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An Efficient Method for the Numerical Solution of Helmholtz Type General Two Point Boundary Value Problems in ODEs

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Abstract In this article, we have proposed and analyzed a computational method for the numerical solution of a general two point Helmholtz type boundary value problems. The proposed method is tested to ensure its computational efficiency. We have obtained computational results that are in good agreement with the theoretical results for the considered model problems. Thus we conclude that the proposed method is computationally efficient and effective.

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1. Introduction

Consider the general Helmholtz type second-order boundary value problem given by

$$u''(x) + K^2u(x) = f(x, u, u'), \quad x \in (a, b),$$

subject to the boundary conditions

$$u(a) = \alpha \text{ and } u(b) = \beta. \tag{1}$$

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where a, b, α, β and K are finite constants.

We assume that $u(x) \in C^6[a, b]$, the set of all functions of x with continuous derivatives up to order 6 in the region R, $f(x, u, u')$ is continuous on $(0, 1) \times R \times R$, $f(x, u, u')$ is sufficiently differentiable w.r.t. x, u and u', $\frac{\partial f}{\partial u} \geqslant 0$, and $|\frac{\partial f}{\partial u'}| \leqslant H$ where H is positive constant. Further some specific assumption on $f(x, u, u')$, to ensure existence and uniqueness will not be considered [2]. Thus the existence and uniqueness of the solution $u(x)$ for the problem (1) is assumed [7, 9, 13].

In the literature, problems of the form (1) are conventionally solved by using finite difference method [5, 7, 10], variational techniques [6], spline techniques [4], shooting method [11] and homotopy method [15] and references therein. So much research has reported on the numerical solution of nonlinear two point boundary value problems in literature, many of them are excellent work. But a concept to develop an efficient method to solve numerically problem (1) cannot be over emphasized and always attracts researcher.

In this article, we develop a finite difference method capable of solving numerically Helmholtz type problems (1). A similar method was reported [12] in study of numerical solution of Helmholtz equation by difference method. Having seen the performance of the method for solution of special boundary value problems, we are motivated and challenged to investigate what will happen if a similar idea is used to derive a method for solution of general two point boundary value problems of Helmholtz type.

This paper is divided into five sections. Section 2 deals with the derivation and development of the algorithm while truncation error and convergence of the algorithm are developed in Section 3. Numerical experiments on model problems are presented in section 4. Section 5 will give some conclusions and an out view for future research work in this area.

2. Development and Derivation of the Method

Many phenomena that occur in chemical, biological, engineering, physical and social sciences can be modeled mathematically in the form of either ordinary or partial differential equations. However it is difficult to obtain exact solution for these differential equations especially if it is nonlinear, by analytical means. So we consider an approximate solution to these problems. There are numerous ways by which an approximate solution can be constructed. In numerical analysis a concept of approximation play very important role. Thus solving approximately these practical problems which modeled as differential equation is one of the main preoccupations in numerical analysis. We define N, the finite number of the nodal points of the interval [a,b], in which the solution of the problem (1) is desired as

$$x_j = a + jh, \quad j = 0, 1, 2 \ldots, N + 1 \tag{2}$$

where the term in right side of expression (2) are defined as the constant step length $h = \dfrac{(b - a)}{N + 1}$. So it is clear that $x_0 = a$ and $x_{N+1} = b$. Suppose we have to determine a number u_j, which is numerical approximation to the numerical value of the theoretical solution $u(x)$ of problem (1) at the nodal point x_j, $j = 1, 2 \ldots, N$. Similarly we can define other notations like f_j i.e. $f(x_j, u_j, u'_j)$ etc.. So using these notations, we can rewrite problem (1) at nodal points x_j as follows,

$$u''_j + K^2 u_j = f_j, \quad j = 1, 2, \ldots, N. \tag{3}$$

If $f(x)$ is the forcing function in problem (1) then the fourth-order discretization of problem as given in [12] is

$$(12+K^2h^2)u_{j+1}-2(12-5K^2h^2)u_j+(12+K^2h^2)u_{j-1}=h^2(f_{j+1}+10f_j+f_{j-1}),\quad j=1,2,\dots,N. \tag{4}$$

We set following approximations,

$$\overline{u}'_j=\frac{u_{j+1}-u_{j-1}}{2h}. \tag{5}$$

$$\overline{u}'_{j+1}=\frac{3u_{j+1}-4u_j+u_{j-1}}{2h}. \tag{6}$$

$$\overline{u}'_{j-1}=\frac{-u_{j+1}+4u_j-3u_{j-1}}{2h}. \tag{7}$$

Define

$$\overline{f}_{j+1}=f(x_{j+1},u_{j+1},\overline{u}'_{j+1}). \tag{8}$$

$$\overline{f}_{j-1}=f(x_{j-1},u_{j-1},\overline{u}'_{j-1}). \tag{9}$$

Let

$$\overline{\overline{u}}'_j=\overline{u}'_j+c(\overline{u}'_{j+1}+\overline{u}'_{j-1})+dh(\overline{f}_{j+1}-\overline{f}_{j-1}). \tag{10}$$

where c and d are free parameters which will be determined in an appropriate condition. Finally we define,

$$\overline{\overline{f}}_j=f(x_j,u_j,\overline{\overline{u}}'_j). \tag{11}$$

Then at each node $x_j, j=1,2,\dots,N$, we discretize equation (3) as,

$$(12+K^2h^2)u_{j+1}-2(12-5K^2h^2)u_j+(12+K^2h^2)u_{j-1}=h^2(\overline{f}_{j+1}+10\overline{\overline{f}}_j+\overline{f}_{j-1}),\quad j=1,2,\dots,N. \tag{12}$$

By application of Taylor series expansion method, from (6) and (8) we have,

$$\overline{f}_{j+1}=f_{j+1}-(\frac{h^2}{3}u_j^{(3)}+\frac{h^3}{3}u_j^{(4)})(\frac{\partial f}{\partial u})_j-\frac{h^3}{3}u_j^{(3)}(\frac{\partial^2 f}{\partial x\partial u})_j+O(h^4). \tag{13}$$

Similarly from (7) and (9) we have,

$$\overline{f}_{j-1}=f_{j-1}-(\frac{h^2}{3}u_j^{(3)}-\frac{h^3}{3}u_j^{(4)})(\frac{\partial f}{\partial u})_j+\frac{h^3}{3}u_j^{(3)}(\frac{\partial^2 f}{\partial x\partial u})_j+O(h^4). \tag{14}$$

Using (5)-(7),(13) and (14) in (10), we have

$$\overline{\overline{u}}'_j = (1 + 2c + 2dK^2h^2)u'_j + \frac{h^2}{6}(1 + 2c + 12dK^2)u_j^{(3)} + O(h^4). \quad (15)$$

Using (15) in (11) and by application of Taylor series expansion method we will obtain,

$$\overline{\overline{f}}_j = f_j + ((2c + 2dK^2h^2)u'_j + \frac{h^2}{6}(1 + 2c + 12dK^2)u_j^{(3)})(\frac{\partial f}{\partial u'})_j. \quad (16)$$

Thus form (13),(14) and (16), we have

$$\overline{f}_{j+1} + 10\overline{\overline{f}}_j + \overline{f}_{j-1} = f_{j+1} + 10f_j + f_{j-1} + \frac{h^2}{3}(5(1 + 2c + 12dK^2) - 2)(u^{(3)}\frac{\partial f}{\partial u'})_j + 20(c + dK^2h^2)(u'\frac{\partial f}{\partial u'})_j + O(h^4). \quad (17)$$

Thus $\overline{f}_{j+1} + 10\overline{\overline{f}}_j + \overline{f}_{j-1}$ will provide $O(h^4)$ approximation for $f_{j+1} + 10f_j + f_{j-1}$ if

$$5(1 + 2c + 12dK^2) - 2 = 0 \quad (18)$$

$$c + dK^2h^2 = 0 \quad (19)$$

Solving (18) and (19), we have

$$d = \frac{3}{10K^2(h^2 - 6)} \quad \text{and} \quad c = \frac{-3h^2}{10(h^2 - 6)}. \quad (20)$$

Thus we have a difference method (12) which is of $O(h^4)$ for numerical solution of problem (1) for the above values of free parameters c and d. If the system of equations (12) are linear generally solved by iterative method otherwise Newton Raphson method. In the numerical section, we will see that the performance of proposed algorithm for a variety of second order boundary value problems.

3. The Local Truncation Error and Convergence

In this section, we consider the error associated to the proposed difference method (12). Let the local truncation error in (12) be T_j and defined as in [8],

$$T_j = \frac{-(12 + K^2h^2)u_{j+1} + 2(12 - 5K^2h^2)u_j - (12 + K^2h^2)u_{j-1}}{h^2} + (\overline{f}_{j+1} + 10\overline{\overline{f}}_j + \overline{f}_{j-1}), \quad j = 1, 2, \ldots, N. \quad (21)$$

Using (17) with parameters defined in (20) and expanding the terms $u_{j\pm1}$ and $f_{j\pm1}$ are in Taylor series about point x_j, simplify the so obtained expression, so we have

$$T_j = \frac{h^4}{360}(K^2h^2 - 18)u_j^{(6)}$$
$$|T_j| \leqslant \frac{h^4}{360}(K^2h^2 - 18)M, \quad j = 1, 2, \ldots, N. \tag{22}$$

where $M = \max |u^{(6)}(x)|$ for all $x \in [a, b]$. Thus local truncation error T_j is bounded.
Consider the general Helmholtz type second-order boundary value problem given by

$$-u''(x) - K^2u(x) + f(x, u, u') = 0, \quad x \in (a, b),$$

Discretize the above equation by difference method (12), we have

$$\begin{aligned} &-(12 + K^2h^2)u_{j+1} + 2(12 - 5K^2h^2)u_j - (12 + K^2h^2)u_{j-1} \\ &\quad + h^2(f(x_{j+1}, u_{j+1}, \overline{u}'_{j+1}) + 10f(x_j, u_j, \overline{\overline{u}}'_j) + f(x_{j-1}, u_{j-1}, \overline{u}'_{j-1})) = 0, \\ &\qquad j = 1, 2, \ldots, N. \end{aligned} \tag{23}$$

Let U is exact solution of (12), so we have

$$\begin{aligned} &-(12 + K^2h^2)U_{j+1} + 2(12 - 5K^2h^2)U_j - (12 + K^2h^2)U_{j-1} \\ &\quad + h^2(f(x_{j+1}, U_{j+1}, \overline{U}'_{j+1}) + 10f(x_j, U_j, \overline{\overline{U}}'_j) + f(x_{j-1}, U_{j-1}, \overline{U}'_{j-1})) + T_j = 0, \\ &\qquad j = 1, 2, \ldots, N. \end{aligned} \tag{24}$$

Let define $\epsilon_j = u_j - U_j$ and subtract (24) from (23). Using quasi linearization technique [3] to linearize $f(x_j, u_j, u'_j)$, so we have

$$\begin{aligned} &-(12 + K^2h^2)\epsilon_{j+1} + 2(12 - 5K^2h^2)\epsilon_j - (12 + K^2h^2)\epsilon_{j-1} + \\ &h^2(\epsilon_{j+1}G_{j+1} + (\overline{u}'_{j+1} - \overline{U}'_{j+1})I_{j+1} + 10(\epsilon_jG_j + (\overline{\overline{u}}'_j - \overline{\overline{U}}'_j)F_j) + \epsilon_{j-1}G_{j-1} + (\overline{u}'_{j-1} - \overline{U}'_{j-1})I_{j-1}) \\ &\qquad = T_j, \quad j = 1, 2, \ldots, N. \end{aligned} \tag{25}$$

where $G = \frac{\partial f}{\partial U}$, $I = \frac{\partial f}{\partial \overline{U}'}$ and $F = \frac{\partial f}{\partial \overline{\overline{U}}'}$. Expand F,G and I in Taylor series about point x_j and using approximations (5)-(10),simplify the expression, we have

$$\begin{aligned} &-(12 + (K^2 + M_j)h^2)\epsilon_{j+1} + 2(12 - 5(K^2 - G_j)h^2 + 20hI_jF_j)\epsilon_j - (12 + (K^2 + m_j)h^2)\epsilon_{j-1} = T_j, \\ &\qquad j = 1, 2, \ldots, N. \end{aligned} \tag{26}$$

where $M_j = G_j + 30dI_jF_j + 10dhG_jF_j + \frac{2I_j + 5(1+2c)F_j}{2h}$ and
$m_j = G_j + 30dI_jF_j - 10dhG_jF_j - \frac{2I_j + 5(1+2c)F_j}{2h}$. It is possible to write (26) in matrix form as,

$$\mathbf{JE} = \mathbf{T} \tag{27}$$

where $\mathbf{J} = [-(12 + (K^2 + m_j)h^2), 2(12 - 5(K^2 - G_j)h^2 + 20hI_jF_j), -(12 + (K^2 + M_j)h^2)]$ tridiagonal matrix,$\mathbf{E} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]^T$ and $\mathbf{T} = [T_1, T_2, \ldots, T_N]^T$.
Let

$$G_* = \min_{x\in[a,b]} \frac{\partial f}{\partial U}, \quad G^* = \max_{x)\in[a,b]} \frac{\partial f}{\partial U}$$

Then

$$0 < G_* \leqslant G_j \leqslant G^* \quad .$$

Let us assume that

$$0 < |\theta| < q_0, \quad q_0 > 0, \qquad ,\forall \quad \theta \in I_0.$$

where $I_0 = \{I_j, F_j; \forall j = 1, 2, \ldots, N\}$. It is easy to verify for sufficiently small h that matrix $\mathbf{J}$ is row diagonally dominant. Let $\mathbf{J}$ be the adjacency matrix of some graph Gr. We may easily prove that graph Gr is connected. From this fact it follows that adjacency matrix $\mathbf{J}$ is irreducible [14]. By the row sum criterion it follows that $\mathbf{J}$ is monotone [7]. Thus positive $\mathbf{J}^{-1}$ exist. Thus from (27), we have

$$\|\mathbf{E}\|_\infty \leqslant \|\mathbf{J}^{-1}\|_\infty \|\mathbf{T}\|_\infty. \tag{28}$$

With the help of (22) and (28), for sufficiently small h, we have

$$\|\mathbf{E}\| \leqslant O(h^4) \tag{29}$$

Thus the proposed difference method (12) converges and the order of the convergence is four.

4. Numerical Experiment

In this section, numerical examples linear and nonlinear were considered, to illustrate our algorithm (12) and to demonstrate computationally its efficiency and accuracy. In tables we have shown maximum absolute error computed on the nodal points in the interval of integration for these examples in their solution. Let u_j is the numerical value of solution calculated by (12) which is an approximate value of the theoretical solution $u(x)$ at the point $x = x_j$. Maximum absolute error is calculated in both solution and derivative of solution by

$$MAE(u) = \max_j |u(x_j) - u_j|, \quad j = 1, 2, \ldots, N.$$

All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC.

Problem 1. Consider the nonlinear boundary value problem

$$u''(x) + K^2u(x) = u(x)u'(x) + f(x), \qquad 0 < x < 1.$$

Table 1. Maximum absolute error in $u(x) = \sin(\pi x)$ for problem 1.

2*	MAE			
	N			
K	4	8	16	32
100	.13411045(-4)	.23245811(-5)	.59604645(-6)	.17881393(-6)
1000	.11920929(-6)	.59604645(-7)	.59604645(-7)	.59604645(-7)
10000	.59604645(-7)	.59604645(-7)	.59604645(-7)	.59604645(-7)

subject to boundary conditions

$$u(0) = 0 \quad , \quad u(1) = 0$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = \sin(\pi x)$. The maximum absolute error in $u(x)$ for different values of K are given *Table 1.*

Problem 2. Consider linear boundary value problem [1],

$$u''(x) + K^2 u(x) = x^2 + \exp(x) + f(x), \qquad 0 < x < 1.$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u(1) = 1.0 + \sin(K)$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = x^2 + \sin(Kx)$. The maximum absolute error in $u(x)$ for different values of K are given *Table 2.*

Problem 3. Consider the nonlinear boundary value problem

$$u''(x) + K^2 u(x) = u^2(x) + f(x), \qquad 0 < x < 1.$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u(1) = \sin(\frac{\pi K}{2})$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = \sin(\frac{\pi K x}{2})$. The maximum absolute error in $u(x)$ for different values of K are given *Table 3.*

Table 2. Maximum absolute error in $u(x) = x^2 + \sin(Kx)$ for problem 2 .

2*	MAE		
	N		
K	32	64	128
2	.11920929(-6)	.11920929(-6)	.11920929(-6)
4	.11920929(-6)	.11920929(-6)	.11920929(-6)
5	.13113022(-5)	.11920929(-6)	.11920929(-6)

Table 3. Maximum absolute error in $u(x) = \sin(\frac{\pi K x}{2})$ for problem 3 .

2*	MAE			
	N			
K	32	64	128	256
4	.10371208(-4)	.59604645 (-7)	.59604645 (-7)	.59604645 (-7)
6	overflows	.95367432(-5)	.59604645(-7)	.59604645(-7)
8	overflows	.92983246(-5)	.59604645(-7)	.59604645(-7)

5. Conclusion

In this article, we have described a novel method that is efficient and convergent for solving two point Helmholtz type boundary value problems in ordinary differential equations. The results we obtained in numerical section for examples show that method is computationally efficient and accurate. It is clear that computational efficiency of the method depends on both h and K. How we can improve the computational efficiency without any effect of K and order of accuracy of the method? Investigation in this specific direction will be done in the future. However our future works will deal with similar extension of the present method to solve boundary value problems in partial differential equations. Work in this direction is in progress.

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