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Solving Fuzzy Impulsive Fractional Differential Equations by Homotopy Perturbation Method

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Abstract. In this paper, we study semi-analytical methods entitled Homotopy pertourbation method (HPM) to solve fuzzy impulsive fractional differential equations based on the concept of generalized Hukuhara differentiability. At the end first of Homotopy pertourbation method is defined and its properties are considered completely. Then econvergence theorem for the solution are proved and we will show that the approximate solution convergent to the exact solution. Some examples indicate that this method can be easily applied to many linear and nonlinear problems.

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1. Introduction

As a result many things can happen in the real world has a fuzzy meaning. Fuzzy set theory is the significant tool for modeling unknown problems and can be found in many branches of regional, physical, mathematical and engineering sciences.The concept of the fuzzy set theory was first proposed by Zadeh, Zimmerman and Kaleva (see[36, 54, 55]). One of the very important branches of the fuzzy theory is fuzzy differential equations. Then Kandel and Byatt [37] have contributed greatly to the development of fuzzy differential equations. Later many scientists, including

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M. Friedman, et al [16] applied numerical methods to solve this equations. Also the idea of fuzzy differential equations has been studied by scientists and engineers such as T. Allaviranloo, et al [2, 3] and Dong Qiu, et al [43, 44]. They have considered new method to solve fuzzy differential equation based on fuzzy Taylor expansion as one of the branches of fuzzy differential equations . The idea of the theory of fuzzy impulsive differential equation has been emerging as an effective tool area of investigation in recent years (see [45]). Mouffak. Benchohra, et al [15] proposed fuzzy solutions for impulsive differential equations. Subsequently, the basic result for fuzzy impulsive differential equation was defined by S. Vengataasalam, et al [48].

Although the fuzzy fractional differential equations have many branches and many applications and in this research we will restrict our attention to fuzzy impulsive fractional differential equations while fuzzy impulsive fractional differential equations are usually hard to solve analytically and the exact solution is rather difficult to be obtained. There are not too many papers on fuzzy impulsive fractional differential equation up to now. Our aim in this paper is to study the semi-analytical methods for solving fuzzy impulsive fractional differential equations. We will use the Homotopy perturbation method (HPM) based on generalized Hukuhara differentiability to solve a nonlinear and linear fuzzy impulsive fractional differential equations given by

$$^{c}D^{\frac{1}{\alpha}}y(t) = f(t, y), \quad t \in J = [0, T], \quad t \neq t_k \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) \quad t = t_k \quad k = 0, 2, \ldots, m \tag{2}$$

$$y(0) = y_0 \tag{3}$$

In this paper the set of all fuzzy real numbers is denoted R_F. It is clear that $R \subset R_F$. Where $k = 1, 2, \ldots, m,\ 0 < \frac{1}{\alpha} \leq 1$, ${}_cD^{\frac{1}{\alpha}}$ denote the Caputo fractional generalized derivative of order $\frac{1}{\alpha}$, and $f : J \times R_F \to R_F$, is continuous fuzzy function, $I_k : R \to R$, is continuous function, $y_0 \in R_F, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta|_{t=t_k} = y(t_k^+) \ominus_{gH} y(t_k^-)$, $y(t_k^+) = lim_{h\to 0^+} y(t_k + h), and\ y(t_k^-) = lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

This paper shows that some difficult fuzzy impulsive fractional differential equations can be easily calculated by fuzzy fractional differential transform method . The paper is organized as follow:

We describe the basic notation and prelimition in Section 2. The HPM is presented in Section 3. We describe the fuzzy fractional impulsive differential equation in Section 4 . In Section 5, we solving fuzzy impulsive fractional differential equation by Homotopy perturbation method based on the concept of generalized Hukuhara differentiability. Some numerical examples are given to clarify the details and efficiency of the method in Section 6. This paper ends with conclusion in Section 7.

2. Preliminaries

In this section, we introduce Definitions , Propositions, Lemmas,Theorems and provided the new Theorem will be needed throughout the paper.

Definition 2.1 We represent an arbitrary fuzzy number by an ordered pair function $(\underline{u}(r), \overline{u}(r))$, which satisfies the following requirements [36]:
a: $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
b: $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
c: $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.
A crisp number θ is simply represented by $\underline{u}(t,r) = \overline{u}(t,r) = \theta$, $0 \leqslant r \leqslant 1$. We recall that for $a < b < c$ which $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ is determined by a, b, c such that $\underline{u}(t,r) = a + (b-c)r$ and $\overline{u}(t,r) = c - (c-b)r$ are left branch and right branch , for all $r \in [0,1]$.

Definition 2.2 ([12])Let κ_c^n be the space of nonempty compact and convex sets of R^n. The Hukuhara H-difference has been introduced as a set C for which $A \ominus_H B = C \iff A = B + C$ and an important property of $\ominus_H$ is that $A \ominus_H A = \{0\}$ for all $A \in \kappa_c^n$ and $(A+B) \ominus_H B = A, for A, B \in \kappa_c^n$.

Definition 2.3 ([9],[12]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in R_F$ is defined as follows:

$$u \ominus_{gH} v = w \iff \begin{cases} u = v + w & (i) \\ v = u + (-1)w & (ii) \end{cases}$$

The condition for the existence of $u \ominus_{gH} v \in R_F$ are given in([9], [12]). Please note that a function $f[a,b] \to R_F$ so called fuzzy-valued function. The r-level representation of fuzzy-valued function f is expressed by $f_r(t) = [\underline{f}(t,r), \overline{f}(t,r)], t \in [a,b], r \in [0,1]$.

Definition 2.4 ([12]) For $0 < r \leq 1$ denote $[u]_r = \{t \in R | u(t) \geqslant r\} = [\underline{u}(t,r), \overline{u}(t,r)]$ and for $r = 0$ by the closur of the support $[u]_0 = cl\{t | t \in R, u(t) > 0\}$ where cl denotes the closure of a subset. The addition $u + v$ and the scaler multiplication ku are defined as having the level cuts

$$[u+v]_r = [u]_r + [v]_r = \{x + y | x \in [u]_r, y \in [v]_r\},$$

$$[ku]_r = k[u]_r = \{kx | x \in [u]_r\}, [0] = \{0\}, \; for \; r \in [0,1].$$

The subtraction of fuzzy numbers $u - v$ is defined as the addition $u + (-1)v$ where $-v = (-1)v$.

Definition 2.5 The Hausdorff distans betwen fuzzy numbers is given by $d : R_F \times R_F \to R^+ \cup \{0\}$ as in [2].

$$d(u,v) = \sup \max \Big(|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)| \Big)$$

Consider $u, v, w, z \in R_F$ and $\lambda \in R$, then the following properties are well-known for metric d

$$1. \quad d(u \oplus w, v \oplus w) = d(u,v),$$

$$2. \quad d(\lambda u, \lambda v) = |\lambda| d(u,v),$$

$$3. \quad d(u \oplus v, w \oplus z) \leq d(u,w) + d(v,z),$$

4. $d(u \ominus v, w \ominus z) \leq d(u, w) + d(v, z)$.

as long as $u \ominus v$ and $w \ominus z$ exist, where $u, v, w, z \in R_F$. Where ,$\ominus$ is the Hukuhara difference.

Theorem 2.1 ([53]) Let $f(x)$ be a fuzzy- valued function on $[a, \infty)$ and it is represented by $(\underline{f}(x,r), \overline{f}(x,r))$. For any fixed $r \in [0,1]$, assume $\overline{f}(x,r)$and $\underline{f}(x,r)$ are Riemann-integrable on $[a,b]$ for every $b \geqslant a$ and assume there are two positive values $\underline{M}(r)$ and $\overline{M}(r)$ such that

$$\int_a^b |\underline{f}(x,r)|dx \leqslant \underline{M}(r) \ ,$$

and

$$\int_a^b |\overline{f}(x,r)|dx \leqslant \overline{M}(r) \ ,$$

for every $b \geqslant a$. Then $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and is a fuzzy number. Further more, we have:

$$\int_a^{\infty} f(x)dx = \Big(\int_a^{\infty} \underline{f}(x)dx, \int_a^{\infty} \overline{f}(x)dx \Big) \ .$$

Definition 2.6 ([47]) A mapping $f : R_{\times} R_F \longrightarrow R_F$ is called fuzzy continuous at point $(t_0, x_0) \in R_{\times} R_F$ provided for any fixed $r \in [0,1]$ and arbitrary $\epsilon > 0$, there exists an $\delta(\epsilon, r) > 0$, such that

$$d([f(t,x)]_r, [f(t_0,x_0)]_r) < \epsilon,$$

whenever $|t - t_0| < \delta$ and $d([x]_r, [x_0]_r) < \delta(\epsilon, r)$ for all $t \in R_, \ x \in R_F$.

Definition 2.7 ([10]) Given a probability space (Ω, A, P), if $\chi : \Omega \rightarrow R_F$ is an fuzzy random variable such that the random variable max $\{|inf\chi_0|, |sup\chi_0|\}$ is integrable, then the fuzzy expected value of χ corresponds to $E(\chi) \in R_F$ such that

$$\Big(E(\chi)\Big)_r = \Big[E(inf\chi_r), E(sup\chi_r)\Big], \quad \forall r \in [0,1].$$

Definition 2.8 Let $L_\iota = \{d(x(s), y(s)) | d(x(s), y(s))$ is metric spaces of fuzzy random variable with $\int (d(x(s), y(s)))^\iota ds < \infty\}$. The all equivalent element in L_ι are identified and the norm $||d(x(s), y(s))||_\iota$ of an element $d(x(s), y(s)) \in L_\iota$ is difined by

$$||d(x(s), y(s))||_\iota = \left(\int \Big(d(x(s), y(s))\Big)^\iota ds \right)^{\frac{1}{\iota}}$$

Our definition agrees with the one in [48]

Definition 2.9 ([12]) Let $x_0 \in]a, b[$ and h be such that $x_0 + h \in]a, b[$, then the gH-derivative of a function $f :]a, b[\rightarrow R_F$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h} [f(x_0 + h) \ominus_{gH} f(x_0)] \tag{4}$$

If $f'_{gH}(x_0) \in R_F$ satisfying (4) exists, we say that f is generlized Hukuhara differentiable at x_0. Also,we say that f is $[(i) - gH] - differentiable$ at x_0 if

$$[f'_{gH}(x_0)]_r = [\underline{f}'_{gH}(x_0, r), \overline{f}'_{gH}(x_0, r)], \quad 0 \le r \le 1. \tag{5}$$

And that f is $[(ii) - gH] - differentiable$ at x_0 if

$$[f'_{gH}(x_0)]_r = [\overline{f}'_{gH}(x_0, r), \underline{f}'_{gH}(x_0, r)], \quad 0 \le r \le 1. \tag{6}$$

Definition 2.10 ([42]) The Beta function $B(a, b)$ in two variables $a, b \in C$ is defined

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{7}$$

C belonging complex numbers and Γ is the gamma function and defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Lemma 2.11 $\forall\ \alpha > 0$ and $\gamma > -1$

$$\int_0^t (t - s)^{\frac{k}{\alpha}-1} s^\gamma ds = \frac{\Gamma(\frac{k}{\alpha})\Gamma(\gamma + 1)}{\Gamma(\frac{k}{\alpha} + \gamma + 1)} t^{\frac{k}{\alpha}+\gamma}.$$

Proof. Lemma 2.6 [49].

Definition 2.12 ([7]) Let $f : [a, b] \to R_F$,the fuzzy Riemann-Liouville integral of fuzzy -valued f is defined as follows:

$$(I_a^\alpha)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x - t)^{1-\alpha}} \mathrm{d}t \quad , x > a$$

for $a \le t \le x$ and $0 < \frac{1}{\alpha} \le 1$;

Theorem 2.2 ([7]) Let $f \in C^F[a, b] \bigcap L^F[a, b]$ is a fuzzy-valued function. The fuzzy Riemann-Liouville integral of fuzzy -valued f can be expressed as follows:

$$(I_a^\alpha f)(x, r) = [(I_a^\alpha \underline{f}(x, r), (I_a^\alpha \overline{f}(x, r)] \quad 0 \le r \le 1$$

where

$$(I_a^\alpha \underline{f})(x, r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\underline{f}(x, r)}{(x - t)^{1-\alpha}} \mathrm{d}t \quad , (I_a^\alpha \overline{f})(x, r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\overline{f}(x, r)}{(x - t)^{1-\alpha}} \mathrm{d}t$$

We denote $C^F[a, b]$ as the space of all continuous fuzzy-valued function on $[a, b]$. Also, we denote the space of all Lebesgue integrable fuzzy-valued function on the bounded interval $[a, b] \subset R$ by $L^F[a, b]$.

Definition 2.13 ([7]) Let $f : [a, b] \to R_F$, $x_0 \in (a, b)$ and $\phi(x) = \int_a^x \frac{f(t)}{(x-t)^{\frac{1}{\alpha}}} \mathrm{d}t$. For all $0 \le r \le 1$, $h > 0$, $f(x)$ is called fuzzy Riemann-Liouville fractional differentiable of order $0 < \frac{1}{\alpha} < 1$, at x_0, if there exists an element $({}^{RL}D_a^{\frac{1}{\alpha}})(x_0) \in E$, such that:

$$(i) \qquad ({}^{RL}D_a^{\frac{1}{\alpha}})(x_0) = \lim_{h\to 0} \frac{\phi(x_0+h) \ominus_{gH} \phi(x_0)}{h} = \lim_{h\to 0} \frac{\phi(x_0) \ominus_{gH} \phi(x_0-h)}{h} \tag{8}$$

$$(ii) \qquad ({}^{RL}D_a^{\frac{1}{\alpha}})(x_0) = \lim_{h\to 0} \frac{\phi(x_0) \ominus_{gH} \phi(x_0+h)}{-h} = \lim_{h\to 0} \frac{\phi(x_0-h) \ominus_{gH} \phi(x_0)}{-h} \tag{9}$$

$$(iii) \qquad ({}^{RL}D_a^{\frac{1}{\alpha}})(x_0) = \lim_{h\to 0} \frac{\phi(x_0+h) \ominus_{gH} \phi(x_0)}{h} = \lim_{h\to 0} \frac{\phi(x_0-h) \ominus_{gH} \phi(x_0)}{-h} \tag{10}$$

$$(iv) \qquad ({}^{RL}D_a^{\frac{1}{\alpha}})(x_0) = \lim_{h\to 0} \frac{\phi(x_0) \ominus_{gH} \phi(x_0+h)}{-h} = \lim_{h\to 0} \frac{\phi(x_0) \ominus_{gH} \phi(x_0-h)}{-h} \tag{11}$$

For sake of simplicity , we say that the fuzzy-valued function f is ${}^{RL}[(i) - gH]-$ differentiable if it is differentiable in case (i) of Dfinition 2.13 and ${}^{RL}[(ii)-gH]$-differentiable if it is differentiable in case (ii) of Dfinition 2.13 and so on for the other case.

Definition 2.14 ([7]) Let $f(x) \in C^F[a, b] \cap L^F[a, b]$, and $x_0 \in (a, b), \frac{1}{\alpha} > 0$, and $(n = [\frac{1}{\alpha}] + 1)$ such that for all $0 \le r \le 1$, then the Caputo definition of fractional differential operator is given by

$$D_{a^+}^{\frac{1}{\alpha}} f(t,r) = \begin{cases} \frac{d^n f(t,r)}{dt^n}, & \frac{1}{\alpha} = n \in Z^+, \\ \frac{1}{\Gamma(n-\frac{1}{\alpha})} \int_a^x \frac{f^{(n)}(t,r)\mathrm{d}t}{(x-t)^{\frac{1}{\alpha}-n+1}}, & m-1 < \frac{1}{\alpha} < m, \quad m \in N. \end{cases} \tag{12}$$

where

$$D_{a^+}^{\frac{1}{\alpha}} \underline{f}(x_0,r) = \frac{1}{\Gamma(n-\frac{1}{\alpha})} \int_a^x \frac{\underline{f}^{(n)}(t,r)\mathrm{d}t}{(x-t)^{\frac{1}{\alpha}-n+1}}, \quad m-1 < \frac{1}{\alpha} < m \ , \quad m \in N. \tag{13}$$

and

$$D_{a^+}^{\frac{1}{\alpha}} \overline{f}(x_0,r) = \frac{1}{\Gamma(n-\frac{1}{\alpha})} \int_a^x \frac{\overline{f}^{(n)}(t,r)\mathrm{d}t}{(x-t)^{\frac{1}{\alpha}-n+1}}, \quad m-1 < \frac{1}{\alpha} < m \ , \quad m \in N. \tag{14}$$

$D_a^{\frac{1}{\alpha}}$ is the Caputo fractional derivative of order $\frac{1}{\alpha}$.

Definition 2.15 Let $f : [a, b] \to R_F$ and $x_0 \in (a, b)$, with $\underline{f}(x, r)$ and $\overline{f}(x, r)$ both differentiable at x_0 for all $r \in [0, 1]$. We say that f is $[(i)-gH]$-differentiable at x_0 if

$$[D_{a|x}^{\frac{1}{\alpha}} f_{i.gH}](x_0,r) = [D_{a|x}^{\frac{1}{\alpha}} \underline{f}(x_0,r), D_{a|x}^{\frac{1}{\alpha}} \overline{f}(x_0,r)] \tag{15}$$

$-f$ is $[(ii)-gH]$-differentiable at x_0 if

$$[D^{\frac{1}{\alpha}}_{a|x} f_{ii.gH}](x_0,r) = [D^{\frac{1}{\alpha}}_{a|x} \overline{f}(x_0,r), D^{\frac{1}{\alpha}}_{a|x} \underline{f}(x_0,r)] \tag{16}$$

3. Analysis of the homotopy perturbation method

To explain the HPM, we consider a general integral equation

$$Lu = 0 \tag{17}$$

where L is a integral operator. Define a convex homotopy $H(y,p)$ by

$$H(y,p) = (1-p)F(y) + pL(y) = 0, \quad p \in [0,1], \tag{18}$$

where $F(y)$is a functional operator with solution y_0 which can be obtained easily. It is obvious that

$$H(y,0) = F(y) = 0, \quad H(y,1) = L(y) = 0, \tag{19}$$

and the process of changing p from $0to1$ is just that of changing y from $y_0$to u. In topology, this is called deformation;$F(y)$ and $L(y)$are called homotopies. According to the HPM, we can use the embedding parameter$p$ as a small parameter, and assume that the solution of Eq(18) can be written as a power series in p:

$$y = u_0 + pu_1 + p^2u_2 + ; \tag{20}$$

when$p \to 1$, the approximate solution of Eq. (17) is obtained with

$$u = \lim_{p\to 1} y = u_0 + u_1 + u_2 + \tag{21}$$

The series (21) is convergent for most cases; however, the rate of convergence depends upon the nonlinear operator$L$ [23].

4. Fuzzy impulsive fractional differential equation

Consider the following fuzzy impulsive fractional differential equation

$$^{c}D^{\frac{k}{\alpha}} y(t) = f(t,y), \quad t \in J = [0,T], \quad t \neq t_k, \; m-1 < \frac{1}{\alpha} < m, \quad m \in N \tag{22}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) \tag{23}$$

$$y(0) = y_0 \tag{24}$$

where $0 < \frac{1}{\alpha} < 1$ is a real number and the operator $_{c}D^{\frac{1}{\alpha}}$ denote the Caputo fractional generalized derivative of order $\frac{1}{\alpha}$, and $f : J \times R_F \to R_F$, is continuous fuzzy function. Also $I : R \to R$ is continuous function. In this section, using fractional differential transform method for fuzzy impulsive fractional differential equation (22) under the conditions (23) and (24) with fuzzy initial conditions is solved under $_{cf}[gH]$-differentiability.

Lemma 4.1 ([8],[35]) The initial value problem (22) under the conditions (23) and (24) is equivalent to one of the following integral equations:

$$y(t) = y_0 \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds, \quad \text{if} \quad t \in [0, t_1] \tag{25}$$

if $y(t)$ be $^{cf}[(i) - gH]$-differentiable,

$$y(t) = y_0 \ominus_{gH} (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds, \quad \text{if} \quad t \in [0, t_1] \tag{26}$$

if $y(t)$ be $^{cf}[(ii) - gH]$-differentiable,

$y(t) =$

$$\begin{cases} y_0 \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds, & \text{if } t \in [0, t_1] \\ y_0 \ominus_{gH} (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus_{gH} (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} & \\ \int_{t_k}^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus_{gH} (-1) \sum_{i=1}^k I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}] \end{cases} \tag{27}$$

if there exists a point $t_1 \in (0, t_{k+1})$ such that $y(t)$ is $[(i) - gH]-$differentiable on $[0, t_1]$ and $[(ii) - gH]-$differentiable on (t_1, t_{k+1}). □

Theorem 4.1 Assume that
(H_1) There exists a constant $0 \leq l$ such that $d(f(t, u), f(t, \overline{u})) \leq ld(u, \overline{u})$, for each $t \in [0, T]$, and each $u, \overline{u} \in R_F$
(H_2) There exists a constant $0 \leq l^*$ such that $d(I_k(u), I_k(\overline{u})) \leq l^* d(u, \overline{u})$, for each $u, \overline{u} \in R_F$, and $k = 1, 2, \ldots, m$.
if

$$[\frac{T^{\frac{k}{\alpha}} l(m+1)}{\Gamma(\frac{k}{\alpha}+1)} + ml^*] < 1 \tag{28}$$

Such that T is very small numbers therefore,Eqs.(22)-(24) has a unique solution on $[0, T]$.
Proof. We transform the problems (22)-(24) into a fixed point problem. Now we introduce $pc(J, R_F) = \{y : J \to R_F : y \in c((t_k, t_{k+1}], R_F), k = 0, 1, \ldots, m$ and there exist $y(t_k^-)$ and $y(t_k^+), k = 1, \ldots, m$ with $y(t_k^-) = y(t_k)\}$, that is a closed and convex subset of the Banach space of all continuous function on $(0, k+1]$. Therefore, pc is a Banach space, too. We suppose that the solution of problems $(22) - (24)$ is in case $[i - gH]-$differentiability and $[ii - gH]-$differentiability is equivalent to integral equation (27). So, we define a mapping $F : pc(J, R_F) \to pc(J, R_F)$, that given by

$$\begin{aligned} F(y)(t) = y_0 &\ominus_{gH} (-1) \tfrac{1}{\Gamma(\frac{k}{\alpha})} \textstyle\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \\ &\ominus_{gH} (-1) \tfrac{1}{\Gamma(\frac{k}{\alpha})} \textstyle\int_{t_k}^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus_{gH} (-1) \sum_{i=1}^k I_i(y(t_i^-)) \end{aligned} \tag{29}$$

Therefore, the fixed point of the operator F is the solution of the problems $(22)-(24)$. We shall use the Banach contraction principle to prove that F has a fixed point. We will show that F is a contraction map. Let $x, y \in pc(J, R_F)$. Then, for each $t \in [0, T]$, and using the Definition 2.8 and Lemma 2.11 we have

$$\begin{aligned}
d(F(x)(t), F(y)(t)) \leq & \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} d(f(s, x(s)), f(s, y(s)))ds \\
& \ominus_{gH}(-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} d(f(s, x(s)), f(s, y(s)))ds \\
& \ominus_{gH}(-1) \sum_{i=1}^{k} d(I(s, x(t_k^-)), I(s, y(t_k^-))) \\
\leq & \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} l d(x(s), y(s))ds \\
& \ominus_{gH}(-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} l d(x(s), y(s))ds \\
& \ominus_{gH}(-1) \sum_{i=1}^{k} l^* d(x(t_k^-), y(t_k^-)) \\
= & \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} l \Big(\int_0^t (d(x(s), y(s)))^\alpha ds \Big)^{\frac{1}{\alpha}} \Big(\frac{\Gamma(\frac{k}{\alpha}) T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)} \Big) \\
& + \frac{1}{\Gamma(\frac{k}{\alpha})} l \Big(\int_0^t (d(x(s), y(s)))^\alpha ds \Big)^{\frac{1}{\alpha}} \Big(\frac{\Gamma(\frac{k}{\alpha}) T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)} \Big) \\
& m l^* \Big(\sum_{i=0}^{k} (d(x(s), y(s)))^\alpha ds \Big)^{\frac{1}{\alpha}} \\
\leq & \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} l ||d(x, y)||_\alpha \Big(\frac{\Gamma(\frac{k}{\alpha}) T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)} \Big) + \frac{1}{\Gamma(\frac{k}{\alpha})} l ||d(x, y)||_\alpha \Big(\frac{\Gamma(\frac{k}{\alpha}) T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)} \Big) \\
& + m l^* ||d(x, y)||_0 \\
= & \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} l ||d(x, y)||_\alpha \Big(T^{\frac{k}{\alpha}} \frac{\Gamma(\frac{k}{\alpha})}{\Gamma(\frac{k}{\alpha}+1)} \Big) \\
& + \frac{1}{\Gamma(\frac{k}{\alpha})} l ||d(x, y)||_\alpha \Big(T^{\frac{k}{\alpha}} \frac{\Gamma(\frac{k}{\alpha})}{\Gamma(\frac{k}{\alpha}+1)} \Big) + m l^* ||d(x, y)||_\alpha \\
= & Ml \quad ||d(x, y)||_\alpha \Big(\frac{T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)} \Big) + l ||d(x, y)||_\alpha \Big(\frac{T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)} \Big) \\
& + m l^* ||d(x, y)||_\alpha \\
= & \quad [\frac{T^{\frac{k}{\alpha}} l(m+1)}{\Gamma(\frac{k}{\alpha}+1)} + m l^*] ||d(x, y)||_\alpha .
\end{aligned}$$

Therefore,

$$||d(F(x), F(y))||_\alpha \leq [\frac{T^{\frac{k}{\alpha}} l(m+1)}{\Gamma(\frac{k}{\alpha}+1)} + m l^*] ||d(x, y)||_\alpha . \tag{30}$$

Consequently by (28), F is a contraetion. As a consequence of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of problems $(22)-(24)$. □

5. Solving fuzzy impulsive fractional differential equation by homotopy perturbation method

Using Lemma (4.1), the solution of problems (22)-(24) is equivalent to integral Eq. (27) .We show how HPM applied to the following integral equations. Now consider the impulsive fractional defferential equation (27) by integration from $[0, k+1]$, we have:

$$y(t)=\begin{cases} y_0 \oplus \frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,y(s))ds, & \text{if } t\in[0,t_1],\\ y_0 \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds & \\ \quad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds & \\ \quad \ominus_{gH}(-1)\sum_{i=1}^{k} I_i(y(t_i^-)), & \text{if } t\in(t_k,t_{k+1}], \end{cases} \tag{31}$$

by impulsive effect, we have:

$$\Delta y(t) = y(0^+)\ominus_{gH} y(0^-) = I_0(y(0^-)), \quad \Longrightarrow y(0^+) = I_0(y(0^-))\oplus y(0^-) \tag{32}$$

By substituting Eq. (32) into Eq. (31) we have

$$y(t)=\begin{cases} I_0(y(0^-)) \oplus y(0^-) \oplus \frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,y(s))ds, & \text{if } t\in[0,t_1],\\ I_0(y(0^-)) \oplus y(0^-) & \\ \qquad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds & \\ \qquad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds & \\ \qquad \ominus_{gH}(-1)\sum_{i=1}^{k} I_i(y(t_i)), & \text{if } t\in(t_k,t_{k+1}], \end{cases} \tag{33}$$

In Eqs. (33) we define $y(t^-) = y(t)$. Thus

$$y(t)=\begin{cases} I_0(y(0)) \oplus y(0) \oplus \frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,y(s))ds, & \text{if } t\in[0,t_1],\\ I_0(y(0)) \oplus y(0) & \\ \qquad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds & \\ \qquad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,y(s))ds & \\ \qquad \ominus_{gH}(-1)\sum_{i=1}^{k} I_i(y(t_i)), & \text{if } t\in(t_k,t_{k+1}], \end{cases} \tag{34}$$

Let

$$L(u)=\begin{cases} u(t) \ominus_{gH} I_0(u(0)) \ominus_{gH} u(0) & \\ \qquad \ominus_{gH}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,u(s))ds, & \text{if } t\in[0,t_1],\\ u(t) \ominus_{gH} I_0(u(0)) \ominus_{gH} u(0) & \\ \qquad \oplus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,u(s))ds & \\ \qquad \oplus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,u(s))ds & \\ \qquad \oplus(-1)\sum_{i=1}^{k} I_i(u(t_i)), & \text{if } t\in(t_k,t_{k+1}], \end{cases} \tag{35}$$

with soluation $u(t) = y(t)$, we can define a homotopy $H(u,p)$ by

$$H(u,0) = u(t) - I_0(u(0)) - u(0) = F(u) \tag{36}$$

$$H(u,1) = L(u) \tag{37}$$

where $F(u)$ is a functional operator with soluation u_0. We choose a convex homotopy

$$H(u,p) = (1-p)F(u) + pL(u) = 0 \tag{38}$$

and continuously trace an implicity define curve from a strating point $H(u_0)$ to a solution $H(y,1)$. The embedding parameter$p$ monotonically increases from zero to one as the trivial problem $F(u) = 0$ is continuously deformed to the original problem$L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter ([23]-[34]):

$$u = u_0 + pu_1 + p^2 u_2 + \dots \tag{39}$$

when $p \to 1$ corresponds to Eq(38) and gives an approximation to the solution of Eq(35) as:

$$y = \lim_{p\to 1} u = u_0 + u_1 + \dots \tag{40}$$

The series (40) converges in most cases and the rate of convergence depends on $L(u)$.Taking $F(u) = u(t) \ominus_{gH} I_0(u(0)) \ominus_{gH} u(0)$ and substituting Eq(39) in to Eq (38) and equating the terms with identical power of p, we obtain

$$p^0 : \begin{cases} u(t) \ominus_{gH} I_0(u(0)) \ominus_{gH} u(0) = 0 \\ \quad \Longrightarrow u_0(t) = I_0(u(0)) \oplus u(0) & \text{if } t \in [0,t_1], \\ u(t) \ominus_{gH} I_0(u(0)) \ominus_{gH} u(0) \oplus (-1)\sum_{i=1}^{k} I_i(u(t_i)) = 0 \\ \quad \Longrightarrow u_0(t) = I_0(u(0)) \oplus u(0) \ominus_{gH} (-1)\sum_{i=1}^{k} I_i(u(t_i)), & \text{if } t \in (t_1, t_{k+1}], \end{cases} \tag{41}$$

$$p^1 : \quad u_1 = \begin{cases} \frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,u_0(s))ds, & \text{if } \quad t \in [0,t_1], \\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1} f(s,u_0(s))ds \ominus_{gH} \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1} f(s,u_0(s))ds, & \text{if } \quad t \in (t_k, t_{k+1}], \end{cases} \tag{42}$$

$$\vdots$$

and in general we have

$$u_0(t) = \begin{cases} u_0(t) = I_0(u(0)) \oplus u(0), & \text{if } \quad t \in [0,t_1], \\ u_0(t) = I_0(u(0)) \oplus u(0) \ominus_{gH} (-1)\sum_{i=1}^{k} I_i(u(t_i)), & \text{if } \quad t \in (t_1, t_{k+1}], \end{cases} \tag{43}$$

$$u_n = \begin{cases} \frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,u_{n-1}(s))ds, & \text{if } \quad t \in [0,t_1], \\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1} f(s,u_{n-1}(s))ds \ominus_{gH} \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1} f(s,u_{n-1}(s))ds, & \text{if } \quad t \in (t_1, t_{k+1}], \end{cases} \tag{44}$$

The relations above have been obtained with the assumption of the convergence of series (39).The conditions for such convergence are discussed in the following theorem:

Theorem 5.1 Let the functions $K = (t-s)^{\frac{k}{\alpha}-1}$ and $f = f(s, u_n(s))$, appearing in Eq.(34), be continuous in the respective domains, i.e. $K, f \in C([a,b] \times [a,b])$. If additionally the following inequality:

$$|\lambda| M_1 < \frac{1}{b-a} \tag{45}$$

is satisfied and as the initial approximation u0, a function continuous in the interval $[a,b]$ is chosen, then series (39), in which the functions u_{n+1} are determined by means of relations (43)-(44), is uniformly convergent in the interval $[a,b]$ for each $p \in [0,1]$.

Proof. Certainly,K and f are bounded; this means that there exist the positive numbers $M1$ and $N1$ such that

$$|K(x,t)| \leq M1 \qquad , |I(u_0(s))| \leq N1 \qquad \forall x, t \in [a,b]. \tag{46}$$

Let $y_0 \in C[a,b]$. Therefore there exists a positive number N_0 such that

$$|y_0(x)| \leq N_0 \quad \forall x \in [a,b]. \tag{47}$$

The assumptions made imply the following estimations:

$$\begin{aligned} &|y_0(x)| = |u_0(x)| \\ &= \begin{cases} |I_0(u(0)) \oplus u(0)| \leq N_0 + N_1, & \text{if } t \in [0, t_1], \\ |I_0(u(0)) \oplus u(0) \ominus_{gH} (-1) \sum_{i=1}^{k} I_i(u(t_i)| \leq N_0 + N_1 + N_2, & \text{if } t \in (t_1, t_{k+1}], \end{cases} \end{aligned} \tag{48}$$

$$|u_1(x)| = \begin{cases} \left| \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, u_0(s)) ds \right| \leq \frac{N_0 (t-0) |\lambda| M}{\Gamma(\frac{k}{\alpha})}, & \text{if} \quad t \in [0, t_1], \\ \Big| \ominus_{gH} (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, u_0(s)) ds \ominus_{gH} & \\ (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} f(s, u_0(s)) ds \Big| & \\ \leq \frac{N_0 |\lambda| M}{\Gamma(\frac{k}{\alpha})} \Big(\sum_{i=1}^{k} (t_i - t_{i-1}) + (t - t_k) \Big), & \text{if} \quad t \in (t_1, t_{k+1}], \end{cases} \tag{49}$$

where $B := N_0 + N_1 + N_0(b-a)(|\lambda| M1)$. In general we have

$$|u_{n+1}(x)| \leq B(b-a)^{n-1} (|\lambda| M1)^{n-1} \quad , x \in [a,b]. \tag{50}$$

In this way, for the series considered, (39) we get, for $p \in [0,1]$,

$$\sum_{n=0}^{\infty} p^n u_n \leq \sum_{n=0}^{\infty} |u_n| \leq a_0 + \sum_{n=1}^{\infty} a_n \tag{51}$$

where

$$a_0 = N_0 + N_1, a_n = B(b-a)^{n-1} (|\lambda| M1)^{n-1} \tag{52}$$

The last series in the above estimation is the convergent geometric series possessing the common ratio $q = (|\lambda| M)(b-a) < 1$. Hence, series considered, (39), is uniformly convergent in the interval $[a,b]$ for each $p \in [0,1]$.

Theorem 5.2 The exact solution of Eqs $(22)-(24)$ could be represented by

$$y(t)=\begin{cases}\sum_{n=1}^{\infty}u_n(t), & \text{if}\quad t\in[0,t_1],\\ \sum_{n=1}^{\infty}u_n(t), & \text{if}\quad t\in(t_1,t_{k+1}],\end{cases}\tag{53}$$

Also, the approximate solution $y(t)$ can be obtained by taking finitely many terms in the series representation of $y(t)$ so,

$$\widehat{y}(t)=\begin{cases}\sum_{n=1}^{N}u_n(t), & \text{if}\quad t\in[0,t_1],\\ \sum_{n=1}^{N}u_n(t), & \text{if}\quad t\in(t_1,t_{k+1}],\end{cases}\tag{54}$$

and $\widehat{y}(t)$ convergence uniformly to the exact solution $y(t)$.

Proof. Let $y(t)$ be solution of Eqs.$(22)-(24)$. From Eq. (40) . $y(t)$ could be expressed by series as follow

$$y(t)=\begin{cases}\sum_{n=1}^{\infty}u_n(t), & \text{if}\quad t\in[0,t_1],\\ \sum_{n=1}^{\infty}u_n(t), & \text{if}\quad t\in(t_1,t_{k+1}],\end{cases}\tag{55}$$

Substiting Eq. (44)into Eq. (55) , where

$$u_0(t)=\begin{cases}u_0(t)=I_0(u(0))\oplus u(0), & \text{if}\quad t\in[0,t_1],\\ u_0(t)=I_0(u(0))\oplus u(0)\ominus_{gH}(-1)\sum_{i=1}^{k}I_i(u(t_i)), & \text{if}\quad t\in(t_1,t_{k+1}],\end{cases}\tag{56}$$

And

$$u_n=\begin{cases}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds, & \text{if}\quad t\in[0,t_1],\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds\ominus_{gH} & \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds, & \text{if}\quad t\in(t_1,t_{k+1}],\end{cases}\tag{57}$$

Also let us $y(t)$ and $\widehat{y}(t)$be exact and approximate solution of the problems $(22)-(24)$. We will show that,

$$d(y(t),\widehat{y}(t))\leq\sum_{n=1}^{N}||d(u_{n-1}(t),\widehat{y_{n-1}}(t))||_{\alpha}\tag{58}$$

Therefore if $y(t)$ and $\widehat{y}(t)$ be exat and approximate solution of the problems $(22)-(24)$,without loss of generality suppose that there exists a point $t_1\in(0,t_{k+1})$ such that $y(t)$ is $[(i)-gH]-$differentiable on $[0,t_1]$ and $[(ii)-gH]-$differentiable on (t_1,t_{k+1}). Using Lemma (4.1), the solution of problems $(22)-(24)$ in this case is equivalent to integral Eq. (27). By using Definition 2.8 and Lemma 2.11 we have,

$$y(t)=\begin{cases}\sum_{n=1}^{\infty}\left(\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds\right), & t\in[0,t_1]\\ \sum_{n=1}^{\infty}\Big(\ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds & \\ \qquad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds\ominus_{gH}\Big), & t\in(t_k,t_{k+1}]\end{cases} \tag{59}$$

and

$$\widehat{y}(t)=\begin{cases}\sum_{n=1}^{N}\left(\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{u_{n-1}}(s))ds\right), & t\in[0,t_1],\\ \sum_{n=1}^{N}\Big(\ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,\widehat{u_{n-1}}(s))ds & \\ \qquad \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{u_{n-1}}(s))ds\Big), & t\in(t_k,t_{k+1}]\end{cases} \tag{60}$$

thus

$$d(y(t),\widehat{y}(t))=$$

$$\begin{cases} d\Big(\sum_{n=1}^{\infty}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds,\sum_{n=1}^{N}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{u_{n-1}}(s))ds\Big)\\ \leq|\sum_{n=1}^{N}\frac{l}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}d(u_{n-1}(s),\widehat{u_{n-1}}(s))ds|\\ \leq\sum_{n=1}^{N}\frac{l}{\Gamma(\frac{k}{\alpha})}\Big(\int_0^t(t-s)^{\frac{k-\alpha}{\alpha}}ds\Big)\Big(\int_0^t(d(u_{n-1}(s),u_{n-1}N(s)))ds)^{\alpha}\Big)^{\frac{1}{\alpha}}\\ =\sum_{n=1}^{N}\frac{l}{\Gamma(\frac{k}{\alpha})}\Big(\frac{\Gamma(\frac{k}{\alpha})T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)\Big(\int_0^t(d(u_{n-1}(s),\widehat{u_{n-1}}(s)))ds)^{\alpha}\Big)^{\frac{1}{\alpha}}\\ =\sum_{n=1}^{N}\frac{lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}||d(u_{n-1}(t),\widehat{u_{n-1}}(t))||_{\alpha},\ t\in[0,t_1]\\ d\Big(\ominus_{gH}(-1)\sum_{n=1}^{\infty}\Big(\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,u_{n-1}(s))ds\ominus_{gH}\Big),\\ \ominus_{gH}(-1)\sum_{n=1}^{N}\Big(\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,\widehat{u_{n-1}}(s))ds\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{u_{n-1}}(s))ds\Big)\Big)\\ \leq d\Big(\sum_{n=1}^{N}\Big(\ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}d(f(s,u_{n-1}(s)),f(s,\widehat{u_{n-1}}(s)))ds\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}d(f(s,u_{n-1}(s)),f(s,\widehat{u_{n-1}}(s)))ds\Big)\Big)\\ \leq d\Big(\sum_{n=1}^{N}\Big(\ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}d(f(s,u_{n-1}(s)),f(s,\widehat{u_{n-1}}(s)))ds\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}d(f(s,u_{n-1}(s)),f(s,\widehat{u_{n-1}}(s)))ds\Big)\Big)\\ \leq\sum_{n=1}^{N}\Big(\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}ld(u_{n-1}(s),\widehat{u_{n-1}}(s))ds\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}ld(u_{n-1}(s),\widehat{u_{n-1}}(s))ds\Big)\\ =\ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\Big(\frac{\Gamma(\frac{k}{\alpha})lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)\Big(\int_0^t(d(u_{n-1}(s),\widehat{u_{n-1}}(s))ds)^{\alpha}\Big)^{\frac{1}{\alpha}}\\ \ominus_{gH}(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\Big(\frac{\Gamma(\frac{k}{\alpha})lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)\Big(\int_0^t(d(u_{n-1}(s),\widehat{u_{n-1}}(s))ds)^{\alpha}\Big)^{\frac{1}{\alpha}}\\ =\sum_{n=1}^{N}\Big(\ominus_{gH}(-1)\frac{MlT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}||u_{n-1}(t),\widehat{u_{n-1}}(t)||_{\alpha}\\ \ominus_{gH}(-1)\frac{MlT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}||u_{n-1}(t),\widehat{u_{n-1}}(t)||_{\alpha}\Big)\\ =\Big(\Big(\frac{MlT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}+\frac{lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)||u_{n-1}(t),\widehat{u_{n-1}}(t)||_{\alpha}\Big)\\ =\sum_{n=1}^{N}\Big((\frac{(M+1)lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)})||u_{n-1}(t),\widehat{u_{n-1}}(t)||_{\alpha}\Big),t\in(t_k,t_{k+1}]\end{cases} \tag{61}$$

Table 1. Numerical results of Example 1 for $\underline{y_{i.gH}}(t)$ *and* $\underline{y_{ii.gH}}(t)$.

r/t	0.0	0.2	0.4	0.6	0.8	0.1	1.2	1.40	1.6	1.8
0.1	0	0.008612	0.048718	0.134252	0.275593	6.472847	6.739047	7.088843	7.529535	8.067877
0.2	0	0.003163	0.017894	0.049310	0.101224	1.353031	1.619232	1.969028	2.40972	2.948062
0.3	0	0.001155	0.006534	0.018005	0.036959	0.669863	0.936063	1.285859	1.72655	2.264893
0.4	0	0.00031	0.001751	0.004825	0.009905	0.470790	0.736991	1.086787	1.527471	2.065821
0.5	0	3.30E-05	0.000187	0.000514	0.001055	0.421958	0.688159	1.037954	1.478646	2.016988
0.6	0	-1.49E-05	-8.40E-05	-0.000231	-0.000475	0.413813	0.680014	1.029810	1.470502	2.008844
0.7	0	-6.24E-05	-0.000353	-0.000972	-0.001995	0.403001	0.669202	1.018998	1.459690	1.998032
0.8	0	-9.53E-05	-0.000540	-0.001486	-0.003051	0.395892	0.662093	1.011889	1.452581	1.990923
0.9	0	-7.74E-05	-0.000438	-0.001206	-0.002476	0.400272	0.666472	1.016268	1.456960	1.995302
1	0	0	0	0	0	0.418116	0.684316	1.034113	1.474805	2.013146

It follows that

$$d(y(t), \widehat{y}(t)) \leq \sum_{n=1}^{N} ||d(u_{n-1}(t), \widehat{u_{n-1}}(t))||_\alpha$$

where M_1, M_2 are constants. Hence $||d(u_{n-1}(t), \widehat{u_{n-1}}(t))||_0 \to 0 \ as \ n \to \infty$, the approximate solution convergence uniformly to the exact solution $y(t)$. The proof for othere case is similar. □

6. Numerical examples

We demonstrate the effectiveness of the fuzzy fractional differential transform method,for solving fuzzy impulsive fractional differential equations by the following some examples.

Example 6.1 Let us consider the fuzzy impulsive fractional equation,

$${}^cD^{\frac{k}{\alpha}} y(t) = \frac{ty^2(t)}{(3+t)(1+y^2(t))}, \ t \in J := [0,1], \quad t \neq \frac{1}{2}, \quad m-1 < \frac{1}{\alpha} < m, \ m \in N, \tag{62}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1^-}{2})|}{2 + |y(\frac{1^-}{2})|}, \tag{63}$$

$$y(0) = [\tilde{0}, \tilde{0}] \tag{64}$$

$$Y_{\frac{1}{\alpha}}(0) = [\tilde{0}, \tilde{0}] \tag{65}$$

Set

$$I_k(t) = \frac{t}{t+2}, t \in [0, \infty) \tag{66}$$

$$f(t, y(t)) = \Big[\frac{t^3(r-1)}{(3+t)(1+t^2)}, \frac{t^3(1-r)}{(3+t)(1+t^2)}\Big] \tag{67}$$

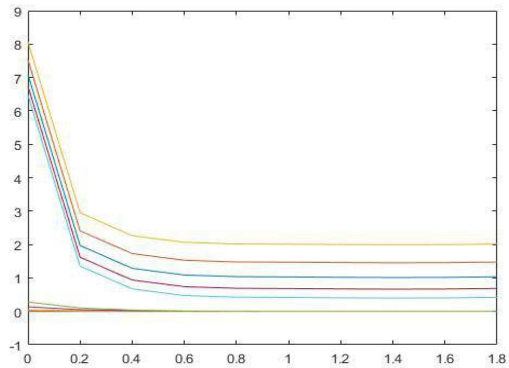
Without loss of generality suppose $y(t)$ is $[(i) - gH]-$differentiable in $[0, 1]$ and $[(ii) - gH]-$differentiable in $(1, 2]$ proof of the other cases are left to the reade. Using the Eqs (43)-(44) and taking $k = 1$ and $\frac{1}{\alpha} = \frac{10}{4}$. The results are showen in Table 1 and Table 2.

7. Conclusions

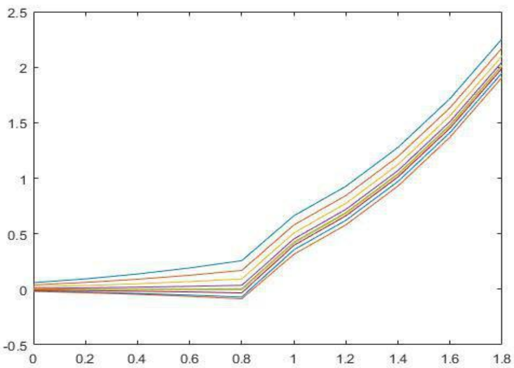
In this paper a new approach by presenting Homotopy perturbation method expansion based on gH-differentiability was introduced. The numerical results show that the present method is an accurate and reliable analytical technique for fuzzy impulsive fractional differential equation. All numerical results are obtained using Software Matlab.

Table 2. Numerical results of Example 1 for $\overline{y_{i.gH}}(t)$ *and* $\overline{y_{ii.gH}}(t)$.

r/t	0.0	0.2	0.4	0.6	0.8	0.1	1.2	1.40	1.6	1.8
0.1	0.059254	0.093470	0.13742	0.191876	0.257574	0.661318	0.927518	1.277314	1.718006	2.256348
0.2	0.03893	0.061417	0.090293	0.126076	0.169244	0.578712	0.844913	1.19471	1.635401	2.173743
0.3	0.021318	0.033627	0.049438	0.069030	0.092665	0.507010	0.773210	1.123007	1.563699	2.102041
0.4	0.008134	0.012831	0.018864	0.026339	0.035358	0.453387	0.719584	1.069380	1.510072	2.048414
0.5	0.001245	0.001964	0.002888	0.004032	0.005413	0.424570	0.6907706	1.040567	1.481259	2.019601
0.6	0.001229	-0.0019395	-0.002850	-0.003980	-0.005342	0.415555	0.681755	1.031551	1.472243	2.010585
0.7	-0.007416	-0.011698	-0.017198	-0.024014	-0.032236	0.395541	0.661742	1.011538	1.452230	1.990572
0.8	-0.016141	-0.025462	-0.037433	-0.052268	-0.070165	0.355179	0.621380	0.971176	1.411868	1.950210
0.9	-0.019648	-0.030994	-0.045566	-0.063624	-0.085408	0.314236	0.580437	0.930233	1.370925	1.909267
1	0	0	0	0	0	0.418116	0.684316	1.034113	1.474805	2.013146



(a)



(b)

Figure 1. Left hand level sets of the gH-derivative of solution of example 1, for $\underline{y_{i.gH}}$ and $\underline{y_{ii.gH}}$.

Figure 2. Right hand level sets of the gH-derivative of solution of example 1, for $\overline{y_{i.gH}}$ and $\overline{y_{ii.gH}}$.

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