\beta-2)t_n}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} [w(t_n) + \frac{s_n}{1+\alpha s_n} + 4u_n], \end{cases} \quad (31)$$

with weight function

$$w(t_n) = \frac{5 - 2\beta - (2 - 8\beta + 2\beta^2)t_n + (1 + 4\beta)t_n^2}{5 - 2\beta - (12 - 12\beta + 2\beta^2)t_n},$$

where  $\alpha, \beta \in R$ ,  $t_n = \frac{f(y_n)}{f(x_n)}$ ,  $s_n = \frac{f(z_n)}{f(y_n)}$ , and  $u_n = \frac{f(z_n)}{f(x_n)}$ .

(ii) Salimi et al. [10]:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - 2\frac{f(y_n)}{f'(x_n)} + \frac{f(y_n)(f(x_n) + (\beta-2)f(y_n))}{f'(x_n)(f(x_n) + \beta f(y_n))} - \frac{f'(x_n)f(y_n)}{f(x_n)(f(x_n) + \beta f(y_n))} \left(\frac{f(y_n)}{f'(x_n)}\right)^2, \\ x_{n+1} = z_n - \frac{f(z_n)\eta(t_n)\psi(u_n)}{f[z_n, y_n] + (z_n - y_n)f[z_n, x_n, x_n]}, \end{cases} \quad (32)$$



with weight function

$$\begin{aligned}\eta(t_n) &= 1 - 4(2 + \beta)t_n^3, \quad \psi(u_n) = 1 + 2u_n, \\ \eta(t_n) &= 1 - \frac{4\beta + 8}{1 + 2t_n}t_n^3, \quad \psi(u_n) = \frac{1 + 3u_n}{1 + u_n}, \\ \eta(t_n) &= \frac{1 + t_n - 4\beta t_n^3}{1 + t_n + 8t_n^3}, \quad \psi(u_n) = 3 - \frac{2}{1 + u_n},\end{aligned}$$

where  $t_n = \frac{f(y_n)}{f(x_n)}$ ,  $u_n = \frac{f(z_n)}{f(x_n)}$ , and  $\beta \in R$ .

We apply the above methods to solve some benchmark test functions drawn from [4]:

$$f_1(x) = x^3 + 4x^2 - 10, \alpha = 1.3652300134140968457608068290 \text{ and } x_0 = 1,$$

$$f_2(x) = x^2 - e^x - 3x + 2, \alpha = 0.25753028543986076045536730494 \text{ and } x_0 = 0,$$

$$f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, \alpha = 1.2076478271309189270094167584 \text{ and } x_0 = -1,$$

$$f_4(x) = \sin(x)e^x + \log(x^2 + 1), \alpha = 0 \text{ and } x_0 = 2,$$

$$f_5(x) = (x - 1)^3 - 2, \alpha = 2.2599210498948731647672106073 \text{ and } x_0 = 3,$$

$$f_6(x) = (x + 2)e^x - 1, \alpha = -0.44285440100238858314132800000 \text{ and } x_0 = 2,$$

$$f_7(x) = \sin^2(x) - x^2 + 1, \alpha = 1.4044916482153412260350868178 \text{ and } x_0 = 2,$$

where  $\alpha$  is a root  $f_k(x) = 0$  for  $k = 1, 2, \dots, 7$  and  $x_0$  is an initial approximation.

The numerical results reported here have been carried out in Matlab R2014a to test our proposed method KBD and also compare with methods WL, TP, SNSP1, SNSP2, and SNSP3. Table 1 and 2 shows the difference of the root  $\alpha$  and the approximate  $x_n$ . The absolute values of the function  $|f(x_n)|$ , number of iteration and the computational order of convergence (COC) is also reported in the tables. Where the COC is defined by [15]

$$\rho \approx \frac{\ln|(x_n + 1 - \alpha)/(x_n - \alpha)|}{\ln|(x_n - \alpha)/(x_n - 1 - \alpha)|}.$$

#### 4. Discussion

The results presented in the tables (1) and (2) shows that our methods KBD1, KBD2, and KBD3 converges more rapidly than the methods proposed by TP, BEM, SNSP1, SNSP2, and SNSP3. It also shows that the new methods introduced in this paper have at least equal performance in terms of number of iteration when compared to the other existing eight-order methods. The total number of function evaluation in each iteration are almost the same expect on functions  $f_3$  and  $f_6$ .

#### 5. Conclusion

A new three-point iterative method for solving nonlinear equations with high efficiency index have been constructed for approximating a simple root of a given nonlinear equation. The method uses only four functions evaluation in each iteration and a numerical comparison with some other known method shows that our proposed method have higher convergence order.

Table 1. Comparison of iterative methods TPM, BEM, SNSP1, SNSP2, and SNSP3 with the new methods KSD1, KSD2, and KSD3.

METHODS	$\beta$	$ x_n - \alpha $	$ f(x_n) $	ITERATION	COC
		$f_1(x) = x^3 + 4x^2 - 10,$		$x_0 = 1$	
TP	-1	1.9043e-10	3.1447e-9	3	8.0000
SNSP1	-1	3.6429e-9	6.015e-8	3	8.0000
SNSP2	0	3.6492e-9	6.0261e-8	3	8.0000
SNSP3	1	1.8385e-9	3.0360e-8	3	8.0000
KBD	-1	1.3574e-11	2.2416e-10	3	8.0000
KBD	0	2.8999e-11	4.7888e-10	3	8.0000
KBD	1	4.4371e-11	7.3270e-10	3	8.0000
		$f_2(x) = x^2 - e^x - 3x + 2,$		$x_0 = 1$	
TP	-1	1.2056e-10	4.5556e-10	3	8.0000
SNSP1	-1	1.7388e-13	6.5713e-13	3	8.0000
SNSP2	0	1.7390e-13	6.5721e-13	3	8.0000
SNSP3	1	8.6921e-14	3.2836e-13	3	8.0000
KBD	-1	4.6899e-11	1.7722e-10	3	8.0000
KBD	0	2.7031e-11	1.0214e-10	3	8.0000
KBD	1	7.1201e-12	2.6905e-11	3	8.0000
		$f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$		$x_0 = -2$	
TP	-1	3.8156e-10	7.7486e-9	6	8.0000
SNSP1	-1	2.0113e-12	4.0877e-11	6	8.0000
SNSP2	0	1.4648e-12	2.9746e-11	6	8.0000
SNSP3	1	6.5533e-13	1.3310e-11	6	8.0000
KBD	-1	1.3323e-15	2.6645e-14	6	8.0000
KBD	0	1.3323e-15	2.6645e-14	6	8.0000
KBD	1	1.3323e-15	2.6645e-14	6	8.0000
		$f_4(x) = \sin(x)e^x + \log(x^2 + 1),$		$x_0 = 2$	
TP	1	3.9092e-11	3.9030e-11	4	8.0000
SNSP1	-1	9.4697e-9	9.4698e-9	3	8.0000
SNSP2	0	1.7737e-8	1.7737e-8	3	8.0000
SNSP3	1	3.3210e-8	3.3210e-8	3	8.0000
KBD	-1	8.3522e-12	8.3520e-12	3	8.0000
KBD	0	1.7008e-11	1.7008e-11	3	8.0000
KBD	1	2.5635e-11	2.5635e-11	3	8.0000

Table 2. Comparison of iterative methods TPM, BEM, SNSP1, SNSP2, and SNSP3 with the new methods KSD1, KSD2, and KSD3.

METHODS	$\beta$	$ x_n - \alpha $	$ f(x_n) $	ITERATION	COC
		$f_5(x) = (x - 1)^3 - 2,$		$x_0 = 3$	
TP	1	4.6838e-12	2.2306e-11	4	8.0000
SNSP1	-1	2.1758e-9	1.0362e-8	3	8.0000
SNSP2	0	4.5065e-9	2.1461e-8	3	8.0000
SNSP3	1	8.1479e-9	3.8802e-8	3	8.0000
KBD	-1	1.0951e-12	5.2160e-11	3	8.0000
KBD	0	2.4047e-12	1.1453e-11	3	8.0000
KBD	1	3.7130e-12	1.7683e-11	3	8.0000
		$f_6(x) = (x + 2)e^x - 1,$		$x_0 = 2$	
TP	1	6.4269e-9	1.0554e-8	6	8.0000
SNSP1	-1	4.5075e-14	7.4e-14	6	8.0000
SNSP2	0	1.0902e-13	1.79e-13	6	8.0000
SNSP3	1	1.7886e-13	2.94e-13	6	8.0000
KBD	-1	1.4455e-9	2.3739e-9	6	8.0000
KBD	0	3.4964e-9	5.7419e-9	6	8.0000
KBD	1	5.5282e-9	9.0783e-9	6	8.0000
		$f_7(x) = \sin^2(x) - x^2 + 1,$		$x_0 = 1$	
TP	1	2.7460e-12	6.8170e-12	3	8.0000
SNSP1	-1	1.6168e-9	4.0136e-9	3	8.0000
SNSP2	0	3.1668e-9	6.5721e-9	3	8.0000
SNSP3	1	5.6023e-9	8.0987e-9	3	8.0000
KBD	-1	4.2966e-13	1.0660e-12	3	8.0000
KBD	0	4.2966e-13	1.0660e-12	3	8.0000
KBD	1	4.2966e-13	1.0660e-12	3	8.0000

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