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An Efficient Neurodynamic Scheme for Solving a Class of Nonconvex Nonlinear Optimization Problems

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Abstract. By p-power (or partial p-power) transformation, the Lagrangian function in nonconvex optimization problem becomes locally convex. In this paper, we present a neural network based on an NCP function for solving the nonconvex optimization problem. An important feature of this neural network is the one-to-one correspondence between its equilibria and KKT points of the nonconvex optimization problem. the proposed neural network is proved to be stable and convergent to an optimal solution of the original problem. Finally, an examples is provided to show the applicability of the proposed neural network.

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1. Introduction

The nonconvex programming problems arise in a wide variety of scientific and engineering applications. At present, there exist several neural networks for solving nonlinear convex optimization problems(see [1, 2, 4]). Now an interesting question is: can neural networks also be used to search for KKT points in noconvex optimization problems? This paper propose a recurrent neural network based on the KKT optimality conditions by using the Fischer-Burmeister NCP function.

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2. Problem formulation and preliminaries

Consider the fallowing nonconvex optimization problem:

$$\begin{aligned} &\min \ f(x) \\ &\text{s.t.} \quad g_j(x) \leqslant b_j, \\ &x \in X, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., m$ are twice continuously differentiable function and $X = \{x \in \mathbb{R}^n | u \leq x \leq v, u \in \mathbb{R}^n, v \in \mathbb{R}^n\}$. We assume that f is positive on X, that g_j, $j = 1, ..., m$, is nonnegative on X, and that b_j, $j = 1, ..., m$, are positive. Throughout the paper, the following notations are used.
$I = \{1, ..., n\}$, $J = \{1, ..., m\}$ and $\mathbb{R}^n_+$ stands for nonnegative quadrant in the n-dimensional real space. int$S$ stands for the interior of a set S.

Theorem 2.1 Second-order sufficiency conditions Suppose that x^* is a feasible point to problem (1) and $x^* \in \text{int}X$. If there exists a Lagrangian multiplier vector $\lambda^* \in \mathbb{R}^m$ such that (x^*, λ^*) is a KKT point pair and the Hessian matrix $\nabla^2_x L(x^*, \lambda^*)$ is positive definite on the tangent subspace $M(x^*) = \{d \in \mathbb{R}^n | d^T \nabla g_j(x^*) = 0, \forall j \in J(x^*)\}$ where $J(x^*) = \{j \in J | \lambda^*_j > 0\}$ then x^* is a strict relative minimum point of problem (1).

Consider the p-power transformation [3] of (1):

$$\begin{aligned} &\min \ [f(x)]^p \\ &\text{s.t.} \quad [g_j(x)]^p \leqslant b_j^p, \quad j \in J, \\ &x \in X, \end{aligned} \tag{2}$$

with $p \geq 1$. Correspondingly, the Lagrangian function of (2) is defined as

$$L_p(x, \mu_p) = [f(x)]^p + \sum_{j=1}^{m} \mu_p^j \{[g_j(x)]^p - b_j^p\},$$

where $\mu_p = (\mu_p^1, ..., \mu_p^m) \geq 0$ is p-power Lagrangian multiplier.

Lemma 2.1 Let x^* be a local optimal solution of (1). Assume that $J(x^*) \neq \emptyset$, x^* is a regular point, and x^* satisfies the second-order sufficiency condition. Then there exists a $q > 0$ such that the Hessian matrix of the p-*power* Lagrangian function, $\nabla^2_x L_p(x^*, \mu^*_p)$, is positive definite when $p > q$.

Definition 2.2 A function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP function if it satisfies

$$\varphi(a, b) = 0 \Leftrightarrow a \geq 0, \ b \geq 0, \ ab = 0$$

In this paprer, we use the Fischer-Burmeister NCP function as following:

$$\psi(a, b) = \sqrt{a^2 + b^2} - a - b \tag{3}$$

Definition 2.3 Let $\Omega \subset \mathbb{R}^l$ be a closed convex set. $P_\Omega : \mathbb{R}^l \to \Omega$ is called a projection function, which is defined by

$$P_\Omega(x) = \arg\min_{v \in \Omega} \|x - v\|$$

Now we consider the p-power problem (2). KKT conditions for this problem can be expressed as

$$u \leq x \leq v,\ \mu_p \geq 0,\ b^p - [g(x)]^p \geq 0,\ \mu_p^T([g(x)]^p - b^p) = 0,$$

$$\nabla_x L_p(x, \mu_p) = 0.$$

3. Neural network model and stability analysis

In this section we present a neural network based on the Fischer-Burmeister NCP function (3). Let

$$\phi_i(x, \mu_p) = \psi((b_i{}^p - [g_i(x)]^p), \mu_p^i),\ i \in J,$$

where ψ defined as (3) and also consider

$$\Phi_1(x, \mu_p) = (\phi_1(x, \mu_p), ..., \phi_m(x, \mu_p))^T.$$

Then we denote

$$\Phi(x, \mu_p) = ((\nabla_x L_p(x, \mu_p))^T, (\Phi_1(x, \mu_p))^T)^T.$$

Lemma 3.1 (x^*, μ_p^*) is a KKT point if and only if $\Phi(x^*, \mu_p^*) = 0$.

Now we present a recurrent neural network for solving $\Phi(x, \mu_p) = 0$ whose state variable is defined by the following system:

$$\frac{dw}{dt} = \lambda\{P_\Omega(w + \alpha\Phi(w)) - w\},\ \lambda > 0, \tag{4}$$

where $w = (x, \mu_p)^T$, $\Omega = X \times \mathbb{R}_m^+$ and α is a scaler. We note that Φ is locally Lipschitz continuous.

Theorem 3.1 For any initial point $w(t_0) = \left(x(t_0)^T, \mu_p(t_0)^T\right)^T \in \Omega$, $\Omega = X \times \mathbb{R}_+^m$, there exists a unique continuous solution $w(t) = \left(x(t)^T, \mu_p(t)^T\right)^T$ for system (4).

Theorem 3.2 The proposed neural network (4) with initial point $w_0 \in \Omega$ is locally asymptotically stable and is locally convergent to one KKT point $w^* = \left(x^{*T}, \mu_p^{*T}\right)^T$, where x^* is the optimal solution of the problem (1).

4. Illustrative example

Example 4.1 Consider the following problem:

$$\begin{aligned}
&\min\ 2 - x_1 x_2 \\
&\text{s.t.}\quad x_1 + 4x_2 \leq 1 \\
&x \in [0, 1]^2
\end{aligned}$$

The optimal solution of this problem is $x^* = (0.5, 0.125)^T$. The Hessian of the Lagrangian function is indefinite. When p=3 we have $\mu_3^* = 0.4692382813$ and the

Hessian of p-power Lagrangian is a positive definite matrix. We apply the neural network (4) to solve the problem with p=3. Simulation results show that the neural network is locally asymptotically stable at the optimum. Figure.1 displays the transient behavior of neural network (4).

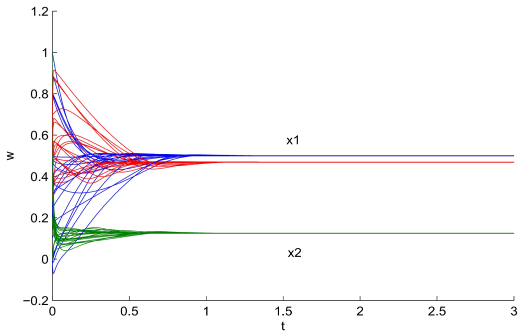

Figure 1. Transient behavior of neural network (4) in Example 4.1

5. Conclusion

In this paper, we have presented a recurrent neural network based on NCP function for solving nonconvex programming problem. It was shown that the neural network is successful in searching for the KKT points especially the local minimum points.

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