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Sturm-Liouville Fuzzy Problem with Fuzzy Eigenvalue Parameter

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Abstract. This study is on the fuzzy eigenvalues and the fuzzy eigenvalues of Sturm-Liouville fuzzy problem with fuzzy eigenvalue parameter. We find fuzzy eigenvalues and the fuzzy eigenvalues of the problem under the approach of Hukuhara differentiability. We solve an example. We draw the graphics of eigenfunctions. We show that eigenfunctions are valid fuzzy functions or not.

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1. Introduction

To solve dynamic problems, fuzzy differential equation is very important topic. Thus, many researchers studied fuzzy differential equation with different methods [1-3,8]. Sturm-Liouville fuzzy differential equation was defined by Gültekin Çitil and Altınışık [6]. Sturm-Liouville fuzzy problem was studied in many papers [4,6,7]. But, eigenvalue parameter was not fuzzy in these papers.

In this paper, we investigate the fuzzy eigenvalues and the fuzzy eigenvalues of Sturm-Liouville fuzzy problem with fuzzy eigenvalue parameter.

The structure of this paper is as follows. In section 2 we give some basic definitions, notations and theorems. In section 3 we define our problem, present the eigenvalues and the eigenfunctions of the problems and solve an example. In section 4 we give conclusion.

2. Preliminaries

Definition 2.1 ([14]) A fuzzy number is a mapping $u: \mathbb{R} \to [0,1]$ satisfying with the following properties:

1. u is normal, $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$.
2. u is convex fuzzy set:
$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \quad \text{for all} \quad x, y \in \mathbb{R},\ \lambda \in [0,1].$$
3. u is upper semi-continuous on $\mathbb{R}$.
4. $supp\ u = \{x \in \mathbb{R} \mid u(x) > 0\}$ is the support of the u and its closure $cl(supp\ u)$ is compact.

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Let $\mathbb{R}_F$ be the space of fuzzy sets.

Definition 2.2 ([12]) Let u be a fuzzy set in $\mathbb{R}_F$. The α-level set of u is

$$[u]^{\alpha} = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}, \quad 0 < \alpha \leq 1.$$

Definition 2.3 ([14]) A fuzzy number u in parametric form is a pair $\left[\underline{u}_{\alpha}, \overline{u}_{\alpha}\right]$ of functions $\underline{u}_{\alpha}$, $\overline{u}_{\alpha}$, $0 \leq \alpha \leq 1$, which satisfy the following requirements:

1. $\underline{u}_{\alpha}$ is bounded non-decreasing left-continuous in $(0,1]$ and right-continuous at $\alpha = 0$.
2. $\overline{u}_{\alpha}$ is bounded non-increasing left-continuous in $(0,1]$ and right-continuous at $\alpha = 0$.
3. $\underline{u}_{\alpha} \leq \overline{u}_{\alpha}$, $0 \leq \alpha \leq 1$.

Definition 2.4 ([10]) For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by

$$[u+v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}, \ [\lambda u]^{\alpha} = \lambda [u]^{\alpha}, \ \forall \alpha \in [0,1].$$

Definition 2.5 ([5]) The metric is defined by

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d\left([u]^{\alpha}, [v]^{\alpha}\right),$$

$$d\left([u]^{\alpha}, [v]^{\alpha}\right) = \max\left\{\left|\underline{u}_{\alpha} - \underline{v}_{\alpha}\right|, \left|\overline{u}_{\alpha} - \overline{v}_{\alpha}\right|\right\}$$

on $\mathbb{R}_F$, where $[u]^{\alpha} = \left[\underline{u}_{\alpha}, \overline{u}_{\alpha}\right]$, $[v]^{\alpha} = \left[\underline{v}_{\alpha}, \overline{v}_{\alpha}\right]$.

Definition 2.6 ([11]) Let be $u, v \in \mathbb{R}_F$, $[u]^{\alpha} = \left[\underline{u}_{\alpha}, \overline{u}_{\alpha}\right]$, $[v]^{\alpha} = \left[\underline{v}_{\alpha}, \overline{v}_{\alpha}\right]$, the product $u.v$ is defined by

$$[u,v]^{\alpha} = [u]^{\alpha}.[v]^{\alpha}, \ \forall \alpha \in [0,1],$$

where

$$[u]^{\alpha}.[v]^{\alpha} = \left[\underline{u}_{\alpha}, \overline{u}_{\alpha}\right]\left[\underline{v}_{\alpha}, \overline{v}_{\alpha}\right] = \left[\underline{w}_{\alpha}, \overline{w}_{\alpha}\right],$$

$$\underline{w}_{\alpha} = \min\left\{\underline{u}_{\alpha}\underline{v}_{\alpha}, \underline{u}_{\alpha}\overline{v}_{\alpha}, \overline{u}_{\alpha}\underline{v}_{\alpha}, \overline{u}_{\alpha}\overline{v}_{\alpha}\right\}, \ \overline{w}_{\alpha} = \max\left\{\underline{u}_{\alpha}\underline{v}_{\alpha}, \underline{u}_{\alpha}\overline{v}_{\alpha}, \overline{u}_{\alpha}\underline{v}_{\alpha}, \overline{u}_{\alpha}\overline{v}_{\alpha}\right\}.$$

Definition 2.7 ([15]) A fuzzy number u is called positive, denoted by $u > 0$ if its membership function $u(x)$ satisfies $u(x) = 0, \ \forall x < 0.$

Definition 2.8 ([12]) Let u and v be two fuzzy sets. If there exists a fuzzy set w such that $u = v + w$, then w is called the H-difference of u and v and it is denoted by $u \ominus v$.

Definition 2.9 ([13]) $f : (a,b) \to \mathbb{R}_F$ and $t_0 \in (a,b)$. If there exists $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$, and the limits hold

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

f is Hukuhara differentiable at t_0.

Theorem 2.1 ([9]) Let $f : (a,b) \to \mathbb{R}_F$ be Hukuhara differentiable and denote

$$[f(t)]^{\alpha} = \left[\underline{f}_{\alpha}(t), \overline{f}_{\alpha}(t)\right].$$

Then, the boundary functions $\underline{f}_{\alpha}$ and $\overline{f}_{\alpha}$ are differentiable,

$$\left[f'(t)\right]^{\alpha} = \left[\underline{f}'_{\alpha}(t), \overline{f}'_{\alpha}(t)\right]$$

and $\underline{f}'_{\alpha}(t) \le \overline{f}'_{\alpha}(t)$.

Definition 2.10 ([6]) If $p'(x) = 0,\ r(x) = 1$ and $Ly = p(x)y'' + q(x)y$ in the fuzzy differential equation $\left(p(x)y'\right)' + q(x)y + \lambda r(x)y = 0,$ where $p(x),\ p'(x),\ q(x),\ r(x)$ are continuous functions and are positive on $[a,b]$, the fuzzy differential equation

$$Ly + \lambda y = 0 \tag{1}$$

is called a fuzzy Sturm-Liouville equation.

Definition 2.11 ([6]) $\left[y\left(x,\lambda_0\right)\right]^{\alpha} = \left[\underline{y}\left(x,\lambda_0\right), \overline{y}\left(x,\lambda_0\right)\right] \ne 0,$ we say that $\lambda = \lambda_0$ is eigenvalue of (1) if the fuzzy differential equation (1) has the nontrivial solutions $\underline{y}\left(x,\lambda_0\right) \ne 0,\ \overline{y}\left(x,\lambda_0\right) \ne 0.$

3. Main results

We investigate the fuzzy eigenvalues and the fuzzy eigenfunctions of the Sturm-Liouville fuzzy problem

$$y'' + [\lambda]^{\alpha} y = 0,\ x \in (a,b), \tag{2}$$

$$[A]^{\alpha} y(a) = [B]^{\alpha} y'(a), \tag{3}$$

$$[C]^{\alpha} y(b) = [D]^{\alpha} y'(b), \tag{4}$$

where $[\lambda]^{\alpha} = \left[\underline{\lambda}_{\alpha}, \overline{\lambda}_{\alpha}\right]$ positive fuzzy eigenvalue parameter,

$$[A]^{\alpha} = \left[\underline{A}_{\alpha}, \overline{A}_{\alpha}\right],\ [B]^{\alpha} = \left[\underline{B}_{\alpha}, \overline{B}_{\alpha}\right],\ [C]^{\alpha} = \left[\underline{C}_{\alpha}, \overline{C}_{\alpha}\right],\ [D]^{\alpha} = \left[\underline{D}_{\alpha}, \overline{D}_{\alpha}\right]$$

are symmetric triangular fuzzy numbers, $[y]^{\alpha} = \left[\underline{y}_{\alpha}, \overline{y}_{\alpha}\right]$ is positive fuzzy function.

Using the Hukuhara differentiability and fuzzy arithmetic, from the fuzzy differential equation (2), we have the equations

$$\underline{y}''_{\alpha} + \underline{\lambda}_{\alpha}\underline{y}_{\alpha} = 0,\ \overline{y}''_{\alpha} + \overline{\lambda}_{\alpha}\overline{y}_{\alpha} = 0.$$

From this, the solutions of the above equations are obtained as

$$\underline{y}_{\alpha}(x) = c_1(\alpha)\cos(\sqrt{\underline{\lambda}_{\alpha}}\,x) + c_2(\alpha)\sin(\sqrt{\underline{\lambda}_{\alpha}}\,x),$$

$$\overline{y}_{\alpha}(x) = c_3(\alpha)\cos\left(\sqrt{\overline{\lambda}_{\alpha}}\,x\right) + c_4(\alpha)\sin\left(\sqrt{\overline{\lambda}_{\alpha}}\,x\right).$$

Let

$$[\phi(x,\lambda)]^{\alpha} = \left[\underline{\phi}_{\alpha}\left(x,\underline{\lambda}_{\alpha}\right), \overline{\phi}_{\alpha}\left(x,\overline{\lambda}_{\alpha}\right)\right]$$

be the solution of the fuzzy differential equation (2) satisfying the condition (3).
Then, we have

$$\underline{y}_{\alpha}(a) = c_1(\alpha)\cos\left(\sqrt{\underline{\lambda}_{\alpha}}\,a\right) + c_2(\alpha)\sin\left(\sqrt{\underline{\lambda}_{\alpha}}\,a\right) = \underline{B}_{\alpha}, \tag{5}$$

$$\underline{y}'_{\alpha}(a) = -c_1(\alpha)\sqrt{\underline{\lambda}_{\alpha}}\sin\left(\sqrt{\underline{\lambda}_{\alpha}}\,a\right) + c_2(\alpha)\sqrt{\underline{\lambda}_{\alpha}}\cos\left(\sqrt{\underline{\lambda}_{\alpha}}\,a\right) = \underline{A}_{\alpha}, \tag{6}$$

$$\overline{y}_\alpha(a) = c_3(\alpha)\cos\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) + c_4(\alpha)\sin\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) = \overline{B}_\alpha, \tag{7}$$

$$\overline{y}'_\alpha(a) = -c_3(\alpha)\sqrt{\overline{\lambda}_\alpha}\sin\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) + c_4(\alpha)\sqrt{\overline{\lambda}_\alpha}\cos\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) = \overline{A}_\alpha. \tag{8}$$

From (5) and (6),

$$c_1(\alpha) = \frac{\underline{B}_\alpha\sqrt{\underline{\lambda}_\alpha}\cos\left(\sqrt{\underline{\lambda}_\alpha}\, a\right) - \underline{A}_\alpha\sin\left(\sqrt{\underline{\lambda}_\alpha}\, a\right)}{\sqrt{\underline{\lambda}_\alpha}},$$

$$c_2(\alpha) = \frac{\underline{A}_\alpha\cos\left(\sqrt{\underline{\lambda}_\alpha}\, a\right) + \underline{B}_\alpha\sqrt{\underline{\lambda}_\alpha}\sin\left(\sqrt{\underline{\lambda}_\alpha}\, a\right)}{\sqrt{\underline{\lambda}_\alpha}}.$$

From (7) and (8),

$$c_3(\alpha) = \frac{\overline{B}_\alpha\sqrt{\overline{\lambda}_\alpha}\cos\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) - \overline{A}_\alpha\sin\left(\sqrt{\overline{\lambda}_\alpha}\, a\right)}{\sqrt{\overline{\lambda}_\alpha}},$$

$$c_4(\alpha) = \frac{\overline{A}_\alpha\cos\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) + \overline{B}_\alpha\sqrt{\overline{\lambda}_\alpha}\sin\left(\sqrt{\overline{\lambda}_\alpha}\, a\right)}{\sqrt{\overline{\lambda}_\alpha}}.$$

Then,

$$[\phi(x,\lambda)]^\alpha = \left[\underline{\phi}_\alpha\left(x,\underline{\lambda}_\alpha\right), \overline{\phi}_\alpha\left(x,\overline{\lambda}_\alpha\right)\right],$$

$$\underline{\phi}_\alpha\left(x,\underline{\lambda}_\alpha\right) = \left(\frac{\underline{B}_\alpha\sqrt{\underline{\lambda}_\alpha}\cos\left(\sqrt{\underline{\lambda}_\alpha}\, a\right) - \underline{A}_\alpha\sin\left(\sqrt{\underline{\lambda}_\alpha}\, a\right)}{\sqrt{\lambda_\alpha}}\right)\cos\left(\sqrt{\underline{\lambda}_\alpha}\, x\right)$$
$$+\left(\frac{\underline{A}_\alpha\cos\left(\sqrt{\underline{\lambda}_\alpha}\, a\right) + \underline{B}_\alpha\sqrt{\underline{\lambda}_\alpha}\sin\left(\sqrt{\underline{\lambda}_\alpha}\, a\right)}{\sqrt{\underline{\lambda}_\alpha}}\right)\sin\left(\sqrt{\underline{\lambda}_\alpha}\, x\right),$$

$$\overline{\phi}_\alpha\left(x,\overline{\lambda}_\alpha\right) = \left(\frac{\overline{B}_\alpha\sqrt{\overline{\lambda}_\alpha}\cos\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) - \overline{A}_\alpha\sin\left(\sqrt{\overline{\lambda}_\alpha}\, a\right)}{\sqrt{\overline{\lambda}_\alpha}}\right)\cos\left(\sqrt{\overline{\lambda}_\alpha}\, x\right)$$
$$+\left(\frac{\overline{A}_\alpha\cos\left(\sqrt{\overline{\lambda}_\alpha}\, a\right) + \overline{B}_\alpha\sqrt{\overline{\lambda}_\alpha}\sin\left(\sqrt{\overline{\lambda}_\alpha}\, a\right)}{\sqrt{\overline{\lambda}_\alpha}}\right)\sin\left(\sqrt{\overline{\lambda}_\alpha}\, x\right).$$

Again, let

$$[\psi(x,\lambda)]^\alpha = \left[\underline{\psi}_\alpha\left(x,\underline{\lambda}_\alpha\right), \overline{\psi}_\alpha\left(x,\overline{\lambda}_\alpha\right)\right],$$

be the solution of the fuzzy differential equation (2) satisfying the condition (4). Then, we have

$$\underline{y}_\alpha(b) = c_1(\alpha)\cos\left(\sqrt{\underline{\lambda}_\alpha}\, b\right) + c_2(\alpha)\sin\left(\sqrt{\underline{\lambda}_\alpha}\, b\right) = \underline{D}_\alpha, \tag{9}$$

$$\underline{y}'_\alpha(b) = -c_1(\alpha)\sqrt{\underline{\lambda}_\alpha}\sin\left(\sqrt{\underline{\lambda}_\alpha}\, b\right) + c_2(\alpha)\sqrt{\underline{\lambda}_\alpha}\cos\left(\sqrt{\underline{\lambda}_\alpha}\, b\right) = \underline{C}_\alpha, \tag{10}$$

$$\overline{y}_\alpha(b) = c_3(\alpha)\cos\left(\sqrt{\overline{\lambda}_\alpha}\, b\right) + c_4(\alpha)\sin\left(\sqrt{\overline{\lambda}_\alpha}\, b\right) = \overline{D}_\alpha, \tag{11}$$

$$\overline{y}'_\alpha(b) = -c_3(\alpha)\sqrt{\overline{\lambda}_\alpha}\sin\left(\sqrt{\overline{\lambda}_\alpha}\, b\right) + c_4(\alpha)\sqrt{\overline{\lambda}_\alpha}\cos\left(\sqrt{\overline{\lambda}_\alpha}\, b\right) = \overline{C}_\alpha. \tag{12}$$

From (9) and (10),

$$c_1(\alpha)=\frac{\underline{D}_\alpha\sqrt{\underline{\lambda}_\alpha}\cos\left(\sqrt{\underline{\lambda}_\alpha}b\right)-\underline{C}_\alpha\sin\left(\sqrt{\underline{\lambda}_\alpha}b\right)}{\sqrt{\underline{\lambda}_\alpha}},$$

$$c_2(\alpha)=\frac{\underline{C}_\alpha\cos\left(\sqrt{\underline{\lambda}_\alpha}b\right)+\underline{D}_\alpha\sqrt{\underline{\lambda}_\alpha}\sin\left(\sqrt{\underline{\lambda}_\alpha}b\right)}{\sqrt{\underline{\lambda}_\alpha}}.$$

From (11) and (12),

$$c_3(\alpha)=\frac{\overline{D}_\alpha\sqrt{\overline{\lambda}_\alpha}\cos\left(\sqrt{\overline{\lambda}_\alpha}b\right)-\overline{C}_\alpha\sin\left(\sqrt{\overline{\lambda}_\alpha}b\right)}{\sqrt{\overline{\lambda}_\alpha}},$$

$$c_4(\alpha)=\frac{\overline{C}_\alpha\cos\left(\sqrt{\overline{\lambda}_\alpha}b\right)+\overline{D}_\alpha\sqrt{\overline{\lambda}_\alpha}\sin\left(\sqrt{\overline{\lambda}_\alpha}b\right)}{\sqrt{\overline{\lambda}_\alpha}}.$$

Then,

$$[\psi(x,\lambda)]^\alpha=\left[\underline{\psi}_\alpha\left(x,\underline{\lambda}_\alpha\right),\overline{\psi}_\alpha\left(x,\overline{\lambda}_\alpha\right)\right],$$

$$\underline{\psi}_\alpha\left(x,\underline{\lambda}_\alpha\right)=\left(\frac{\underline{D}_\alpha\sqrt{\underline{\lambda}_\alpha}\cos\left(\sqrt{\underline{\lambda}_\alpha}b\right)-\underline{C}_\alpha\sin\left(\sqrt{\underline{\lambda}_\alpha}b\right)}{\sqrt{\underline{\lambda}_\alpha}}\right)\cos\left(\sqrt{\underline{\lambda}_\alpha}x\right)$$
$$+\left(\frac{\underline{C}_\alpha\cos\left(\sqrt{\underline{\lambda}_\alpha}b\right)+\underline{D}_\alpha\sqrt{\underline{\lambda}_\alpha}\sin\left(\sqrt{\underline{\lambda}_\alpha}b\right)}{\sqrt{\underline{\lambda}_\alpha}}\right)\sin\left(\sqrt{\underline{\lambda}_\alpha}x\right),$$

$$\overline{\psi}_\alpha\left(x,\overline{\lambda}_\alpha\right)=\left(\frac{\overline{D}_\alpha\sqrt{\overline{\lambda}_\alpha}\cos\left(\sqrt{\overline{\lambda}_\alpha}b\right)-\overline{C}_\alpha\sin\left(\sqrt{\overline{\lambda}_\alpha}b\right)}{\sqrt{\overline{\lambda}_\alpha}}\right)\cos\left(\sqrt{\overline{\lambda}_\alpha}x\right)$$
$$+\left(\frac{\overline{C}_\alpha\cos\left(\sqrt{\overline{\lambda}_\alpha}b\right)+\overline{D}_\alpha\sqrt{\overline{\lambda}_\alpha}\sin\left(\sqrt{\overline{\lambda}_\alpha}b\right)}{\sqrt{\overline{\lambda}_\alpha}}\right)\sin\left(\sqrt{\overline{\lambda}_\alpha}x\right).$$

The eigenvalues of the fuzzy boundary value problem (2)-(4) if and only if are consist of the zeros of functions $\underline{W}\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)\left(x,\underline{\lambda}_\alpha\right)$ and $\overline{W}\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)\left(x,\overline{\lambda}_\alpha\right)$ [6].

Since

$$\underline{W}\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)\left(x,\underline{\lambda}_\alpha\right)=\underline{\phi}_\alpha\left(x,\underline{\lambda}_\alpha\right)\underline{\psi}_\alpha\left(x,\underline{\lambda}_\alpha\right)-\underline{\phi}'_\alpha\left(x,\underline{\lambda}_\alpha\right)\underline{\psi}_\alpha\left(x,\underline{\lambda}_\alpha\right),$$
$$\overline{W}\left(\overline{\phi}_\alpha,\overline{\psi}_\alpha\right)\left(x,\overline{\lambda}_\alpha\right)=\overline{\phi}_\alpha\left(x,\overline{\lambda}_\alpha\right)\overline{\psi}_\alpha\left(x,\overline{\lambda}_\alpha\right)-\overline{\phi}'_\alpha\left(x,\overline{\lambda}_\alpha\right)\overline{\psi}_\alpha\left(x,\overline{\lambda}_\alpha\right),$$

making the necessary operations, we obtain

$$\underline{W}\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)\left(\underline{\lambda}_\alpha\right)=\frac{1}{\sqrt{\underline{\lambda}_\alpha}}\left\{\sqrt{\underline{\lambda}_\alpha}\cos\left(\sqrt{\underline{\lambda}_\alpha}(a-b)\right)\left(\underline{B}_\alpha\underline{C}_\alpha-\underline{A}_\alpha\underline{D}_\alpha\right)\right.$$
$$\left.-\sin\left(\sqrt{\underline{\lambda}_\alpha}(a-b)\right)\left(\underline{A}_\alpha\underline{C}_\alpha+\underline{\lambda}_\alpha\underline{B}_\alpha\underline{D}_\alpha\right)\right\},$$

$$\overline{W}\left(\overline{\phi}_\alpha,\overline{\psi}_a\right)\left(\overline{\lambda}_a\right)=\frac{1}{\sqrt{\overline{\lambda}_a}}\left\{\sqrt{\overline{\lambda}_a}\cos\left(\sqrt{\overline{\lambda}_a}(a-b)\right)\left(\overline{B}_a\overline{C}_a-\overline{A}_a\overline{D}_a\right)\right.$$
$$\left.-\sin\left(\sqrt{\overline{\lambda}_\alpha}(a-b)\right)\left(\overline{A}_\alpha\overline{C}_\alpha+\overline{\lambda}_\alpha\overline{B}_\alpha\overline{D}_\alpha\right)\right\}.$$

Example 3.1 Consider the eigenvalues and the eigenfunctions of the fuzzy problem

$$y''+[\lambda]^{\alpha}y=0, \tag{13}$$

$$[1]^{\alpha}y(0)=y'(0), \tag{14}$$

$$[2]^{\alpha}y(1)=[0]^{\alpha}y'(1) \tag{15}$$

where

$$[0]^{\alpha}=[-1+\alpha,1-\alpha],\ [1]^{\alpha}=[\alpha,2-\alpha],\ [2]^{\alpha}=[1+\alpha,3-\alpha],$$

$[y]^{\alpha}=\left[\underline{y}_{\alpha},\overline{y}_{\alpha}\right]$ is positive fuzzy function.

Let

$$[\phi(x,\lambda)]^{\alpha}=\left[\underline{\phi}_{\alpha}\left(x,\underline{\lambda}_{\alpha}\right),\overline{\phi}_{\alpha}\left(x,\overline{\lambda}_{\alpha}\right)\right],$$

$$\underline{\phi}_{a}\left(x,\underline{\lambda}_{a}\right)=\cos\left(\sqrt{\underline{\lambda}_{a}}x\right)+\left(\frac{a}{\sqrt{\underline{\lambda}_{w}}}\right)\sin\left(\sqrt{\underline{\lambda}_{a}}x\right),$$

$$\overline{\phi}_{a}\left(x,\overline{\lambda}_{a}\right)=\cos\left(\sqrt{\overline{\lambda}_{a}}x\right)+\left(\frac{2-a}{\sqrt{\overline{\lambda}_{a}}}\right)\sin\left(\sqrt{\overline{\lambda}_{a}}x\right)$$

be the solution of the fuzzy differential equation (13) satisfying the condition (14) and

$$[\psi(x,\lambda)]^{a}=\left[\underline{\psi}_{a}\left(x,\underline{\lambda}_{a}\right),\overline{\psi}_{a}\left(x,\overline{\lambda}_{a}\right)\right],$$

$$\underline{\psi}_{\alpha}\left(x,\underline{\lambda}_{\alpha}\right)=\left(\frac{(-1+\alpha)\sqrt{\underline{\lambda}_{\alpha}}\cos\left(\sqrt{\underline{\lambda}_{\alpha}}\right)-(1+\alpha)\sin\left(\sqrt{\underline{\lambda}_{\alpha}}\right)}{\sqrt{\underline{\lambda}_{\alpha}}}\right)\cos\left(\sqrt{\underline{\lambda}_{\alpha}}x\right)$$

$$+\left(\frac{(1+\alpha)\cos\left(\sqrt{\underline{\lambda}_{\alpha}}\right)+(-1+\alpha)\sqrt{\underline{\lambda}_{\alpha}}\sin\left(\sqrt{\underline{\lambda}_{\alpha}}\right)}{\sqrt{\underline{\lambda}_{\alpha}}}\right)\sin\left(\sqrt{\underline{\lambda}_{\alpha}}x\right),$$

$$\overline{\psi}_{\alpha}\left(x,\overline{\lambda}_{\alpha}\right)=\left(\frac{(1-\alpha)\sqrt{\overline{\lambda}_{\alpha}}\cos\left(\sqrt{\overline{\lambda}_{\alpha}}\right)-(3-\alpha)\sin\left(\sqrt{\overline{\lambda}_{\alpha}}\right)}{\sqrt{\overline{\lambda}_{\alpha}}}\right)\cos\left(\sqrt{\overline{\lambda}_{\alpha}}x\right)$$

$$+\left(\frac{(3-\alpha)\cos\left(\sqrt{\overline{\lambda}_{\alpha}}\right)+(1-\alpha)\sqrt{\overline{\lambda}_{\alpha}}\sin\left(\sqrt{\overline{\lambda}_{\alpha}}\right)}{\sqrt{\overline{\lambda}_{\alpha}}}\right)\sin\left(\sqrt{\overline{\lambda}_{\alpha}}x\right)$$

be the solution of the fuzzy differential equation (13) satisfying the condition (15).
From this, we obtain

$$\underline{W}\left(\underline{\phi}_{\alpha},\underline{\psi}_{\alpha}\right)\left(\underline{\lambda}_{\alpha}\right)=\cos\left(\sqrt{\underline{\lambda}_{\alpha}}\right)\left(1+2\alpha-\alpha^{2}\right)+\frac{\sin\left(\sqrt{\underline{\lambda}_{\alpha}}\right)}{\sqrt{\underline{\lambda}_{\alpha}}}\left(\alpha+\alpha^{2}+\underline{\lambda}_{\alpha}(\alpha-1)\right),$$

$$\overline{W}\left(\overline{\phi}_{\alpha},\overline{\psi}_{\alpha}\right)\left(\overline{\lambda}_{\alpha}\right)=\cos\left(\sqrt{\overline{\lambda}_{\alpha}}\right)\left(1+2\alpha-\alpha^{2}\right)+\frac{\sin\left(\sqrt{\overline{\lambda}_{\alpha}}\right)}{\sqrt{\overline{\lambda}_{\alpha}}}\left((2-\alpha)(3-\alpha)+\overline{\lambda}_{\alpha}(1-\alpha)\right).$$

Computing the values $\underline{\lambda}_{\alpha}$ satisfying the equation

$$\underline{W}\left(\underline{\phi}_{\alpha},\underline{\psi}_{\alpha}\right)\left(\underline{\lambda}_{\alpha}\right)=0$$

for each $\alpha\in[0,1]$, we get infinitely many values as

$\alpha=0\Rightarrow\lambda_{0,1}=0.860334,\ \underline{\lambda}_{0,2}=3.42562,\ \underline{\lambda}_{0,3}=6.4373,\ \underline{\lambda}_{0,4}=9.52933,\ldots$

$\alpha=0.2 \Rightarrow \underline{\lambda}_{0.2,1}=1.11126,\ \underline{\lambda}_{0.2,2}=3.59271,\ \underline{\lambda}_{0.2,3}=6.53924,\ \underline{\lambda}_{0.2,4}=9.60059,\ldots$

$\alpha=0.5 \Rightarrow \underline{\lambda}_{0.5,1}=1.45177,\ \underline{\lambda}_{0.5,2}=3.92146,\ \underline{\lambda}_{0.5,3}=6.77379,\ \underline{\lambda}_{0.5,4}=9.77371,\ldots$

$\alpha=0.8 \Rightarrow \underline{\lambda}_{0.8,1}=1.79362,\ \underline{\lambda}_{0.8,2}=4.43323,\ \underline{\lambda}_{0.8,3}=7.28305,\ \underline{\lambda}_{0.8,4}=10.2246,\ldots$

$\alpha=1 \Rightarrow \underline{\lambda}_{1,1}=2.02876,\ \lambda_{1,2}=4.91318,\ \lambda_{1,3}=7.97867,\ \underline{\lambda}_{0.5,4}=11.0855,\ldots$

We show that this values are $\underline{\lambda}_{\alpha,k}$, $k=1,2,...,$ for each $\alpha\in[0,1]$.

Again, computing the values $\overline{\lambda}_{\alpha}$ satisfying the equation

$$\overline{W}(\overline{\phi}_{\alpha},\overline{\psi}_{\alpha})(\overline{\lambda}_{\alpha})=0$$

for each $\alpha\in[0,1]$, we get infinitely many values as

$\alpha=0 \Rightarrow \overline{\lambda}_{0,1}=2.9435,\ \overline{\lambda}_{0,2}=6.14365,\ \overline{\lambda}_{0,3}=9.32479,\ \overline{\lambda}_{0,4}=12.4894,\ldots$

$\alpha=0.2 \Rightarrow \overline{\lambda}_{0.2,1}=2.81708,\ \overline{\lambda}_{0.2,2}=6.04784,\ \overline{\lambda}_{0.2,3}=9.25532,\ \overline{\lambda}_{0.2,4}=12.4358,\ldots$

$\alpha=0.5 \Rightarrow \overline{\lambda}_{0.5,1}=2.57385,\ \overline{\lambda}_{0.5,2}=5.82593,\ \overline{\lambda}_{0.5,3}=9.0853,\ \overline{\lambda}_{0.5,4}=12.3017,\ldots$

$\alpha=0.8 \Rightarrow \overline{\lambda}_{0.8,1}=2.26141,\ \overline{\lambda}_{0.8,2}=5.38696,\ \overline{\lambda}_{0.8,3}=8.65855,\ \overline{\lambda}_{0.8,4}=11.9215,\ldots$

$\alpha=1 \Rightarrow \overline{\lambda}_{1,1}=2.02876,\ \overline{\lambda}_{1,2}=4.91318,\ \overline{\lambda}_{1,3}=7.97867,\ \overline{\lambda}_{1,4}=11.0855,\ldots$

We show that this values are $\overline{\lambda}_{\alpha,k}$, $k=1,2,...,$ for each $\alpha\in[0,1]$.

Then, the eigenvalues are $\left[\lambda_k\right]^{\alpha}=\left[\underline{\lambda}_{\alpha,k},\overline{\lambda}_{\alpha,k}\right]$ with associated eigenfunctions

$$\left[\phi\left(x,\lambda_k\right)\right]^{\alpha}=\left[\underline{\phi}_{\alpha}\left(x,\underline{\lambda}_{\alpha,k}\right),\overline{\phi}_{\alpha}\left(x,\overline{\lambda}_{\alpha,k}\right)\right],$$

$$\underline{\phi}_{\alpha}\left(x,\underline{\lambda}_{\alpha,k}\right)=\cos\left(\sqrt{\underline{\lambda}_{\alpha,k}}\,x\right)+\left(\frac{\alpha}{\sqrt{\underline{\lambda}_{\alpha,k}}}\right)\sin\left(\sqrt{\underline{\lambda}_{\alpha,k}}\,x\right),$$

$$\overline{\phi}_{\alpha}\left(x,\overline{\lambda}_{\alpha,k}\right)=\cos\left(\sqrt{\overline{\lambda}_{\alpha,k}}\,x\right)+\left(\frac{2-\alpha}{\sqrt{\overline{\lambda}_{\alpha,k}}}\right)\sin\left(\sqrt{\overline{\lambda}_{\alpha,k}}\,x\right)$$

and

$$\left[\psi\left(x,\lambda_k\right)\right]^{\alpha}=\left[\underline{\psi}_{\alpha}\left(x,\underline{\lambda}_{\alpha,k}\right),\overline{\psi}_{\alpha}\left(x,\overline{\lambda}_{\alpha,k}\right)\right],$$

$$\underline{\psi}_{\alpha}\left(x,\underline{\lambda}_{\alpha,k}\right)=\left(\frac{(-1+\alpha)\sqrt{\underline{\lambda}_{\alpha,k}}\cos\left(\sqrt{\underline{\lambda}_{\alpha,k}}\right)-(1+\alpha)\sin\left(\sqrt{\underline{\lambda}_{\alpha,k}}\right)}{\sqrt{\underline{\lambda}_{\alpha,k}}}\right)\cos\left(\sqrt{\underline{\lambda}_{\alpha,k}}\,x\right)$$
$$+\left(\frac{(1+\alpha)\cos\left(\sqrt{\underline{\lambda}_{\alpha,k}}\right)+(-1+\alpha)\sqrt{\underline{\lambda}_{\alpha,k}}\sin\left(\sqrt{\underline{\lambda}_{\alpha,k}}\right)}{\sqrt{\underline{\lambda}_{\alpha,k}}}\right)\sin\left(\sqrt{\underline{\lambda}_{\alpha,k}}\,x\right),$$

$$\overline{\psi}_{\alpha}\left(x,\overline{\lambda}_{\alpha,k}\right)=\left(\frac{(1-\alpha)\sqrt{\overline{\lambda}_{\alpha,k}}\cos\left(\sqrt{\overline{\lambda}_{\alpha,k}}\right)-(3-\alpha)\sin\left(\sqrt{\overline{\lambda}_{\alpha,k}}\right)}{\sqrt{\overline{\lambda}_{\alpha,k}}}\right)\cos\left(\sqrt{\overline{\lambda}_{\alpha,k}}\,x\right)$$
$$+\left(\frac{(3-\alpha)\cos\left(\sqrt{\overline{\lambda}_{\alpha,k}}\right)+(1-\alpha)\sqrt{\overline{\lambda}_{\alpha,k}}\sin\left(\sqrt{\overline{\lambda}_{\alpha,k}}\right)}{\sqrt{\overline{\lambda}_{\alpha,k}}}\right)\sin\left(\sqrt{\overline{\lambda}_{\alpha,k}}\,x\right).$$

As

$$\frac{\partial \underline{\phi}_\alpha\left(x,\underline{\lambda}_{\alpha,k}\right)}{\partial\alpha}\geq 0,\quad \frac{\partial \overline{\phi}_\alpha\left(x,\overline{\lambda}_{\alpha,k}\right)}{\partial\alpha}\leq 0,\quad \underline{\phi}_\alpha\left(x,\underline{\lambda}_{\alpha,k}\right)\leq\overline{\phi}_\alpha\left(x,\overline{\lambda}_{\alpha,k}\right),$$

$$\frac{\partial \underline{\psi}_\alpha\left(x,\underline{\lambda}_{\alpha,k}\right)}{\partial\alpha}\geq 0,\quad \frac{\partial \overline{\psi}_\alpha\left(x,\overline{\lambda}_{\alpha,k}\right)}{\partial\alpha}\leq 0,\quad \underline{\psi}_\alpha\left(x,\underline{\lambda}_{\alpha,k}\right)\leq\overline{\psi}_\alpha\left(x,\overline{\lambda}_{\alpha,k}\right),$$

$\left[\phi\left(x,\lambda_k\right)\right]^\alpha$ and $\left[\psi\left(x,\lambda_k\right)\right]^\alpha$ are valid $\alpha-$level sets.

Let draw the graphics of $\left[\phi\left(x,\lambda_k\right)\right]^\alpha$ and $\left[\psi\left(x,\lambda_k\right)\right]^\alpha$ for $\alpha=0.5$ and $k=1$.

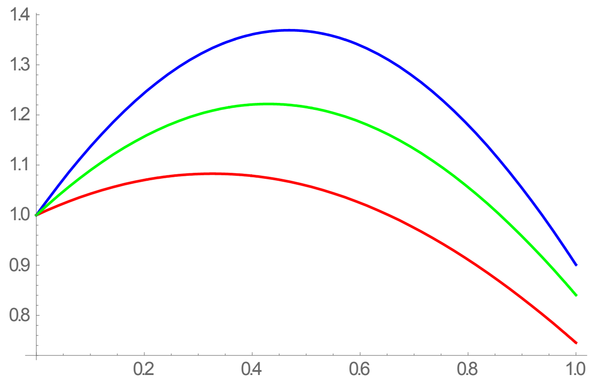

Figure 1. $\underline{\phi}_{0.5}\left(x,\underline{\lambda}_{0.5,1}\right)$ (red), $\overline{\phi}_{0.5}\left(x,\overline{\lambda}_{0.5,1}\right)$ (blue), $\underline{\phi}_1\left(x,\underline{\lambda}_{1,1}\right)=\overline{\phi}_1\left(x,\overline{\lambda}_{1,1}\right)$ (green).

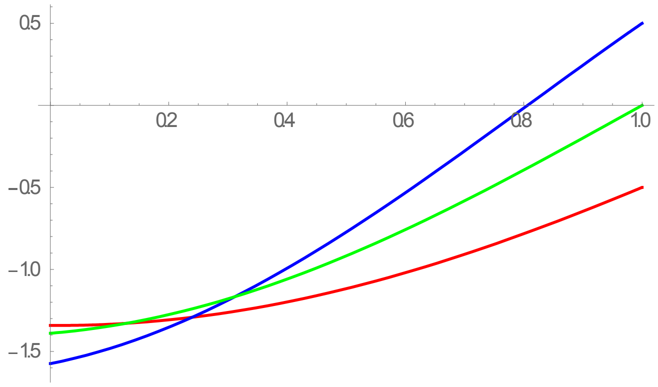

Figure 2. $\underline{\psi}_{0.5}\left(x,\underline{\lambda}_{0.5,1}\right)$ (red), $\overline{\psi}_{0.5}\left(x,\overline{\lambda}_{0.5,1}\right)$ (blue), $\underline{\psi}_1\left(x,\underline{\lambda}_{1,1}\right)=\overline{\psi}_1\left(x,\overline{\lambda}_{1,1}\right)$ (green).

We can see that $\left[\phi\left(x,\lambda_k\right)\right]^\alpha$ is a valid α-level set in Figure 1 and $\left[\psi\left(x,\lambda_k\right)\right]^\alpha$ is a valid $\alpha-$level set for $x\in\left(0.239161,1\right)$ in Figure 2. But, since the solution function $[y]^\alpha$ is positive fuzzy function, $\left[\psi\left(x,\lambda_k\right)\right]^\alpha$ is not solution. That is, the eigenfunction is $\left[\phi\left(x,\lambda_k\right)\right]^\alpha$ with associated the eigenvalue $\left[\lambda_k\right]^\alpha$ for $\alpha=0.5$ and $k=1$.

4. Conclusion

In this study, we investigate the fuzzy eigenvalues and the fuzzy eigenfunctions of Sturm-Liouville fuzzy problem with fuzzy eigenvalue parameter under the approach of Hukuhara differentiability. Since eigenvalue parameter is fuzzy, the eigenvalues depend on α. Thus, different eigenvalues and eigenfunctions are obtained for each α, $0 \le \alpha \le 1$. Also, eigenfunctions must be valid α – level sets.

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