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Numerical Solution of the Lane-Emden Equation Based on DE Transformation via Sinc Collocation Method

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Abstract. In this paper, numerical solution of general Lane-Emden equation via collocation method based on Double Exponential **(DE)** transformation is considered. The method converts equation to the nonlinear Volterra integral equation. Numerical examples show the accuracy of the method. Also, some remarks with respect to run-time, computational cost and implementation are discussed.

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1. Introduction

The Lane-Emden type equations are obtained from Emden-Fowler equation which is of the form [3]:

$$y^{''} + \frac{2}{t}y^{'} + g(t)f(y(t)) = h(t), \quad 0 < t < \infty, \tag{1}$$

with conditions:

$$y(0) = a,\ y^{'}(0) = 0, \tag{2}$$

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where a is a constant and $f(y), g(t), h(t)$ are given functions, $y(t)$ is unknown function and must be determined.

By selecting $f(y) = y^m, h(t) = 0, g(t) = 1$, the Lane-Emden type equations are obtained:

$$y^{''} + \frac{2}{t}y^{'} + y^m(t) = 0, \quad 0 < t < \infty. \tag{3}$$

Lane-Emden equation is linear for $m = 0, 1$, and nonlinear otherwise. Exact solutions exist only for $m = 0, 1, 5$ that are given in Bender [1], respectively, by

$$\begin{aligned} m &= 0, \qquad y(t) = 1 - \frac{1}{3!}t^2, \\ m &= 1, \qquad y(t) = \frac{\sin(t)}{t}, \\ m &= 5, \qquad y(t) = (1 + \frac{t^2}{3})^{-1/2}. \end{aligned} \tag{4}$$

The Lane-Emden type equations have significant application in nonlinear science and are frequently used to model the several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical law of the thermodynamics. A discussion of the formulation of these models can be found in Chandrasekhar [4], Shawagfeh [19], Horedt [12], Biles [3] and Davis [5].

Many different methods are usually use to solve Eq. (1). He developed the variational iteration method (VIM) [11]. Liao solved Lane-Emden type equations by applying a homotopy analysis method [13]. Youseffi used Legendre wavelets to obtain approximated solution [25]. Wazwaz in [23, 24] used Adomian decomposition method to solve Eq.(1), also many authors introduced a modification of ADM for solving Lane-Emden singular problems. Bender et al used perturbation method based on existence of small parameter [1]. Sinc collocation method in [18] was applied by single exponential and derivation form of function y to approximate the solution.

Sinc method is a powerful numerical tool for finding fast and accurate solution in various areas of problems [14, 20]. Double exponential transformation, abbreviated as **DE** was first proposed by Takahasi and Mori [22] in 1974. It has come to be widely used in applications. Also, it is known that the double exponential transformation gives an optimal result for numerical evaluation of definite integral of an analytic function [16]. However, Sugihara [15, 21] has recently found that the errors in the Sinc numerical methods are $O(\exp(-cN/logN))$ with some $c > 0$.

The main difficulty arises in the singularity of the Eq. (1) at $t = 0$. Because of the singularity, the solution may be not differentiable at $t = 0$ and system of equations in this case becomes ill-conditioned as the number of basis functions increases. Discussion in [17] shows using smoothing transformation in cooperate with sinc approximation is generally an effective tool with derivative singularity at endpoints. Furthermore, it can be examined numerically that in this method, the system of equations is well-conditioned.

The main purpose of the present research is to consider the numerical solution of Lane-Emden integral equations corresponding to ordinary differential equation based on double exponential transformation by converting Eq. (1) to an integral equation and investigating computational cost and implementation of the algorithm.

The layout of the paper is as follows: in section 2, we give some basic definitions, assumptions and preliminaries of the sinc approximations and related topics. In section 3, the proposed method to solve the corresponding Lane-Emden integral equation is applied. Finally, section 4, contains the details of the proposed algorithm and numerical implementation and some experimental results.

2. Basic definitions and preliminaries

Let f be a function defined on $\mathbb{R}$ and $h > 0$ is step size then the Whittaker cardinal defined by the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x), \tag{5}$$

whenever this series convergence, and

$$S(j,h)(x) = \frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h}, j = 0, \pm 1, \pm 2, .. \tag{6}$$

where $S(j,h)(x)$ is known as $j-th$ Sinc function evaluated at x.

Moreover, let us consider $H^1(D_d)$ be the family of all functions g analytic in D_d, such that

$$\begin{aligned} N_1(g, D_d) &= \lim_{\epsilon\to 0} \textstyle\int_{\partial D_{d(\epsilon)}} |g(t)||dt| < \infty, \\ D_{d(\epsilon)} &= \{t \in C, \quad |Ret| < \tfrac{1}{\epsilon}, \quad |Imt| < d(1-\epsilon)\}. \end{aligned}$$

We recall the following definitions from [14, 20], that will become instrumental in establishing our useful formulas:

Definition 2.1 A function g is said to be decay double exponentially, if there exist constants α and C, such that:

$$|g(t)| \leqslant C \exp(-\alpha \exp|t|), \quad t \in (-\infty, \infty)$$

equivalently, a function g is said to be decay double exponentially with respect to conformal map ϕ, if there exist positive constants α and C such that:

$$|g(\phi(t))\phi'(t)| \leqslant C \exp(-\alpha \exp|t|), \quad t \in (-\infty, \infty).$$

Here, we suppose that $K_\phi^\alpha(D_d)$ denote the family of functions g where $g(\phi(t))\phi'(t)$ belongs to $H^1(D_d)$ and decays double exponentially with respect to ϕ. If f belongs to $K_\phi^\alpha(D_d)$ with respect to ϕ, then we have the following formulas for definite and indefinite integrals based on DE transformation.

Theorem 2.1 [10, 14, 20] Let $f \in K_\phi^\alpha(D_d)$ then DE formula for indefinite integration is:

$$
\begin{aligned}
\int_a^x f(x)dx = h \sum_{j=-N}^{N} f(\phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi} Si(\frac{\pi\phi^{-1}(x)}{h} - j\pi)) \\
+ O(\frac{logN}{N}\exp(-\frac{\pi dN}{log(\pi dN/\alpha)})),
\end{aligned}
\tag{7}
$$

where

$$
\phi(t) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh t) + \frac{a+b}{2}, \tag{8}
$$

$$
\phi'(t) = \frac{b-a}{2}\frac{\pi/2\cosh(t)}{\cosh^2(\pi/2\sinh(t))}, \tag{9}
$$

also $Si(t)$ is the Sine integral defined by:

$$
Si(t) = \int_0^t \frac{\sin w}{w} dw,
$$

and the mesh size h satisfies $h = \frac{1}{N}\log(\pi dN/\alpha)$.

3. Main idea

We consider Eq. (1) as follows:

$$
ty^{''} + 2y^{'} = -tg(t)f(y(t)) + th(t), \quad 0 < t < \infty, \tag{10}
$$

By integrating (10) with respect to s from $(0, t)$ with $t < T$ we get:

$$
\int_0^t sy^{''}(s)ds + 2\int_0^t y^{'}(s)ds = \int_0^t \left[-sg(s)f(y(s)) + sh(s)\right] ds. \tag{11}
$$

By using the initial conditions (2) we obtain

$$
[ty(t)]^{'} = a + \int_0^t \{-sg(s)f(y(s)) + sh(s)\} ds. \tag{12}
$$

By Integrating again and simplifying, it results

$$
\begin{aligned}
y(t) =& a + \frac{1}{t}\int_0^t (t-s)\{-sg(s)f(y(s)) + sh(s)\} ds \\
=& a + \int_0^t \left(\frac{s^2}{t} - s\right)\{g(s)f(y(s)) - h(s)\} ds,
\end{aligned}
\tag{13}
$$

which is a nonlinear Volterra integral equation of the second kind with weakly singular kernel. Eq. (13) can be written as:

$$y(t) = F(t) + \int_0^t K(t,s)f(y(s))ds, \tag{14}$$

where,

$$\begin{aligned} K(t,s) &= \left(\frac{s^2}{t} - s\right) g(s), \\ F(t) &= a - \int_0^t K(t,s)h(s)ds. \end{aligned} \tag{15}$$

To apply DE transformation for approximation of Eq. (14), first, we apply Theorem 1 to the Volterra integral part of Eq. (14), so we get:

$$\int_0^t K(t,s)f(y(s))ds \simeq h \sum_{j=-N}^{N} K(t,\phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si(\frac{\pi\phi^{-1}(t)}{h} - j\pi))f(y_j), \tag{16}$$

where y_j denotes an approximation value of $y(t_j)$ and

$$h = \frac{1}{N}\log(\pi dN/\alpha). \tag{17}$$

If we substitute (16) in the right-hand side of (14), we obtain:

$$y(t) \simeq F(t) + h \sum_{j=-N}^{N} K(t,\phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si(\frac{\pi\phi^{-1}(t)}{h} - j\pi))f(y_j). \tag{18}$$

There are $2N+1$ unknowns $y_j,\ j = -N..N$ to be determined. In order to determine these variables we use collocation method at Sinc points as follows:

$$\phi(t_k) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh hk) + \frac{a+b}{2}, \quad k = -N..N. \tag{19}$$

So, we have the following nonlinear system of $2N+1$ unknowns:

$$y(t_k) = F(t_k) + h \sum_{j=-N}^{N} K(x_k,\phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si(\frac{\pi\phi^{-1}(t)}{h} - j\pi))f(y_j). \tag{20}$$

By solving above system of nonlinear equations, we obtain approximate solution $y_j,\ j = -N..N$ which corresponds to the exact solution $y(t_j)$ at the sinc points t_j. To obtain an approximation in arbitrary t we use a method similar to the Nyström method for the Volterra integral equations as:

$$y_N(t) = F(t) + h \sum_{j=-N}^{N} K(t,\phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si(\frac{\pi\phi^{-1}(t)}{h} - j\pi))f(y_j). \tag{21}$$

System (20) can be shown in the matrix form

$$\tilde{\mathbf{u}}_{\mathbf{1}} = \mathbf{p} + A\tilde{\mathbf{u}}_{\mathbf{2}}, \tag{22}$$

where,

$$\begin{aligned} A_{kj} &= [K(t_k, \phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si((k-j)\pi))], \\ \tilde{\mathbf{u}}_{\mathbf{1}} &= [y_j]^t, \quad j = -N..N, \\ \tilde{\mathbf{u}}_{\mathbf{2}} &= [f(y_j)]^t, \quad j = -N..N, \\ \mathbf{p} &= [F(t_k)]^t, \quad k = -N..N. \end{aligned} \tag{23}$$

4. Numerical experiments

At first, we give the following algorithm to compute numerical solution of Eq. (1):
Algorithm1:
Step1: Input $a, T, N, \alpha, f(y), h(t), g(t), \phi(t)$
Step2: Execute nested loops
$z := 1$
Take h by relation(17)
for $k = -N..N$ do
$t_k = \phi(kh)$
$ss := 0$;
$eq[z] := 0$
for $j = -N..N$ do
$ss := ss + K(t_k, \phi(jh))\phi'(jh)(\frac{1}{2} + \frac{1}{\pi}Si((k-j)\pi))$
end do
$eq[z] := y_k - h * ss - F(t_k)$.
$z := z + 1$
end do
Step3: Solve nonlinear system of equations $eq[z] = 0,\ z = 1..2N+1$ by Newton method.
Step4: Output $y_j,\ j = -N..N$ and (21) to approximate the solution.

In this section, based on Algorithm 1, two examples are presented to illustrate the effectiveness and importance of proposed method. All programs have been provided by Maple 13. Also, in order to show the error and the accuracy of approximation, we apply the following criteria:
1) Absolute error between the exact and approximated solution (L_∞error norm)is defined by

$$\|.\|_\infty = Max_{i=-N..N}|y(t_i) - y_N(t_i)| \tag{24}$$

2) Run time of program which is showed by T(s),(s means second).

Example 4.1 Consider the Lane-Emden equation with $m = 1$:

Table 1. Results of Example 1 by Sinc collocation method for $m = 1$.

N	$T(s)$	h	$\|.\|_\infty$
5	1.90	0.17087	1.65E-2
10	2.00	0.15475	6.78E-3
15	1.72	0.13019	6.13E-3
30	6.53	0.08820	1.06E-3
60	25.87	0.05565	2.95E-4

Table 2. Results of Example 1 by Sinc collocation method.

N	$N = 5$	$N = 10$	$N = 15$	$N = 25$
y_N	4.77	5.00	4.98	4.999999
	2.70E-3	1.27E-6	1.92E-8	3.23E-13
y_{N-1}	4.55	4.99	4.95	4.9999997
	2.24E-3	5.56E-7	2.44E-8	1.80E-13
y_{N-2}	4.22	4.96	4.90	4.999998
	2.78E-3	4.33E-6	1.24E-7	1.30E-14

$$y'' + \frac{2}{t}y' + y = 0, \quad 0 < t < 5, \tag{25}$$

with condition $y(0) = 1$, $y'(0) = 0$.

To obtain results, we take several numbers of basic functions, such as $N = 5, 10, 15, 30, 60$. Also, in order to have better results we concentrate on the mentioned criteria as run-time (column $T(s)$ in Table 1(second)) and Norm infinity (column $\|.\|_\infty$). The results in Sinc collocation method are shown in Table 1.

This table indicates that as N increases the errors decrease. Although, Sinc approximation is of order exponential but column $\|.\|_\infty$ shows error decreases slowly. It is due to singularity at $t = 0$.

For full discussion, we compare results in table 2 at three points y_N, y_{N-1}, y_{N-2} for different values of N. As seen in table 2, by decreasing N, the error between exact value and approximated value decreases rapidly. For example with $N = 25$ at $t = 4.999999$ error is $3.23E - 13$ which is very noticeable.

Also, we must notice the size of linear system in the case $N = 60$ is 121×121. By considering the run time of program in this case, an important property of this method is remarkable.

Figure 1. shows the exact and approximate solution of this example with $m = 1$ and $N = 3$.

Example 4.2 Consider the following Lane-Emden equation with $m = 3$:

$$y'' + \frac{2}{t}y' + y^3(t) = 0, \quad 0 < t < 1, \tag{26}$$

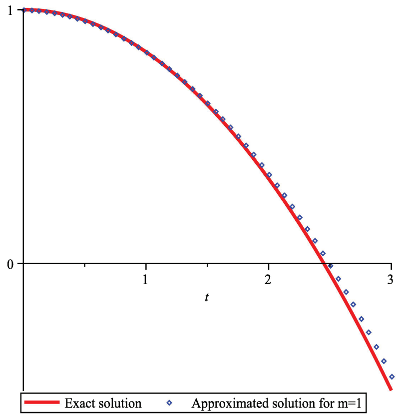

Figure 1. Exact and approximate solution of Example 1 for $m = 1$ and $N = 3$.

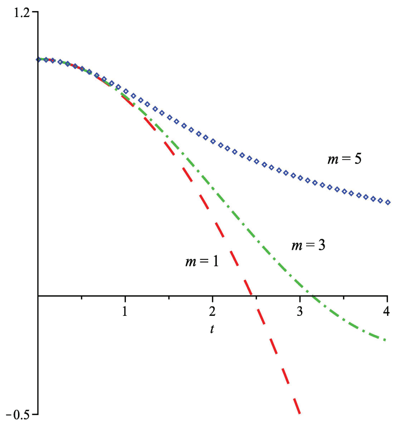

Figure 2. Approximate solution of Example 2 with $m = 3$ in comparison with $m = 1$ and $m = 5$.

with conditions:

$$y(0) = 0,\ y^{'}(0) = 0. \tag{27}$$

the closed-form of the solution is not known, but it is shown that the approximated closed-form solution $u(t) = sech(\frac{t}{\sqrt{3}})$ is attained [2].

Table 3 shows error in three points y_{-N}, y_0, y_N for different values of N. In this example $\|.\|_\infty = 10E-3$, but error in each cell between exact and closed form is very small. The solutions are more accurate than the results in [18].

Figure 2, shows approximate solution for $m = 3$ in comparison with the exact solution by $m = 1$ and $m = 5$.

Table 3. Results of Example 2 by Sinc collocation method.

N	$N=5$	$N=7$	$N=10$
y_{-N}	0.0001	0.0024	0.00001
	1.73E-5	2.23E-7	6.53E-8
y_0	0.5	0.5	0.5
	2.44E-5	5.49E-6	2.63E-7
y_N	0.9823	0.9758	0.9998
	1.42E-5	3.2E-7	4.43E-9
$\|.\|_\infty$	1.79E-3	1.34E-3	1.33E-3

However, results show that the method is practically well. Also, Sinc collocation method gives better accuracy at the computational cost, also the implementing and coding are very easy.

5. Conclusion

We applied the sinc collocation method based on double exponential transformation to Lane-Emden type differential equations. Sinc collocation method in run time has good reliability and efficiency even when singularity occurs at end points. In addition, based on [6–9] this method is portable to other area of problems and easy to programming.

References

[1] C. M. Bender, K. A. Milton, S. S. Pinsky and L. M. Simmons, A new perturbative approach to nonlinear problems, J. Math. Phys, **30** (1989) 1447-1455.

[2] D. Benkoa, D. C. Bilesb, M. P. Robinsonc and J. S. Sprakerc, Numerical approximation for singular second order differential equations, Math. Comput. Model, **49** (2009) 1109-1114.

[3] D. C. Biles, M. P. Robinson and J. S. Spraker, A generalization of the Lane Emden equation, J. Math. Anal. Appl, **273** (2002) 654-666.

[4] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, University of Chicago Press, Chicago, (1939).

[5] H. T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, (1962).

[6] A. Fahim and M. A. Fariborzi Araghi, Numerical solution of Convection-Diusion equations with memory term based on sinc method, Computational Methods for Dierential Equations, **6** (2018) 380-395.

[7] A. Fahim, M. A. Fariborzi Araghi, J. Rashidinia and M. Jalalvand, Numerical solution of Volterra partial integro-differential equations based on sinc-collocation method, Advances in Difference Equations, doi:10.1186/s13662-017-1416-7.

[8] M. A. Fariborzi Araghi and Gh. Kazemi Gelian, Numerical solution of integro differential equations based on double exponential transformation in the sinc-collocation method, App. Math. and Comp. Intel, **1** (2012) 48-55.

[9] M. A. Fariborzi Araghi and Gh. Kazemi Gelian, Numerical solution of nonlinear Hammerstien integral equations via Sinc collocation method based on Double Exponential Transformation, Mathematical Science, **30 (1)** (2013), doi:10.1186/2251-7456-7-30.

[10] S. Haber, Two formulas for numerical indefinite integration, Math. Comp, **60** (1993) 279-296. .

[11] Jh. He, Variational approach to the Lane-Emden equation, Appl. Math. Comput, **143** (2003) 539-541.

[12] G. P. Horedt, Polytropes Applications in Astrophysics and Related Fields, Kluwer Academic Publishers, Dordrecht, (2004).

[13] S. Liao, A new analytic algorithm of Lane-Emden type equations, Appl. Math. Comput, **142** (2003) 1-16.

[14] J. Lund and k. Bowers, Sinc Methods for Quadrature and Differential Equations, SIAM, Philadelphia, (1992).

[15] M. Mori and M. Sugihara, The double exponential transformation in numerical analysis, J. Comput. Appl. Math, **127** (2001) 287-296.

[16] M. Muhammad, A. Nurmuhammad, M. Mori and M. Sugihara, Numerical solution of integral

equations by means of the Sinc collocation based on the DE transformation, J. Comput. Appl. Math, **177** (2005) 269-286.
[17] T. Okayama, T. Matsuo and M. Sugihara, Sinc collocation method for weakly singular Fredholm integral equations of the second kind, J. Comput. Appl. Math, **234** (2010) 1211-1227.
[18] K. Parand, A. Pirkhedri, Sinc-Collocation method for solving Astro- physics equations, A. New. Astron, **15** (2010) 533-537.
[19] N. Shawagfeh, Non perturbative approximate solution for Lane Emden equation, J. Math. Phys. **34** (1993) 4364-4369.
[20] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer, (1993).
[21] M. Sugihara, Optimality of the double exponential formula - functional analysis approach, Numer. Math, **75** (1997) 379-395.
[22] H. Takahasi and M. Mori, Double exponetial formulas for numerical integration, Publ. Res. Inst. Math. Sci, **9** (1974) 721-741.
[23] A. M. Wazwaz, A new algorithm for solving differential equations of Lane Emden type, Appl. Math. Comput, **118** (2001) 287-310.
[24] A. M. Wazwaz, Adomian decomposition method for a reliable treatment of the Emden-Fowler equation, Appl. Math. Comput, **166** (2005) 638-651.
[25] SA. Yousefi, Legendre wavelet method for solving differential equations of Lane- Emden type, Appl. Math. Comput, **181** (2006) 1417-1422.