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An L^p-L^q-Version Of Morgan's Theorem For The Generalized Fourier Transform Associated with a Dunkl Type Operator

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Abstract The aim of this paper is to prove new quantitative uncertainty principle for the generalized Fourier transform connected with a Dunkl type operator on the real line. More precisely we prove An L^p-L^q-version of Morgan's theorem.

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1. Introduction

The uncertainty principle is a cornerstone in quantum phsysics. However, its principles play an equally monumental role in harmonic analysis. To put it in one sentence: A nonzero function and its Fourier transform cannot both be sharply localized. While Heisenberg gave a clear physical interpretation of the uncertainty principal in 1927 in [8]. As description of this, one has Hardy's theorem [7], Morgan's theorem[9]. These theorems have been generalized to many other situations (see, for example, [1, 2, 5]). In this paper we establish an analogous of L^p-L^q-version of Morgan's theorem for the generalized Fourier transform $\mathcal{F}_\Lambda$ associated with associated with a Dunkl type operator Λ introduit and study in [3]. We prove that

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for $1 \leqslant p, q \leqslant \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p_Q(\mathbb{R})$$

and

$$e^{b|\lambda|^\eta} \mathcal{F}_\Lambda(f)(\lambda) \in L^q_Q(\mathbb{R})$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}.$$

The structure of the paper is as follows: In section 2 we set some notations and collect some basic results about the first singular differential-difference operator Λ and the generalized Fourier transform associated with Λ. In section 3 we state and prove an L^p-L^q-version of Morgan's theorem for the generalized Fourier transform associated with Λ.

2. The Harmonic Analysis Associated with Λ

In this section we provide some facts about harmonic analysis related to Λ on the real line. We cite here, as briefly as possible, some properties. For more details we refer to [3]. Throughout this paper we assume that $\alpha > \frac{-1}{2}$ and let

- $$Q(x) = \exp\left(-\int_0^x q(t)dt\right), \quad x \in \mathbb{R} \tag{1}$$
- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where
$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad if \quad p < \infty,$$
and $\|f\|_{\infty,\alpha} = \|f\|_\infty = esssup_{x\in\mathbb{R}}|f(x)|$.
- $L^1_Q(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\alpha} < \infty$, where Q is given by (1)
- $\mathcal{M}$ the map defined by $\mathcal{M}f(x) = Q(x)f(x)$ is an isometry from L^p_Q onto L^p_α

We consider the first singular differential-difference operator Λ defined on $\mathbb{R}$

$$\Lambda f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x} + q(x)f(x) \tag{2}$$

where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$. For $q = 0$ we regain the Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}.$$

2.1 *Generalized Fourier Transform*

The following statements are proved in [3]

(1) For each $\lambda \in \mathbb{C}$, the differential-difference equation

$$\Lambda u = i\lambda u, \quad u(0) = 1$$

admits a unique $\mathcal{C}^\infty$ solution on $\mathbb{R}$, denoted by Ψ_λ, given by

$$\Psi_\lambda(x) = Q(x)e_\alpha(i\lambda x), \tag{3}$$

where e_α denotes the one-dimensional Dunkl kernel defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z) \quad (z \in \mathbb{C}),$$

and j_α being the normalized spherical Bessel function of index α given by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\ \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}). \tag{4}$$

(2) For all $x \in \mathbb{R},\ \lambda \in \mathbb{C}$ and $n = 0, 1, ...$ we have

$$\mid \frac{\partial^n}{\partial \lambda^n} \Psi_\lambda(x) \mid \leqslant Q(x)|x|^n e^{|Im\ \lambda||x|}. \tag{5}$$

In particular

$$\mid \Psi_\lambda(x) \mid \leqslant Q(x)e^{|Im\ \lambda||x|}. \tag{6}$$

(3) For all $x \in \mathbb{R},\ \lambda \in \mathbb{C}$, we have the Laplace type integral representation

$$\Psi_\lambda(x) = a_\alpha Q(x)\int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}(1+t)e^{i\lambda x t}dt, \tag{7}$$

where $a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

The generalized Fourier transform associated with Λ for a function in $L^1_Q(\mathbb{R})$ is defined by

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda}(x)x^{2\alpha+1}dx. \tag{8}$$

(1) Let $f \in L^1_Q(\mathbb{R})$ such that $\mathcal{F}_\Lambda(f) \in L^1_\alpha$. Then for allmost $x \in \mathbb{R}$ we have the inversion formula

$$f(x)\,(Q(x))^2 = m_\alpha \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Psi_\lambda(x)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)}(\Gamma(\alpha+1))^2}.$$

(2) For every $f \in L^2_Q(\mathbb{R})$, we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 (Q(x))^2 |x|^{2\alpha+1} dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda.$$

(3) The generalized Fourier transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_Q(\mathbb{R})$ onto $L^2_\alpha(\mathbb{R})$.

3. An L^p-L^q-Version of Morgan's Theorem for $\mathcal{F}_\Lambda$

We start by getting the following lemma of Phragmen-Lindlöf type using the same technique as in [4, 6]. We need this lemma to prove the main result of this paper.

Suppose that $\rho \in]1, 2[$, $q \in [1, \infty]$, $\sigma > 0$ and $B > \sigma \sin\left(\frac{\pi}{2}(\rho - 1)\right)$. If g is an entire function on $\mathbb{C}$ verifiying:

$$|g(x+iy)| \leqslant C.e^{\sigma|y|^\rho} \tag{9}$$

and

$$e^{B|x|^\rho} g|_{\mathbb{R}} \in L^q_Q(\mathbb{R}) \tag{10}$$

for all $x, y \in \mathbb{R}$ then $g = 0$.

Let $1 \leqslant p, q \leqslant \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p_Q(\mathbb{R}) \tag{11}$$

and

$$e^{b|\lambda|^\eta} \mathcal{F}_\Lambda(f)(\lambda) \in L^q_Q(\mathbb{R}) \tag{12}$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta - 1)\right)\right)^{\frac{1}{\eta}}. \tag{13}$$

Proof The function

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{-\lambda}(x) x^{2\alpha+1} dx.$$

is well defined, entirely on $\mathbb{C}$ and from (8) and (6), we have

$$\begin{aligned} |\mathcal{F}_\Lambda(f)(\lambda)| &= |\int_{\mathbb{R}} f(x) \Psi_{-\lambda}(x) x^{2\alpha+1} dx|, \\ &\leqslant \int_{\mathbb{R}} |f(x)| Q(x) e^{|x||\zeta|} x^{2\alpha+1} dx, \\ &= \int_{\mathbb{R}} |\mathcal{M}f(x)| e^{|x||\zeta|} x^{2\alpha+1} dx, \quad \forall \lambda = \xi + i\zeta \in \mathbb{C} \end{aligned}$$

Applying Hölder inequality and using (15), we get

$$|\mathcal{F}_\Lambda(f)(\lambda)| \leqslant | \left(\int_{\mathbb{R}} \left(\mathcal{M}f(x)|e^{a|x|^\gamma}\right)^p x^{2\alpha+1}dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \left(|e^{-a|x|^\gamma} e^{|x||\zeta|}\right)^{p'} x^{2\alpha+1}dx\right)^{\frac{1}{p'}},$$

$$\leqslant C \left(\int_{\mathbb{R}} \left(|e^{-a|x|^\gamma} e^{|x||\zeta|}\right)^{p'} x^{2\alpha+1}dx\right)^{\frac{1}{p'}}.$$

where p' is the conjugate exponent of p.
Let $C \in I =](b\eta)^{\frac{-1}{\eta}} \sin\left(\frac{\pi}{2}(\eta-1)\right)^{\frac{1}{\eta}}, (a\gamma)^{\frac{1}{\gamma}}[$.
Applying the convex inequality

$$|ty| \leqslant (\frac{1}{\gamma})|t|^\gamma + (\frac{1}{\eta})|y|^\eta$$

to the positive numbers $C|x|$ and $\frac{|\zeta|}{C}$, we obtain

$$|x||\zeta| \leqslant (\frac{C^\gamma}{\gamma})|x|^\gamma + (\frac{1}{\eta C^\eta})|\zeta|^\eta$$

and the following relation holds

$$\int_{\mathbb{R}} e^{-ap'|x|^\gamma} e^{p'|x||\zeta|} x^{2\alpha+1}dx \leqslant e^{\frac{p'|\zeta|^\eta}{\eta C^\eta}} \int_{\mathbb{R}} e^{-p'(a-\frac{C^\gamma}{\gamma})|x|^\gamma} x^{2\alpha+1}dx.$$

Since $C \in I$, then $a > \frac{C^\gamma}{\gamma}$, and thus the integral

$$\int_{\mathbb{R}} e^{-p'(a-\frac{C^\gamma}{\gamma})|x|^\gamma} x^{2\alpha+1}dx$$

is finite. Moreover

$$|\mathcal{F}_\Lambda(f)(\lambda)| \leqslant Const.e^{\frac{p'|\zeta|^\eta}{\eta C^\eta}}, \quad forall \lambda \in \mathbb{C}. \tag{14}$$

By virtue of relations (15), (16), (14) and Lemma 3, we obtain that $\mathcal{F}_{\alpha,n}f = 0$. Then $f = 0$ by Theorem 2.1. ■

4. Conclusion

In this paper, using a generalized Fourier transform associated with a Dunkl type operator, we obtained an L^p-L^q-version of Morgan's. We proved that if $1 \leqslant p, q \leqslant \infty$, $a > 0$, $b > 0$, $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$, then for all measurable function f on $\mathbb{R}$, the conditions

$$e^{a|x|^\gamma} f \in L^p_Q(\mathbb{R}) \tag{15}$$

and

$$e^{b|\lambda|^\eta} \mathcal{F}_\Lambda(f)(\lambda) \in L^q_Q(\mathbb{R}) \tag{16}$$

imply $f = 0$ if

$$(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}. \tag{17}$$

The demonstration of this result is based on the lemma of Phragmen-Lindlöf type.

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