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# Numerical Solution of the Parabolic Equations by Variational Iteration Method and Radial Basis Functions

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**Abstract.** In this work, we consider the parabolic equation: $u_t - u_{xx} = 0$. The purpose of this paper is to introduce the method of variational iteration method and radial basis functions for solving this equation. Also, the method is implemented to three numerical examples. The results reveal that the technique is very effective and simple.

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**Keywords:** Radial basis functions; Variational iteration method; Parabolic equations; Partial Differential Equations.

**Index to information contained in this paper**



## 1. Introduction

Over the last decades several analytically/approximate methods have been developed to solve nonlinear equations. For initial-value problems in ordinary differential equations, some of these technique include perturbation [15–17], variational [9–11], decomposition [2–4] methods, etc [19]. In the recent years, radial basis function collocation has become a useful alternative to finite difference and finite element methods for solving elliptic partial differential equations. RBF collocation methods have been shown numerically (see for example [14]) and theoretically ([7, 8]) to be very accurate even for a small number of collocation points. The variational iteration method was proposed by He in 1998, and was successfully applied to autonomous ordinary differential equation [12], to nonlinear partial differential

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equations with variable coefficients [13], and recently to nonlinear fractional differential equations [18], and other fields [1, 5]. In this paper we present a good approximation for solving a parabolic equation.

## 2. Preliminaries radial basis functions and variational iteration method

Radial Basis Functions (RBFs) are popular for interpolating scattered data since the associated system of linear equations is guaranteed to be invertible under very mild conditions on the location of the data points. For example, the thin-plate spline used in this library only requires that the points are not co-linear. In particular, Radial Basis Functions do not require that the data lie on any sort of regular grid.
A radial basis function (RBF) is a function of the form:

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \Phi(x - x_i), \tag{1}$$

where: $s$ is the radial basis function (RBF for short) and $p$ is a low degree polynomial, typically linear or quadratic and the $\lambda_i$'s are the RBF coefficients and $\Phi$ is a real valued function called the basic function and the $x_i$ 's are the RBF centres.

The RBF consists of a weighted sum of a radially symmetric basic function $\phi$ located at the centres $x_i$ and a low degree polynomial $p$. Given a set of $N$ points $x_i$ and values $f_i$, the process of finding an interpolating RBF, $s$, such that,

$$s(x_i) = f_i, \quad i = 1, 2, ..., N,$$

is called fitting. The fitted RBF is defined by the $\lambda_i$, the coefficients of the basic function in the summation, together with the coefficients of the polynomial term $p(x)$.

For a fixed point $x_j \in \mathbb{R}^d$ , a radial basis function is defined:

$$\phi_j(x) = \phi(\| (x - x_j) \|) \tag{2}$$

which is function only dependents on the distance between $x_j \in \mathbb{R}^d$ and the point $x_j$. We have GA, MQ, IMQ and IQ-RBF whit the following forms, respectively

$$\phi(r) = \exp(-cr^2) \tag{3}$$

$$\phi(r) = \sqrt{r^2 + c^2} \tag{4}$$

$$\phi(r) = (\sqrt{r^2 + c^2})^{-1} \tag{5}$$

$$\phi(r) = (r^2 + c^2)^{-1} \tag{6}$$

where $c$ is a shape parameter which should be considered suitably also the Euclidean distance is considered for the RBF. [6]

**Variational iteration method**:
To illustrate the basic concepts of the variational iteration method, we consider the following general nonlinear system:

$$Lu(x) + Nu(x) = f(x) \tag{7}$$

where $L$ is a linear operator part while $N$ is the nonlinear operator part, and $f(x)$ is a known analytic function. According to the variational iteration method, a correction functional can be constructed as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + N\tilde{u}(\tau) - f(\tau)\}d\tau \tag{8}$$

where $\lambda$ is a general multiplier [13], which can be identified optimally via the variational thory [13], the subscript $n$ denotes the $n$th approximation, and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.
The initial guess can be freely chosen with possible unknown constants; it can also be solved from its corresponding linear homogeneous equation

$$Lu_0(x) = 0. \tag{9}$$

## 3. Parabolic equations

The computational stage of all numerical methods for solving problem of any complexity generally involves a great deal of arithmetic. It is usual therefore to arrange, whenever possible, for one solution to suffice for a variety of different problems. This can be done by expressing all equations in terms of non-dimensional variables. Then all problems with the same non-dimensional mathematical formulation can be dealt with by means of one solution. The problems need not, be dimensionally different, but merely variations of the same type of problem, as we would have with the calculation of the periods of oscillation of springs of different lengths $l$ supporting different masses $m$ and having different stiffnesses $s$.
This non-dimensionalizing process is illustrated below with the parabolic equation

$$\frac{\partial U}{\partial T} = k\frac{\partial^2 U}{\partial X^2}, \qquad k \ \ constant \tag{10}$$

the solution of which gives the temperature $U$ at a distance $X$ from one end of a thin uniform rod after a time $T$. Let $L$ represent the length of the rod and $U_0$ some particular temperature such as the maximum or minimum temperature at zero time. Put

$$x = \frac{X}{L} \quad and \quad u = \frac{U}{U_0} \tag{11}$$

Then

$$\frac{\partial U}{\partial X} = \frac{\partial U}{\partial x}.\frac{dx}{dX} = \frac{\partial U}{\partial x}.\frac{1}{L}$$

and

$$\frac{\partial^2 U}{\partial X^2} = \frac{\partial}{\partial X}(\frac{\partial U}{\partial X}) = \frac{\partial}{\partial x}(\frac{1}{L}.\frac{\partial U}{\partial X}).\frac{dx}{dX} = \frac{1}{L^2}.\frac{\partial^2 U}{\partial x^2}$$

so equation (10) transforms to

$$\frac{\partial (uU_0)}{\partial T} = \frac{k}{L^2}.\frac{\partial^2 (uU_0)}{\partial x^2}$$

i.e.

$$\frac{1}{kL^{-2}}.\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial x^2}$$

Writing $t = \frac{kT}{L^2}$ and applying the function of a function rule to the left side yields

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{12}$$

as the non-dimensional form of (10).
It should be noted that the number representing the length of the rod is 1. [20]

To illustrate the basic concepts of this technique, consider the approximation solution as:

$$\tilde{u}(x_i, t_i) \equiv \sum_{j=1}^{N} c_j \phi_j(x_i, t_i) \tag{13}$$

which $c_j$'s are constants and the $\phi_j$'s are radial basis functions. where
$\phi(r) = (\sqrt{r^2 + c^2})^{-1}$ is a radial basis function on $r = \|(x,t)\|$.
With substituting equation (13) into equation (8), by considering $N = 0$ and $f(x,t) = u_t - u_{xx}$, we have:

$$\sum_{j=1}^{N} c_j^{(k+1)} \phi_j = \sum_{j=1}^{N} c_j^{(k)} [\phi_j + \int_0^t (c_j \phi_j - f) d\tau] \tag{14}$$

For simplicity we consider

$$\alpha_j(x_i, t_i) = \phi_j(x_i, t_i) + \int_0^t c_j [(\phi_j)_t(x, \tau) - (\phi_j)_{xx}(x, \tau) - g(x, \tau)] d\tau$$

which is a system of equations.

For solving, we assume $N = N_1 + N_2$ that $N_1$ denotes the number of boundary points and $N_2$ shows the number of interior points.
Suppose that the following sets contain a collocation of scattered nodes in every levels of interpolation

$$\Xi_1 = \{(x_i, t_i) \in \bar{\Omega} \times [0, T_1], \quad i = 1, ..., m\}, \quad T > T_1. \tag{15}$$

$$\Xi_k = \{(x_i, t_i + (k-1)T_1); \ (x_i, t_i) \in \Xi_1, \quad i = 1, ..., m, k = 2, 3, ...\} \tag{16}$$

and the problem has a solution in $\bar{\Omega} \times [(k-1)T_1, kT_1]$. We have a linear system of equations:

$$H\Lambda = q$$

where $H = [h_{ij}]$, $\Lambda = [\lambda_j]$.

## 4. Numerical examples

To illustrate the effectiveness of the present method, several test examples are consider in this section. We draw Figures in .... step, and obtain good error graphs for this approximation that are shown in Figures.

**Example 4.1** Consider the following partial differential equation:

$$f(x,t) = \frac{-3t^2 x}{(4+t^3)^2} \tag{17}$$

with initial conditions,

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = \frac{1}{4+t^3} \\ u(x,0) = \frac{x}{4} \end{cases} \tag{18}$$

The exact solution is:

$$u(x,t) = \frac{x}{4+t^3} \tag{19}$$

By considering,

$$\phi(r) = (\sqrt{r^2 + c^2})^{-1} \tag{20}$$

with $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$, we obtain good error graphs:

**Example 4.2** We assume,

$$f(x,t) = \frac{-3t^2(x+2)}{(2+t^3)^2} \tag{21}$$

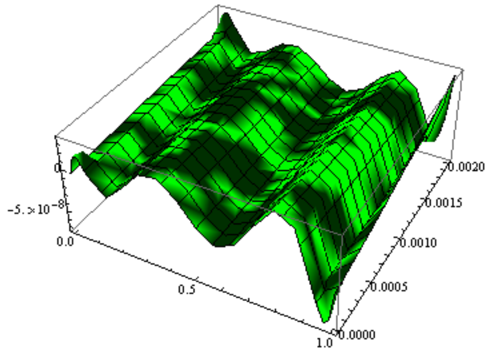


Figure 1. Error function ($k = 1$).

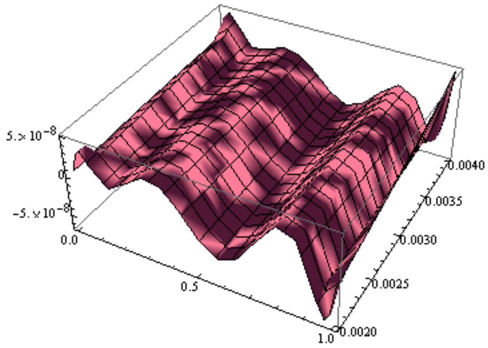


Figure 2. Error function ($k = 2$).

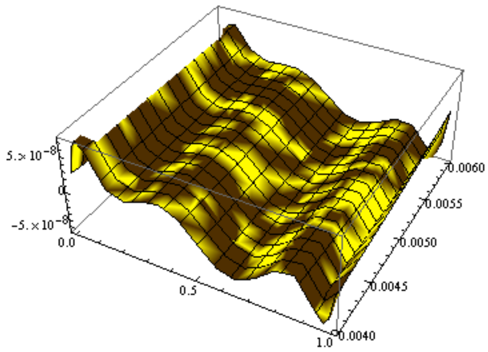


Figure 3. Error function ($k = 3$).

with initial conditions,

$$\begin{cases} u(0,t) = \frac{2}{2+t^3} \\ u(1,t) = \frac{3}{2+t^3} \\ u(x,0) = \frac{x+2}{2} \end{cases} \tag{22}$$

The exact solution is:

$$u(x,t) = \frac{x+2}{2+t^3} \tag{23}$$

As stated before, using equation (20), and by taking $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$ we have Figures for function,

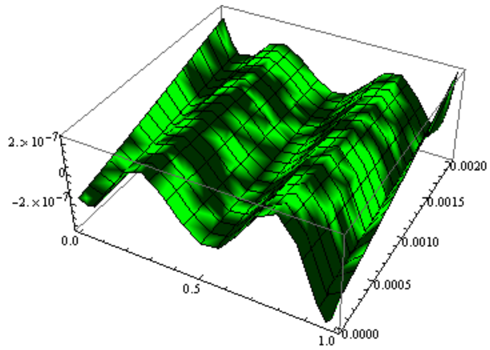


Figure 4. Error function ($k = 1$).

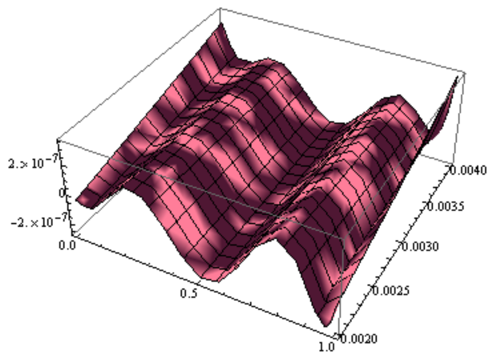


Figure 5. Error function ($k = 2$).

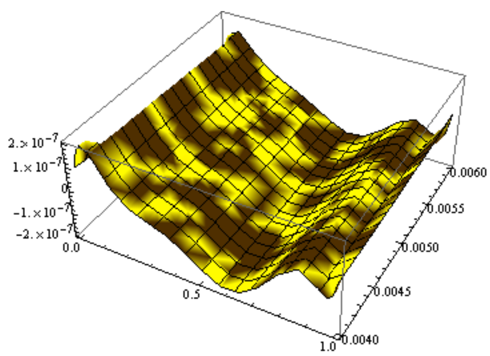


Figure 6. Error function ($k = 3$).

We can be applied the present method to other problem.

**Example 4.3** Consider the following problem:

$$f(x,t) = \frac{-3t^2 \sin(x)}{(2+t^3)^2} + \frac{\sin(x)}{2+t^3} \tag{24}$$

by using initial conditions:

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = \frac{\sin(1)}{2+t^3} \\ u(x,0) = \frac{\sin(x)}{2} \end{cases} \tag{25}$$

The exact value is:

$$u(x,t) = \frac{\sin(x)}{2+t^3} \tag{26}$$

Using the method and according to the equation (20), by taking $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$ see Figures for function,

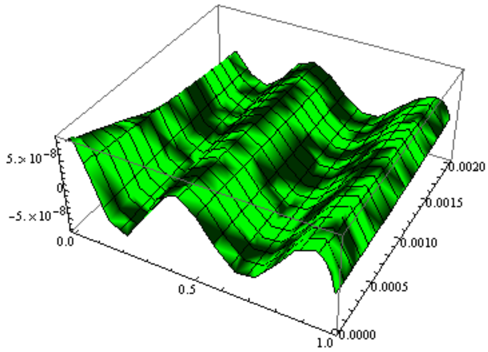


Figure 7. Error function ($k = 1$).

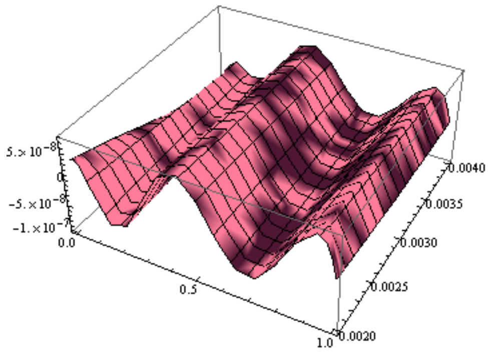


Figure 8. Error function ($k = 2$).

## 5. Conclusions

Variational iteration method and radial basis function have been known as a powerful tool for solving many equations. In this article, we have presented a general framework of the **VIM** and **RBF** method for the partial differential equations like the parabolic equations. The present work shows the validity and great potential of this technique for solving the parabolic equations. All of examples show that the results of the method are in excellent agreement with those obtained by other

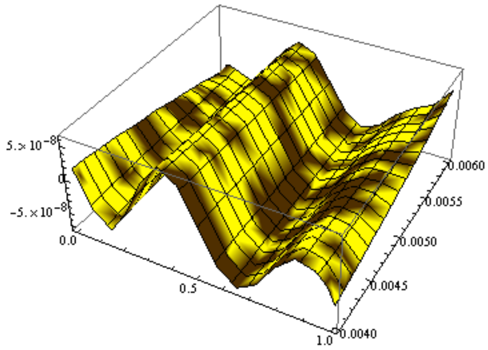


Figure 9. Error function ($k = 3$).

methods. In this type of problems, if we take care in selection of approximation radial basis functions and their shape parameter, we can obtain more accurate solution with less error.

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