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Operator-valued bases on Hilbert spaces

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Abstract. In this paper we develop a natural generalization of Schauder basis theory, we term operator-valued basis or simply *ov*-basis theory, using operator-algebraic methods. We prove several results for *ov*-basis concerning duality, orthogonality, biorthogonality and minimality. We prove that the operators of a dual *ov*-basis are continuous. We also define the concepts of Bessel, Hilbert *ov*-basis and obtain some characterizations of them. We study orthonormal and Riesz *ov*-bases for Hilbert spaces. Finally we consider the stability of *ov*-bases under small perturbations. We generalize a result of Paley-Wiener [4] to the situation of *ov*-basis.

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1. Introduction

Throughout this paper, $\mathcal{H}, \mathcal{K}$ are separable Hilbert spaces and I, J, J_j denote the countable (or finite) index sets and $\{W_j\}_j$ is a sequence of closed subspaces of $\mathcal{K}$ and $B(\mathcal{H}, W_j)$ denote the collection of all bounded linear operators from $\mathcal{H}$ into W_j and $\Lambda_j \in B(\mathcal{H}, W_j)$ for all $j \in J$. Also $\mathcal{R}_T$ and $\mathcal{N}_T$ denote the range and null spaces of an operator $T \in B(\mathcal{H}, \mathcal{K})$ respectively. Recently, W. Sun [3] introduced a generalized frame and a generalized Riesz basis for a Hilbert space and discussed some properties of them. In this paper we introduce the concept of the operator-valued basis and then we redefined the concepts of the orthonormal operator-valued basis and operator-valued Riesz basis

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for a Hilbert space. we develop the basis theory to the situation of operator-valued basis theory in Hilbert spaces.

Definition 1.1 Let $\Lambda_j \in B(\mathcal{H}, W_j)$ be an onto operator for all $j \in J$. Then the family $\Lambda = \{\Lambda_j\}_{j\in J}$ is called an operator-valued basis or simply *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if for any $f \in \mathcal{H}$ there exists an unique sequence $\{g_j : g_j \in W_j\}_{j\in J}$ such that

$$f = \sum_{j\in J} \Lambda_j^* g_j, \tag{1}$$

with the convergence being in norm. If series (1) is unconditionally convergent, Λ is called an unconditional *ov*-basis. We call this family an *ov*-basis for $\mathcal{H}$ with respect to $\mathcal{K}$ if $W_j = \mathcal{K}$ for all $j \in J$.

Theorem 1.2 Let $\{\Lambda_j\}_{j\in J}$ be an*ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then

$$\dim \mathcal{H} = \sum_{j\in J} \dim W_j$$

.

Proof. Let $\{e_{ij}\}_{i\in J_j}$ be an orthonormal basis for W_j for all $j \in J$. We show that $\{\Lambda_j^* e_{ij}\}_{j\in J, i\in J_j}$ is a basis for $\mathcal{H}$. Since $\{e_{ij}\}_{i\in J_j}$ is an orthonormal basis for W_j, hence every $g_j \in W_j$ has a unique expansion of the form $g_j = \sum_{i\in J_j} < g_j, e_{ij} > e_{ij}$. This implies that also every $f \in \mathcal{H}$ has a unique expansion of the form

$$f = \sum_{j\in J}\sum_{i\in J_j} < g_j, e_{ij} > \Lambda_j^* e_{ij}.$$

This shows that $\dim \mathcal{H} = \sum_{j\in J} \dim W_j$. ∎

Corollary 1.3 Let $\{\Lambda_j\}_{j\in J}, \{\Gamma_i\}_{i\in I}$ be *ov*-bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}, \{V_i\}_{i\in I}$ respectively. Then $\sum_{j\in J} \dim W_j = \sum_{i\in I} \dim V_i$.

2. Characterizations of *ov*-bases

Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then every $f \in \mathcal{H}$ has a unique expansion of the form $f = \sum_{j\in J} \Lambda_j^* g_j$. It is clear that each $g_j \in W_j$ is a linear operator of f. If we denote this linear operator by $\Gamma_j : \mathcal{H} \to W_j$, then $g_j = \Gamma_j f$, and we have $f = \sum_{j\in J} \Lambda_j^* \Gamma_j f$. The sequence $\{\Gamma_j\}_{j\in J}$ is called the dual *ov*-basis of Λ. In the next theorem we show that the operators of a dual *ov*-basis are continuous.

Theorem 2.1 Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, and let $\{\Gamma_j\}_{j\in J}$ be the dual *ov*-basis of Λ, then $\Gamma_j \in B(\mathcal{H}, W_j)$, for all $j \in J$. Moreover, if $\Gamma_j \neq 0$ for some $j \in J$, then $\|\Gamma_j\|\|\Lambda_j\| \geqslant 1$.

Proof. Define the space

$$\mathcal{A} = \Big\{\{g_j\}_{j\in J} |\ g_j \in W_j,\ \sum_{j\in J} \Lambda_j^* g_j \text{ is convergent}\Big\},$$

with the norm defined by

$$\left\|\{g_j\}_{j\in J}\right\| = \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} \Lambda_i^* g_i\right\| < \infty.$$

It is clear that $\mathcal{A}$ endowed with this norm, is a normed space with respect to the pointwise operations. We will show that the space $\mathcal{A}$ is a complete. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}$. If $u_n = \{g_{nj}\}_{j\in J}$, then given any $\varepsilon > 0$, there exists a number N such that

$$\sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} (\Lambda_i^* g_{ni} - \Lambda_i^* g_{mi})\right\| < \varepsilon \tag{2}$$

for all $m, n \geqslant N$. Now for all $j \in J$ and $m, n \geqslant N$ we have

$$\|\Lambda_j^* g_{nj} - \Lambda_j^* g_{mj}\| \leqslant \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} (\Lambda_i^* g_{ni} - \Lambda_i^* g_{mi})\right\| < \varepsilon.$$

This shows that $\{\Lambda_j^* g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Since Λ_j is onto hence by Theorem 4.13 of [2] the sequence $\{g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in W_j and thus convergent. Let $g_j \in W_j$ such that $g_j = \lim_{n\to\infty} g_{nj}$ and $u = \{g_j\}_{j\in J}$. From (2), by letting $m \to \infty$, we obtain

$$\sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} (\Lambda_i^* g_{ni} - \Lambda_i^* g_i)\right\| \leqslant \varepsilon \tag{3}$$

for all $n \geqslant N$. Since for all finite non-empty subset $F \subset J$ we have

$$\begin{aligned}\left\|\sum_{i\in J-F} \Lambda_i^* g_i\right\| &\leqslant \left\|\sum_{i\in J-F} (\Lambda_i^* g_{Ni} - \Lambda_i^* g_i)\right\| + \left\|\sum_{i\in J-F} \Lambda_i^* g_{Ni}\right\| \\ &\leqslant \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} (\Lambda_i^* g_{Ni} - \Lambda_i^* g_i)\right\| + \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} \Lambda_i^* g_{Ni}\right\|\end{aligned}$$

thus $u \in \mathcal{A}$. Moreover (3) implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is convergent to u in $\mathcal{A}$. Therefore $\mathcal{A}$ is a Banach space. Define the mapping

$$T : \mathcal{A} \to \mathcal{H} \quad \text{with} \quad T(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j.$$

Since Λ is a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ hence T is linear, one-to-one and onto. On the other hand, since

$$\|T(\{g_j\}_{j\in J})\| = \left\|\sum_{j\in J} \Lambda_j^* g_j\right\| \leqslant \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\|\sum_{i\in F} \Lambda_i^* g_i\right\| = \|\{g_j\}_{j\in J}\|.$$

Thus T is continuous and the open mapping theorem then guarantees that T^{-1} is also continuous. This shows that $\mathcal{A}$ and $\mathcal{H}$ are Banach spaces isomorphic. Now suppose that $f = \sum_{j\in J} \Lambda_j^* g_j$ is a fixed, arbitrary element of $\mathcal{H}$ and let $j \in J$. Since Λ_j is onto thus by

Theorem 4.13 of [2] there is a $m_j > 0$ such that $m_j\|g\| \leqslant \|\Lambda_j^* g\|$ for all $g \in W_j$. Moreover, we have

$$\|\Gamma_j f\| = \|g_j\| \leqslant \frac{\|\Lambda_j^* g_j\|}{m_j} \leqslant \frac{\sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \left\| \sum_{i\in F} \Lambda_i^* g_i \right\|}{m_j} = \frac{2\|T^{-1}f\|}{m_j} \leqslant \frac{2\|T^{-1}\|\|f\|}{m_j}.$$

This shows that each Γ_j is continuous and $\|\Gamma_j\| \leqslant \frac{2\|T^{-1}\|}{m_j}$. For the remaining inequality assume that $0 \neq g_j = \Gamma_j f$ for some $f \in \mathcal{H}$, then we have

$$\|g_j\| = \|\Gamma_j \Lambda_j^* g_j\| \leqslant \|\Gamma_j\|\|\Lambda_j\|\|g_j\|,$$

which implies that $\|\Gamma_j\|\|\Lambda_j\| \geqslant 1$. ■

Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be the dual *ov*-basis of $\{\Lambda_j\}_{j\in J}$. Then F-partial sum operator of $\{\Lambda_j\}_{j\in J}$ defined by

$$S_F : \mathcal{H} \to \mathcal{H} \quad \text{with} \quad S_F f = \sum_{j\in F} \Lambda_j^* \Gamma_j f,$$

for all finite subset $F \subset J$. By Theorem 2.1, S_F is a bounded operator and

$$1 \leqslant \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \|S_F\| < \infty. \tag{4}$$

A family of operators $\{\Lambda_j \in B(\mathcal{H}, W_j) : \ j \in J\}$ is called a complete sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if $\mathcal{H} = \overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$. It is easy to check that $\{\Lambda_j\}_{j\in J}$ is a complete sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if and only if $\{f : \ \Lambda_j f = 0, j \in J\} = \{0\}$.

Theorem 2.2 Let $\{\Lambda_j\}_{j\in J}$ be a complete sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if there exists a constant M such that

$$\Big\| \sum_{i\in F} \Lambda_i^* g_i \Big\| \leqslant M \Big\| \sum_{i\in G} \Lambda_i^* g_i \tag{5}$$

for all finite subsets $F \subset G \subset J$ and arbitrary vectors $g_j \in W_j, j \in G$.

Proof. First suppose that $\{\Lambda_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $M = \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \|S_F\|$, then for all finite subsets $F \subset G \subset J$ and arbitrary vectors $g_j \in W_j$ we have

$$\|\sum_{j\in F} \Lambda_j^* g_j\| = \|S_F\big(\sum_{j\in G} \Lambda_j^* g_j\big)\| \leqslant M\|\sum_{j\in G} \Lambda_j^* g_j\|.$$

To prove the opposite implication take $f \in \mathcal{H}$. By hypothesis, there exist finite subsets $F_n \subset F_{n+1} \subset J$ and vectors $g_{nj} \in W_j$ for all $n \in \mathbb{N}, j \in F_n$ such that $f = \lim_{n\to\infty} \sum_{j\in F_n} \Lambda_j^* g_{nj}$. For notational convenience, put $g_{nj} = 0$ for $j \notin F_n$, then for every $m > n$ and $j \in F_n$ we have

$$\begin{aligned}\|\Lambda_j^*(g_{nj} - g_{mj})\| &\leqslant M\Big\|\sum_{i\in F_n}\Lambda_i^*(g_{ni} - g_{mi})\Big\| \\ &\leqslant M^2\Big\|\sum_{i\in F_m}\Lambda_i^*(g_{ni} - g_{mi})\Big\| \\ &= M^2\Big\|\sum_{i\in F_n}\Lambda_i^* g_{ni} - \sum_{i\in F_m}\Lambda_i^* g_{mi}\Big\| \to 0 \quad (n\to\infty).\end{aligned}$$

This shows that $\{\Lambda_j^* g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Since Λ_j is onto hence by Theorem 4.13 [2] the sequence $\{g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in W_j and thus convergent. Let $g_j \in W_j$ such that $g_j = \lim_{n\to\infty} g_{nj}$, then $f = \sum_{j\in J}\Lambda_j^* g_j$. Now we show that this representation is unique. If $\sum_{j\in J}\Lambda_j^* g_j = 0$, then for any finite subset $F \subset J$ and $j \in F$ we have

$$\|\Lambda_j^* g_j\| \leqslant M\Big\|\sum_{i\in F}\Lambda_i^* g_i\Big\| \to 0.$$

This shows that $\|\Lambda_j^* g_j\| = 0$. Since Λ_j^* is one-to-one on W_j, hence $g_j = 0$ which this completes the proof. ■

Corollary 2.3 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual *ov*-basis $\{\Gamma_j\}_{j\in J}$. Then $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and

$$f = \sum_{j\in J}\Gamma_j^*\Lambda_j f \qquad \forall f \in \mathcal{H}.$$

Proof. First we prove that $\mathcal{H} = \overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in J}$. To see this, let $f \perp \overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in J}$. Then

$$\|\Gamma_j f\|^2 =< f, \Gamma_j^*\Gamma_j f >= 0,$$

which implies that $\Gamma_j f = 0$ for all $j \in J$. We also have

$$f = \sum_{j\in J}\Lambda_j^*\Gamma_j f = 0.$$

Thus $\mathcal{H} = \overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in J}$. We now prove that $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. For this, we show that $S_F^* f \to f$ for all $f \in \mathcal{H}$. First assume that f is a finite linear combination of $\{\Gamma_j^* g_j : \ g_j \in W_j, \ j \in J\}$, say $f = \sum_{j\in G}\Gamma_j^* g_j$ and let $F \supseteq G$ be a finite arbitrary set. Then by hypothesis for any $i, j \in J$ we have $\Gamma_j\Lambda_i^*\Gamma_i = \delta_{ij}\Gamma_i$ hence $\Gamma_i^*\Lambda_i\Gamma_j^* = \delta_{ij}\Gamma_i^*$. It follows that

$$S_F^* f = \sum_{j\in G} S_F^*\Gamma_j^* g_j = \sum_{j\in G}\sum_{i\in F}\Gamma_i^*\Lambda_i\Gamma_j^* g_j = \sum_{j\in G}\Gamma_j^* g_j = f.$$

Now if $f \in \mathcal{H}$, then given $\varepsilon > 0$ we can find $g = \sum_{j\in G}\Gamma_j^* g_j$ such that $\|f - g\| < \frac{\varepsilon}{M+1}$, where $M = \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \|S_F\|$. We also have

$$\|S_F^* f - f\| \leqslant \|S_F^* f - S_F^* g\| + \|g - f\| \leqslant (\|S_F\| + 1)\|f - g\| < \varepsilon$$

for every finite set $F \supseteq G$. Thus every $f \in \mathcal{H}$ has at least one representation of the form $f = \sum_{j\in J} \Gamma_j^* \Lambda_j f$. We show that this representation is unique. Assume that $\sum_{j\in J} \Gamma_j^* g_j = 0$ then by hypothesis for any $i, j \in J$ we have $\Gamma_j \Lambda_i^* \Lambda_i = \delta_{ij} \Lambda_i$ thus $\Lambda_i^* \Lambda_i \Gamma_j^* = \delta_{ij} \Lambda_i^*$. It follows that

$$\Lambda_i^* g_i = \Lambda_i^* \Lambda_i \big(\sum_{j\in J} \Gamma_j^* g_j\big) = 0.$$

Since Λ_i^* is one-to-one on W_i, therefore $g_i = 0$ for all $i \in J$. This completes the proof. ■

Definition 2.4 Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be sequences of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

- (i) Let Λ_j be onto for all $j \in J$, then $\{\Gamma_j\}_{j\in J}$ is called a *ov*-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$, if $\Gamma_i \Lambda_j^* g_j = \delta_{ij} g_j$ for all $i, j \in J$, $g_j \in W_j$.
- (ii) $\{\Lambda_j\}_{j\in J}$ is called minimal, if for each $j \in J$

$$\Lambda_j^*(W_j) \cap \overline{\operatorname{span}}\{\Lambda_k^*(W_k)\}_{\substack{k\in J,\\ k\neq j}} = \{0\}.$$

- (iii) We say that $\{\Lambda_j\}_{j\in J}$ is ω-independent if whenever $\sum_{j\in J} \Lambda_j^* g_j = 0$ for some sequence $\{g_j : g_j \in W_j\}_{j\in J}$, then necessarily $g_k = 0$ for all $k \in J$.

Since $\Lambda_j^* \Lambda_j \Gamma_i^* = \delta_{ij} \Lambda_j^*$ for all $i, j \in J$ and Λ_j^* is one-to-one on W_j hence if $\{\Gamma_j\}_{j\in J}$ is a *ov*-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ is also a *ov*-biorthogonal sequence of $\{\Gamma_j\}_{j\in J}$.

Proposition 2.5 Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let Λ_j be onto for all $j \in J$, then $\{\Lambda_j\}_{j\in J}$ is minimal if and only if it is ω-independent.

Proof. First assume that $\{\Lambda_j\}_{j\in J}$ is not ω-independent, then there is a sequence $\{g_j : g_j \in W_j\}_{j\in J}$ with $g_k \neq 0$ for some $k \in J$, such that $\sum_{j\in J} \Lambda_j^* g_j = 0$. It follows $\Lambda_k^* g_k = \sum_{\substack{j\in J,\\ j\neq k}} \Lambda_j^*(-g_j)$ which implies that $\Lambda_k^* g_k \in \overline{\operatorname{span}}\{\Lambda_j^*(W_j)\}_{\substack{j\in J,\\ j\neq k}}$. That is, $\{\Lambda_j\}_{j\in J}$ is not minimal. The other implication is obvious. ■

Proposition 2.6 Every *ov*-basis for a Hilbert space possesses a unique *ov*-biorthogonal sequence.

Proof. By definition, the dual *ov*-basis of a *ov*-basis is a *ov*-biorthogonal sequence of it. Moreover, if $\{\Gamma_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be *ov*-biorthogonal sequences of *ov*-basis $\{\Lambda_j\}_{j\in J}$, then for all $f \in \mathcal{H}$ and $i, j \in J$ we have $\Psi_i \Lambda_j^* \Gamma_j f = \delta_{ij} \Gamma_j f$, which implies that

$$\Lambda_i^* \Psi_i f = \sum_{j\in J} \Lambda_i^* \Psi_i \Lambda_j^* \Gamma_j f = \sum_{j\in J} \delta_{ij} \Lambda_i^* \Gamma_j f = \Lambda_i^* \Gamma_i f.$$

Since Λ_i^* is one-to-one on W_i, hence $\Gamma_i = \Psi_i$. ■

Proposition 2.7 Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, and let Λ_j be onto for all $j \in J$. Then

(*i*) $\{\Lambda_j\}_{j\in J}$ has a *ov*-biorthogonal sequence, if and only if $\{\Lambda_j\}_{j\in J}$ is minimal.
(*ii*) The *ov*-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$ is unique if and only if $\{\Lambda_j\}_{j\in J}$ is complete.

Proof. For the proof of (*i*) suppose that $\{\Gamma_j\}_{j\in J}$ is a *ov*-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$, and let $f \in \Lambda_k^*(W_k) \cap \overline{\text{span}}\{\Lambda_j^*(W_j)\}_{\substack{j\in J,\\ j\neq k}}$ for any given $k \in J$. Then there exists a sequence $\{g_j :\ g_j \in W_j\}_{j\in J}$ such that $f = \Lambda_k^* g_k = \sum_{\substack{j\in J,\\ j\neq k}} \Lambda_j^* g_j$. We also have

$$g_k = \Gamma_k \Lambda_k^* g_k = \sum_{\substack{j\in J,\\ j\neq k}} \Gamma_k \Lambda_j^* g_j = \sum_{\substack{j\in J,\\ j\neq k}} \delta_{kj} g_j = 0,$$

which implies that $f = 0$. That is, $\{\Lambda_j\}_{j\in J}$ is minimal. For the opposite implication in (*i*), suppose that $\{\Lambda_j\}_{j\in J}$ is minimal, and let $\mathcal{H}_0 = \overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$. From Proposition 2.5 it follows that $\{\Lambda_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}_0$ with respect to $\{W_j\}_{j\in J}$. Let $\{\Gamma_j'\}_{j\in J}$ be dual *ov*-basis of $\{\Lambda_j\}_{j\in J}$. If we define $\Gamma_j = \Gamma_j' P$ for all $j \in J$, where P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$. Then $\{\Gamma_j\}_{j\in J}$ is a *ov*-biorthogonal sequence for $\{\Lambda_j\}_{j\in J}$.

(*ii*) Let $\{\Gamma_j\}_{j\in J}$ be a *ov*-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$. If $\{\Lambda_j\}_{j\in J}$ is not complete, then the sequence $\{\Psi_j\}_{j\in J}$ defined by $\Psi_j = \Gamma_j + \Lambda_j(Id_{\mathcal{H}} - P)$ for all $j \in J$ is a *ov*-biorthogonal sequence for $\{\Lambda_j\}_{j\in J}$. For the other implication in (*ii*), assume that $\{\Lambda_j\}_{j\in J}$ is complete. If $\sum_{j\in J} \Lambda_j^* g_j = 0$ for any given sequence $\{g_j :\ g_j \in W_j\}_{j\in J}$, then for every $k \in J$ we have

$$g_k = \sum_{j\in J} \delta_{kj} g_j = \sum_{j\in J} \Gamma_k \Lambda_j^* g_j = \Gamma_k(\sum_{j\in J} \Lambda_j^* g_j) = 0.$$

This shows that $\{\Lambda_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Now the conclusion follows from Proposition 2.6. ■

Theorem 2.8 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $T : \mathcal{H} \to \mathcal{U}$ be a bounded linear operator such that $\Gamma_j = \Lambda_j T^*$ for all $j \in J$. Then $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ if and only if T is invertible.

Proof. Let T be invertible and let $g \in \mathcal{U}$, then we can write $g = Tf$ for some $f \in \mathcal{H}$. Since $\{\Lambda_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ hence $f \in \mathcal{H}$ has an unique expansion of the form $f = \sum_{j\in J} \Lambda_j^* g_j$ where $g_j \in W_j$ for all $j \in J$. It follows that

$$g = Tf = \sum_{j\in J} T\Lambda_j^* g_j = \sum_{j\in J} \Gamma_j^* g_j$$

which implies that $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$. Now we assume $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are *ov*-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ respectively. Since for every sequence $\{g_j :\ g_j \in W_j\}_{j\in J}$ we have $T(\sum_{j\in J} \Lambda_j^* g_j) = \sum_{j\in J} \Gamma_j^* g_j$. Therefore T is invertible. ■

Definition 2.9 Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be *ov*-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ respectively. Then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are said to be equivalent if for any given sequence $\{g_j :\ g_j \in W_j\}_{j\in J}$ the series $\sum_{j\in J} \Lambda_j^* g_j$ is convergent if and only if the series $\sum_{j\in J} \Gamma_j^* g_j$ is convergent.

Theorem 2.10 Two *ov*-bases $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ are equivalent if and only if there exists a bounded linear invertible operator $T : \mathcal{H} \to \mathcal{U}$

such that $\Gamma_j = \Lambda_j T^*$.

Proof. Assume that $T : \mathcal{H} \to \mathcal{U}$ be the bounded linear invertible operator such that $\Gamma_j = \Lambda_j T^*$ for all $j \in J$. Then the sufficiency follows from the fact that for every sequence $\{g_j : g_j \in W_j\}_{j \in j}$ we have

$$\sum_{j\in J} \Gamma_j^* g_j = T\Big(\sum_{j\in J} \Lambda_j^* g_j\Big) \quad \text{and} \quad \sum_{j\in J} \Lambda_j^* g_j = T^{-1}\Big(\sum_{j\in J} \Gamma_j^* g_j\Big).$$

Now suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are equivalent ov-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$. If $f \in \mathcal{H}$ with unique expansion $f = \sum_{j\in J} \Lambda_j^* g_j$, then the series $\sum_{j\in J} \Gamma_j^* g_j$ converges to an element $Tf \in \mathcal{U}$. Therefore, Tf is well defined. Since Λ_j^* is one-to-one on W_j for all $j \in J$, hence it is easy to check that T is linear, bijective and $\Gamma_j = \Lambda_j T^*$. To show that T is a bounded invertible operator, we define operators T_F by $T_F f = \sum_{j\in F} \Gamma_j^* g_j$ for every non-empty finite subset $F \subset J$. Then $Tf = \lim_F T_F f$ for every $f \in \mathcal{H}$. Since by Theorem 2.8 each T_F is bounded thus the Banach-Steinhaus Theorem implies that T is bounded. Moreover the open mapping Theorem guarantees that T is invertible. ■

Theorem 2.11 The ov-biorthogonal sequences associated with equivalent ov-bases are equivalent.

Proof. Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be equivalent ov-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ and let, $\{\Psi_j\}_{j\in J}$ and $\{\Phi_j\}_{j\in J}$ be ov-biorthogonal sequences for them respectively. By assumption there exists a bounded invertible operator $T : \mathcal{H} \to \mathcal{U}$ such that $\Gamma_j = \Lambda_j T^*$. For any $f \in \mathcal{H}$ we have

$$f = T^{-1}Tf = T^{-1}\Big(\sum_{j\in J} \Gamma_j^* \Phi_j Tf\Big) = T^{-1}\Big(\sum_{j\in J} T\Lambda_j^* \Phi_j Tf\Big) = \sum_{j\in J} \Lambda_j^* \Phi_j Tf.$$

By Proposition 2.6 it follows that $\Psi_j = \Phi_j T$ for all $j \in J$. that is $\{\Psi_j\}_{j\in J}$ and $\{\Phi_j\}_{j\in J}$ are equivalent. ■

For each sequence $\{W_j\}_{j\in J}$ of closed subspaces of $\mathcal{K}$, we define the Hilbert space associated with $\{W_j\}_{j\in J}$ by

$$\Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2} = \Big\{\{g_j\}_{j\in J} | g_j \in W_j \text{ and } \sum_{j\in J} \|g_j\|^2 < \infty\Big\}. \quad (6)$$

with inner product given by

$$< \{f_k\}_{k\in J}, \{g_k\}_{k\in J} >= \sum_{j\in J} < f_j, g_j > . \quad (7)$$

Definition 2.12 Let $\{\Lambda_j\}_{j\in J}$ be a ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. We say that $\{\Lambda_j\}_{j\in J}$ is a Bessel ov-basis if whenever $\sum_{j\in J} \Lambda_j^* g_j$ converges, then $\{g_j\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$. It is called a Hilbert ov-basis, if the series $\sum_{j\in J} \Lambda_j^* g_j$ is convergent for all $\{g_j\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$.

Theorem 2.13 Let $\{\Lambda_j\}_{j\in J}$ be a ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a Bessel ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if there exists a constant

$A > 0$ such that

$$A\sum_{j\in F}\|g_j\|^2 \leqslant \Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2$$

for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$.

Proof. The sufficiency is trivial. Assume that $\{\Lambda_j\}_{j\in J}$ is a Bessel *ov*-basis and consider the space

$$\mathcal{A} = \Big\{\{g_j\}_{j\in J} |\ g_j \in W_j,\ \sum_{j\in J}\Lambda_j^* g_j \text{ is convergent}\Big\}.$$

Clearly $\mathcal{A}$ is a subspace of $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$. We show that $\mathcal{A}$ is closed. To see this, let $\{g_{nj}\}_{j\in J}$ be a sequence in $\mathcal{A}$ such that converges to some $\{g_j\}_{j\in J} \in \big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$, then $g_{nj} \to g_j$ for all $j \in J$. Let F be an arbitrary finite subset of J and $n \in \mathbb{N}$, then we have

$$\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\| \leqslant \Big\|\sum_{j\in F}\Lambda_j^*(g_{nj} - g_j)\Big\| + \Big\|\sum_{j\in F}\Lambda_j^* g_{nj}\Big\|.$$

It follows that $\sum_{j\in J}\Lambda_j^* g_j$ is Cauchy and hence convergent in $\mathcal{H}$, which implies that $\mathcal{A}$ is closed. Now define the operator $T : \mathcal{A} \to \mathcal{H}$ by

$$T(\{g_j\}_{j\in J}) = \sum_{j\in J}\Lambda_j^* g_j.$$

Then, it is obvious that T is linear, one-to-one. To show that T is a bounded operator, we define the bounded operators $T_F : \mathcal{A} \to \mathcal{H}$ by $T_F(\{g_j\}_{j\in J}) = \sum_{j\in F}\Lambda_j^* g_j$. Then $T_F \to T$ pointwise. Since each T_F is bounded the Banach-Steinhaus Theorem follows that T is bounded. Now by Theorems 4.13 and 4.15 of [2] there exists a constant $A > 0$ such that

$$A\sum_{j\in F}\|g_j\|^2 \leqslant \Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2$$

for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$. ■

Theorem 2.14 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a Hilbert *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if there exists a constant $B > 0$ such that

$$\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2 \leqslant B\sum_{j\in F}\|g_j\|^2$$

for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$.

Proof. Suppose that $\{\Lambda_j\}_{j\in J}$ is a Hilbert *ov*-basis then the Banach-Steinhaus Theorem guarantees that the operator $T : \big(\sum_{j\in J}\oplus W_j\big)_{\ell^2} \to \mathcal{H}$ defined by $T(\{g_j\}_{j\in J}) = \sum_{j\in J}\Lambda_j^* g_j$ is bounded. Therefore there exists a constant $B > 0$ such that

$$\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2 \leqslant B\sum_{j\in F}\|g_j\|^2$$

for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$. The opposite conclusion is trivial. ■

Theorem 2.15 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual *ov*-basis $\{\Gamma_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a Bessel *ov*-basis if and only if $\{\Gamma_j\}_{j\in J}$ is a Hilbert *ov*-basis.

Proof. First suppose that $\{\Lambda_j\}_{j\in J}$ is a Bessel *ov*-basis, then $\{\Gamma_j f\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$ for all $f \in \mathcal{H}$. Fix $F \subset J$ with $|F| < \infty$ and let $f = \sum_{j\in F} \Gamma_j^* g_j$. Then we have

$$\Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|^4 = |< f, \sum_{j\in F}\Gamma_j^* g_j >|^2 \leqslant \Big(\sum_{j\in F}\|\Gamma_j f\|\|g_j\|\Big)^2$$
$$\leqslant \Big(\sum_{j\in J}\|\Gamma_j f\|^2\Big)\Big(\sum_{j\in F}\|g_j\|^2\Big).$$

This shows that $\{\Gamma_j\}_{j\in J}$ is a Hilbert *ov*-basis. For the other implication, assume that $\{\Gamma_j\}_{j\in J}$ is a Hilbert *ov*-basis. Fix $F \subset J$ with $|F| < \infty$ and let $f = \sum_{j\in F} \Lambda_j^* g_j$, then $g_j = \Gamma_j f$ for all $j \in F$. By Theorem 2.14 there exists a constant $B > 0$ such that

$$\Big\|\sum_{j\in F}\Gamma_j^*\Gamma_j f\Big\|^2 \leqslant B\sum_{j\in F}\|\Gamma_j f\|^2 = B < f, \sum_{j\in F}\Gamma_j^*\Gamma_j f >$$
$$\leqslant B\|f\|\Big\|\sum_{j\in F}\Gamma_j^*\Gamma_j f\Big\|.$$

Hence,

$$\Big\|\sum_{j\in F}\Gamma_j^*\Gamma_j f\Big\| \leqslant B\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|.$$

We also have

$$\sum_{j\in F}\|g_j\|^2 = \sum_{j\in F}\|\Gamma_j f\|^2 =< f, \sum_{j\in F}\Gamma_j^*\Gamma_j f >$$
$$\leqslant \Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|\Big\|\sum_{j\in F}\Gamma_j^*\Gamma_j f\Big\| \leqslant B\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2.$$

Now applying Theorem 2.13 the result follows at once. ■

3. Orthonormal *ov*-bases and Riesz *ov*-bases

In this section we give some characterizations of orthonormal *ov*-bases and Riesz *ov*-bases in Hilbert spaces. For more details about the theory and applications of orthonormal *ov*-bases we refer the readers to [1].

Definition 3.1 Let $\{\Xi_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then (*i*) $\{\Xi_j\}_{j\in J}$ is called an orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if:

$$\Xi_i\Xi_j^*g_j = \delta_{ij}g_j \quad \forall i,j\in J,\ g_j\in W_j.$$

(*ii*) $\{\Xi_j\}_{j\in J}$ is called an orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if it is a complete orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Corollary 3.2 Let $\{\Xi_j\}_{j\in J}$ be an orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then Ξ_j is onto and $\|\Xi_j\| = 1$ for all $j\in J$.

Proof. For any $j\in J$ and $g\in W_j$, we have $\Xi_j\Xi_j^*g = g$ which implies that Ξ_j is onto. We further have $\Xi_j\Xi_j^*\Xi_j = \Xi_j$. This shows that $\Xi_j^*\Xi_j$ is an orthogonal projection from $\mathcal{H}$ onto $\mathcal{R}_{\Xi_j^*}$ and hence $\|\Xi_j^*\Xi_j\| = 1$. This yields

$$\|\Xi_j\|^2 = \sup_{\|f\|=1}\|\Xi_jf\|^2 = \sup_{\|f\|=1} < \Xi_jf, \Xi_jf >= \sup_{\|f\|=1}\|\Xi_j^*\Xi_jf\|^2 = 1$$

■

***Example* 3.3** Let $\mathcal{H} = \mathcal{K} = \mathbb{C}^{N+1}$ and let $\{e_k\}_{k=1}^{N+1}$ be the standard basis of $\mathbb{C}^{N+1}$. For each $1\leqslant j\leqslant N+1$ define the subspace $W_j\subset\mathcal{K}$ and the operator $\Xi_j:\mathcal{H}\to W_j$ by

$$W_j = \text{span}\{\sum_{\substack{k=1\\k\neq j}}^{N+1}e_k\}, \quad \Xi_j(\{z_i\}_{i=1}^{N+1}) = \frac{z_j}{\sqrt{N}}\sum_{\substack{k=1\\k\neq j}}^{N+1}e_k.$$

Then $\{\Xi_j\}_{j=1}^{N+1}$ is an orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j=1}^{N+1}$.

Corollary 3.4 Orthonormal *ov*-systems are ω-independent.

Proof. This follows immediately from the definition. ■

Theorem 3.5 Let $\{\Xi_j\}_{j\in J}$ be an orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the series $\sum_{j\in J}\Xi_j^*g_j$ converges if and only if $\{g_j\}_{j\in J}\in\left(\sum_{j\in J}\oplus W_j\right)_{\ell^2}$ and in that case

$$\Big\|\sum_{j\in J}\Xi_j^*g_j\Big\|^2 = \sum_{j\in J}\|g_j\|^2.$$

Proof. For any finite subset $F\subset J$ we have $\big\|\sum_{j\in F}\Xi_j^*g_j\big\|^2 = \sum_{j\in F}\|g_j\|^2$. From this the result follows. ■

Theorem 3.6 (Bessel's inequality) Let $\{\Xi_j\}_{j\in J}$ be an orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then

$$\sum_{j\in J}\|\Xi_jf\|^2 \leqslant \|f\|^2$$

for all $f\in\mathcal{H}$.

Proof. Let $f \in \mathcal{H}$. Fix $F \subset J$ with $|F| < \infty$. Then By Theorem 3.5 we have

$$\Big\| f - \sum_{j\in F} \Xi_j^* g_j \Big\|^2 = \|f\|^2 - \sum_{j\in F} < \Xi_j f, g_j > - \sum_{j\in F} < g_j, \Xi_j f > + \sum_{j\in F} \|g_j\|^2$$
$$= \|f\|^2 - \sum_{j\in F} \|\Xi_j f\|^2 + \sum_{j\in F} \|\Xi_j f - g_j\|^2$$

for arbitrary vectors $\{g_j : g_j \in W_j\}_{j\in F}$. In particular, if $g_j = \Xi_j f$, then

$$\|f - \sum_{j\in F} \Xi_j^* \Xi_j f\|^2 = \|f\|^2 - \sum_{j\in F} \|\Xi_j f\|^2.$$

From this we have $\sum_{j\in F} \|\Xi_j f\|^2 \leqslant \|f\|^2$, which implies that $\sum_{j\in J} \|\Xi_j f\|^2 \leqslant \|f\|^2$. ■

Corollary 3.7 Let $\{\Xi_j\}_{j\in J}$ be an orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then for all $f \in \mathcal{H}$ the series $\sum_{j\in J} \Xi_j^* \Xi_j f$ convergent and

$$\|f - \sum_{j\in J} \Xi_j^* \Xi_j f\|^2 \leqslant \|f - \sum_{j\in J} \Xi_j^* g_j\|^2$$

for every $\{g_j\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$.

Theorem 3.8 Let $\Xi = \{\Xi_j\}_{j\in J}$ be an orthonormal *ov*-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the following conditions are equivalent:

(*i*) Ξ is an orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.
(*ii*) $f = \sum_{j\in J} \Xi_j^* \Xi_j f \quad \forall f \in \mathcal{H}$.
(*iii*) $\|f\|^2 = \sum_{j\in J} \|\Xi_j^* \Xi_j f\|^2 \quad \forall f \in \mathcal{H}$.
(*iv*) $\|f\|^2 = \sum_{j\in J} \|\Xi_j f\|^2 \quad \forall f \in \mathcal{H}$.
(*v*) $< f, g >= \sum_{j\in J} < \Xi_j f, \Xi_j g > \quad \forall f, g \in \mathcal{H}$.
(*vi*) If $\Xi_j f = 0$ for all $j \in J$, then $f = 0$.

Proof. The implication $(i) \Rightarrow (ii)$ follows immediately from Corollary 3.7. To prove $(ii) \Rightarrow (iii)$ assume that $f \in \mathcal{H}$. Since Ξ is an orthonormal *ov*-system, hence $(\Xi_j^* \Xi_j)^2 f = \Xi_j^* \Xi_j f$ for all $j \in J$. This yields

$$\|f\|^2 =< \sum_{j\in J} \Xi_j^* \Xi_j f, f >= \sum_{j\in J} \|\Xi_j^* \Xi_j f\|^2,$$

which implies (iii). The implications $(iii) \Rightarrow (iv) \Rightarrow (v)$ are clear. To prove $(v) \Rightarrow (vi)$ assume that $\Xi_j f = 0$ for all $j \in J$, then we have $\|f\|^2 = \sum_{j\in J} \|\Xi_j f\|^2 = 0$. It follows that $f = 0$. To prove $(vi) \Rightarrow (i)$ suppose that $f \perp \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j\in J}$, then for every $j \in J$ we have $\|\Xi_j f\|^2 =< f, \Xi_j^* \Xi_j f >= 0$ which implies that $f = 0$. Therefore $\mathcal{H} = \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j\in J}$. ■

Theorem 3.9 Let $\{\Xi_j\}_{j\in J}$ be an orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $T : \mathcal{H} \to \mathcal{U}$ be a bounded linear operator such that $\Xi'_j = \Xi_j T^*$ for all $j \in J$. Then $\{\Xi'_j\}_{j\in J}$ is an orthonormal *ov*-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ if and only if T is unitary.

Proof. First suppose that $\{\Xi'_j\}_{j\in J}$ is an orthonormal ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then by Theorem 3.8 for every $g \in \mathcal{U}$ we have

$$\|T^*g\|^2 = \sum_{j\in J} \|\Xi_j T^*g\|^2 = \sum_{j\in J} \|\Xi'_j g\|^2 = \|g\|^2.$$

Hence T is co-isometry. We also see from Theorem 2.10 that T is unitary. Now if T is unitary then we have

$$\|g\|^2 = \|T^*g\|^2 = \sum_{j\in J} \|\Xi_j T^*g\|^2 = \sum_{j\in J} \|\Xi'_j g\|^2$$

for all $g \in \mathcal{U}$. From this follows that $\{\Xi'_j\}_{j\in J}$ is an orthonormal ov-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$. ■

Corollary 3.10 Let $\{\Xi_j\}_{j\in J}$ be an orthonormal ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the orthonormal ov-bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ are precisely the sets $\{\Xi_j T\}_{j\in J}$, where $T : \mathcal{H} \to \mathcal{H}$ is an unitary operator.

Corollary 3.11 Let $\{W_j\}_{j\in J}$ be a family of closed subspaces of $\mathcal{H}$ such that

$$\sum_{j\in J} \|\pi_{W_j} f\|^2 = \|f\|^2 \qquad \forall f \in \mathcal{H},$$

where π_{W_j} is the orthogonal projections from $\mathcal{H}$ onto W_j. Then $\{\pi_{W_j}\}_{j\in J}$ is an orthonormal ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. For each $j \in J$ and $g_j \in W_j$ we have

$$\|g_j\|^2 = \sum_{i\in J} \|\pi_{W_i} g_j\|^2 = \|g_j\|^2 + \sum_{\substack{i\in J\\ i\neq j}} \|\pi_{W_i} g_j\|^2$$

which shows that $\pi_{W_i} g_j = \delta_{ij} g_j$. It follows that $\{\pi_{W_j}\}_{j\in J}$ is an orthonormal ov-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Now the result follows from the Theorem 3.8. ■

In the following, we give some characterizations of Riesz ov-bases in Hilbert spaces.

Definition 3.12 A sequence of operators $\{\Lambda_j \in B(\mathcal{H}, W_j) : \ j \in J\}$ is called a Riesz ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if there is an orthonormal ov-basis $\{\Xi_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and a bounded invertible linear operator T on $\mathcal{H}$ such that $\Lambda_j = \Xi_j T^*$ for all $j \in J$.

Corollary 3.13 If $\{\Lambda_j\}_{j\in J}$ is a Riesz ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then

$$0 < \inf_{j\in J} \|\Lambda_j\| \leqslant \sup_{j\in J} \|\Lambda_j\| < \infty.$$

Proof. According to the definition we can write $\{\Lambda_j\}_{j\in J} = \{\Xi_j T^*\}_{j\in J}$, where T is a bounded bijective operator and $\{\Xi_j\}_{j\in J}$ is an orthonormal ov-basis. By Corollary 3.2 for every $j \in J$ we have

$$\|T^{-1}\|^{-1} \leqslant \|\Lambda_j\| \leqslant \|T\|.$$

From this the result follows. ■

Theorem 3.14 If $\{\Lambda_j\}_{j\in J} = \{\Xi_j T^*\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then $\{\frac{\Lambda_j}{\|\Lambda_j\|}\}_{j\in J}$ is also a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. Define a mapping $S : \mathcal{H} \to \mathcal{H}$ by $Sf = \sum_{j\in J} \frac{\Xi_j^*\Xi_j f}{\|\Lambda_j\|}$. By Theorem 3.8 and Corollary 3.13 we have

$$\|T\|^{-1}\|f\| \leqslant \|Sf\| \leqslant \|T^{-1}\|\|f\|,$$

which implies that S is bounded and injective. Since S is self-adjoint hence S is invertible. Moreover, the operator $\Theta = TS$ is also bounded, invertible and we have

$$\begin{aligned}\Xi_j\Theta^* = \Xi_j ST^* &= \Big(\sum_{i\in J} \frac{\Xi_j\Xi_i^*\Xi_i}{\|\Lambda_j\|}\Big)T^* \\ &= \Big(\sum_{i\in J} \frac{\delta_{ji}\Xi_i}{\|\Lambda_j\|}\Big)T^* = \frac{\Xi_j T^*}{\|\Lambda_j\|} = \frac{\Lambda_j}{\|\Lambda_j\|},\end{aligned}$$

for any $j \in J$. Consequently $\{\frac{\Lambda_j}{\|\Lambda_j\|}\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. ■

Corollary 3.15 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual *ov*-basis $\{\Gamma_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if $\{\Gamma_j\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. This follows immediately from the definition and Theorem 2.11. ■

To check Riesz *ov*-baseness of a family of operators $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, we derive the following useful characterization.

Theorem 3.16 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual *ov*-basis $\{\Gamma_j\}_{j\in J}$. Then the following conditions are equivalent:

(i) The sequence $\{\Lambda_j\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.
(ii) There is an equivalent inner product on $\mathcal{H}$, with respect to which the sequence $\{\Gamma_j\}_{j\in J}$ becomes an orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. (i) ⇒ (ii) Assume that $\{\Lambda_j\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$, and write it in the form $\{\Xi_j T^*\}_{j\in J}$ as in the definition. Define a new inner product $< .,. >_T$ on $\mathcal{H}$ by

$$< f, g >_T = < T^*f, T^*g > \qquad \forall f, g \in \mathcal{H}.$$

If $\|.\|_T$ is the norm generated by this inner product, then for all $f \in \mathcal{H}$ we have

$$\|T^{-1}\|^{-1}\|f\| \leqslant \|f\|_T \leqslant \|T\|\|f\|,$$

which implies that the new inner product is equivalent to the original one. By Theorem 2.11 for any $g \in \mathcal{K}$ and arbitrary vector $g_j \in W_j, i, j \in J$ we have

$$\begin{aligned}< \Gamma_i\Gamma_j^* g_j, g > &= < \Gamma_j^* g_j, \Gamma_i^* g >_T = < T^*\Gamma_j^* g_j, T^*\Gamma_i^* g > \\ &= < \Xi_j^* g_j, \Xi_i^* g > = < \Xi_i\Xi_j^* g_j, g > = < \delta_{ij} g_j, g > .\end{aligned}$$

Now the Corollary 3.15 follows that $\{\Gamma_j\}_{j\in J}$ is an orthonormal ov-basis for $\mathcal{H}$ with inner product $<.,.>_T$ with respect to $\{W_j\}_{j\in J}$.

$(ii) \Rightarrow (i)$ Suppose that $<.,.>_1$ is an equivalent inner product on $\mathcal{H}$ with respect to which $\{\Gamma_j\}_{j\in J}$ is an orthonormal ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Therefore there exist positive constants m, M such that

$$m\|f\| \leqslant \|f\|_1 \leqslant M\|f\| \qquad \forall f \in \mathcal{H}.$$

By Theorem 3.5 we obtain

$$\begin{aligned}\frac{1}{M^2}\sum_{j\in F}\|g_j\|^2 &= \frac{1}{M^2}\Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|_1^2 \leqslant \Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|^2\\ &\leqslant \frac{1}{m^2}\Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|_1^2 = \frac{1}{m^2}\sum_{j\in F}\|g_j\|^2,\end{aligned}$$

for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$. Now let $\{\Xi_j\}_{j\in J}$ be an arbitrary orthonormal ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and define the mapping

$$T : \mathcal{H} \to \mathcal{H}, \qquad \text{with} \qquad T\Xi_j^* g_j = \Gamma_j^* g_j \qquad \forall g_j \in W_j,\ \ j \in J.$$

Let $f \in \mathcal{H}$ with $f = \sum_{j\in J}\Xi_j^* g_j$, then we have

$$\frac{1}{M^2}\|f\|^2 = \frac{1}{M^2}\sum_{j\in J}\|g_j\|^2 \leqslant \|T(f)\|^2 \leqslant \frac{1}{m^2}\sum_{j\in J}\|g_j\|^2 = \frac{1}{m^2}\|f\|^2.$$

It follows that T is invertible and $T\Xi_j^*\Xi_j = \Gamma_j^*\Xi_j$, which from this $\Xi_j T^* = \Gamma_j$ holds for all $j \in J$. Thus $\{\Gamma_j\}_{j\in J}$ is a Riesz ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. From this the result follows at once. ■

The next theorem was proved by Sun in [3] we prove this theorem with another way.

Theorem 3.17 Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the following conditions are equivalent:

- (i) The sequence $\{\Lambda_j\}_{j\in J}$ is a Riesz ov-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.
- (ii) The family $\{\Lambda_j\}_{j\in J}$ is a complete sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and there exist positive constants A, B such that for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$, we have

$$A\sum_{j\in F}\|g_j\|^2 \leqslant \Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2 \leqslant B\sum_{j\in F}\|g_j\|^2.$$

Proof. $(i) \Rightarrow (ii)$ Assume that $\{\Lambda_j\}_{j\in J}$ is a Riesz ov-basis for $\mathcal{H}$, and write it in the form $\{\Xi_j T^*\}_{j\in J}$ as in the definition. Then for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$ we have

$$\frac{1}{\|T^{-1}\|^2}\sum_{j\in F}\|g_j\|^2 = \frac{1}{\|T^{-1}\|^2}\Big\|\sum_{j\in F}\Xi_j^* g_j\Big\|^2 \leqslant \Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2 \leqslant \|T\|^2\sum_{j\in F}\|g_j\|^2.$$

$(ii) \Rightarrow (i)$ Let $\{\Xi_j\}_{j\in J}$ be an arbitrary orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and define the mapping

$$T : \mathcal{H} \to \mathcal{H}, \qquad \text{with} \qquad T\Xi_j^* g_j = \Lambda_j^* g_j \quad \forall g_j \in W_j, \ \ j \in J.$$

Suppose that $f \in \mathcal{H}$ with $f = \sum_{j\in J} \Xi_j^* g_j$, then we have

$$A\|f\|^2 = A\sum_{j\in J} \|g_j\|^2 \leqslant \|T(f)\|^2 \leqslant B\sum_{j\in J} \|g_j\|^2 = B\|f\|^2.$$

From this and completeness of $\{\Lambda_j\}_{j\in J}$ follows that T is invertible and $T\Xi_j^*\Xi_j = \Lambda^*\Xi_j$, which implies that $\Xi_j T^* = \Lambda_j$ for all $j \in J$. ∎

Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. If $f = \sum_{j\in J} \Lambda_j^* g_j$, then the coordinate representation of $f \in \mathcal{H}$ relative to the *ov*-basis Λ is $[f]_\Lambda = \{g_j\}_{j\in J}$.

Let $\Xi = \{\Xi_j\}_{j\in J}, \Xi' = \{\Xi_i'\}_{i\in I}$ be orthonormal *ov*-bases for $\mathcal{H}$ and $\mathcal{U}$ respectively. Then the matrix representation of the linear map $T : \mathcal{H} \to \mathcal{U}$ relative to the orthonormal *ov*-bases Ξ, Ξ' is the matrix $[T] = \{T_{ij}\}_{i\in I, j\in J}$ whose (i, j) entry is $T_{ij} = \Xi_i' T\Xi_j^*$ for all $i \in I, j \in J$. For any $f \in \mathcal{H}$ we also have

$$[Tf]_{\Xi'} = [T][f]_\Xi.$$

Moreover, if S, T are linear maps on $\mathcal{H}$ represented by matrices $[S], [T]$ respectively, then $S + T$ and ST is represented by the matrices $[S] + [T]$ and $[S][T]$ respectively. Further T is a invertible operator if and only if $[T]$ is invertible.

Let $\Lambda = \{\Lambda_j\}_{j\in J} = \{\Xi_j T^*\}_{j\in J}$ be a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the analysis operator Θ_Λ of Λ is defined by

$$\Theta_\Lambda : \mathcal{H} \to \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2} \quad \text{with} \quad \Theta_\Lambda f = \{\Lambda_j f\}_{j\in J} \quad \forall f \in \mathcal{H}.$$

It can easily be shown that Θ_Λ is linear, bounded and $\|\Theta_\Lambda\| \leqslant \|T\|$. The synthesis operator Θ_Λ^* which is the adjoint operator of Θ_Λ is given by

$$\Theta_\Lambda^* : \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2} \to \mathcal{H} \quad \text{with} \quad \Theta_\Lambda^* g = \sum_{j\in J} \Lambda_j^* g_j \quad \forall g = \{g_j\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}.$$

***Example* 3.18** For every sequence of closed subspaces $\{W_j\}_{j\in J}$ of $\mathcal{K}$ the sequence $\{\Xi_j\}_{j\in J}$ defined by

$$\Xi_j g = g_j \quad \forall j \in J, \ \ g = \{g_j\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$$

is an orthonormal *ov*-basis for $\big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$ with respect to $\{W_j\}_{j\in J}$ which is called the standard orthonormal *ov*-basis of it.

Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the matrix representing of the linear operator $\Theta_\Lambda\Theta_\Lambda^*$ relative to the standard orthonormal *ov*-basis of $\big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$ is the matrix $[\Theta_\Lambda\Theta_\Lambda^*] = \{\Lambda_i\Lambda_j^*\}_{i\in I, j\in J}$ which is called the Gram matrix associated with Λ.

Theorem 3.19 Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the following conditions are equivalent:

(*i*) The sequence $\{\Lambda_j\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

(*ii*) The family $\{\Lambda_j\}_{j\in J}$ is complete sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and its Gram matrix $\{\Lambda_i\Lambda_j^*\}_{i\in I,j\in J}$ defines a bounded, invertible operator on $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$.

Proof. (*i*) ⇒ (*ii*) Assume that $\{\Lambda_j\}_{j\in J} = \{\Xi_j T^*\}_{j\in J}$ is a Riesz *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. If $G = \{G_{ij}\}_{i,j\in J}$ denotes the matrix of the invertible operator T^*T relative to $\{\Xi_j\}_{j\in J}$, then

$$G_{ij} = \Xi_i T^* T \Xi_j^* = \Lambda_i \Lambda_j^*.$$

Therefore the Gram matrix of $\{\Lambda_j\}_{j\in J}$ is G.

(*ii*) ⇒ (*i*) Suppose that Gram matrix of $\{\Lambda_j\}_{j\in J}$ defines a bounded, invertible operator on $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$. Let $\{\Xi_j\}_{j\in J}$ be an arbitrary orthonormal *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and define the mapping

$$T:\mathcal{H}\to\mathcal{H},\qquad \text{with}\qquad T\Xi_j^* g_j = \sum_{i\in J}\Xi_i^*\Lambda_i\Lambda_j^* g_j \qquad \forall g_j\in W_j,\ \ j\in J.$$

It is straightforward that T is linear, bounded and invertible. Suppose that $f\in\mathcal{H}$ with $f=\sum_{j\in J}\Xi_j^* g_j$, then we have

$$\begin{aligned} < Tf, f > &= \sum_{j\in J}\sum_{i\in J} < T\Xi_j^* g_j, \Xi_i^* g_i > = \sum_{j\in J}\sum_{i\in J}\sum_{k\in J} < \Xi_i\Xi_k^*\Lambda_k\Lambda_j^* g_j, g_i > \\ &= \sum_{j\in J}\sum_{i\in J} < \Lambda_i\Lambda_j^* g_j, g_i > = \big\|\sum_{j\in J}\Lambda_j^* g_j\big\|^2. \end{aligned}$$

Thus T is positive and self-adjoint. Since T is positive, it has a unique positive square-root. Let P denote the square-root of T, then the above calculation follows that

$$\frac{1}{\|T^{-1}\|}\sum_{j\in J}\|g_j\|^2 \leqslant \big\|\sum_{j\in J}\Lambda_j^* g_j\big\|^2 = \big\|P\big(\sum_{j\in J}\Xi_j^* g_j\big)\big\|^2 \leqslant \|T\|^2\sum_{j\in J}\|g_j\|^2.$$

Now the result follows from Theorem 3.17. ■

4. Stability of *ov*-bases under perturbations

Stability of bases is important in practice and is therefore studied widely by many authors, e.g., see [4]. In this section we study the stability of *ov*-bases for a Hilbert space $\mathcal{H}$. First we generalized a result of Paley-Wiener [4] to the situation of *ov*-basis.

Theorem 4.1 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ such that

$$\big\|\sum_{j\in F}(\Lambda_j^* g_j - \Gamma_j^* g_j)\big\| \leqslant \lambda\big\|\sum_{j\in F}\Lambda_j^* g_j\big\|$$

for some constant $0 \leqslant \lambda < 1$ and each finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$. Then $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. By assumption the series $\sum_{j\in J}(\Lambda_j^* g_j - \Gamma_j^* g_j)$ is convergent whenever the series $\sum_{j\in J} \Lambda_j^* g_j$ is convergent for all arbitrary vectors $g_j \in W_j$. If we define the mapping

$$T : \mathcal{H} \to \mathcal{H}, \qquad \text{with} \quad T\Lambda_j^* g_j = \Lambda_j^* g_j - \Gamma_j^* g_j \quad \forall g_j \in W_j,\ \ j \in J.$$

Then T is a bounded operator and $\|T\| \leqslant \lambda < 1$. Thus the operator $Id_{\mathcal{H}} - T$ is invertible and we have $(Id_{\mathcal{H}} - T)\Lambda_j^*\Lambda_j = \Gamma_j^*\Lambda_j$, consequently $\Lambda_j^*\Lambda_j(Id_{\mathcal{H}} - T^*) = \Lambda_j^*\Gamma_j$. Since Λ_j^* is one-to-one on W_j, thus $\Lambda_j(Id_{\mathcal{H}} - T^*) = \Gamma_j$. Now the conclusion follows from Theorem 2.8. ■

Corollary 4.2 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual *ov*-basis $\{\Psi_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ such that

$$\sum_{j\in J} \|\Lambda_j - \Gamma_j\| \|\Psi_j\| < 1.$$

Then $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. If $\lambda = \sum_{j\in J} \|\Lambda_j - \Gamma_j\| \|\Psi_j\|$, then $0 \leqslant \lambda < 1$. Fix $F \subset J$ with $|F| < \infty$ and let $f = \sum_{j\in F} \Lambda_j^* g_j$ for arbitrary vectors $g_j \in W_j$. Then we compute

$$\begin{aligned}\Big\| \sum_{j\in F}(\Lambda_j^* g_j - \Gamma_j^* g_j)\Big\| &= \Big\| \sum_{j\in F}(\Lambda_j^* - \Gamma_j^*)\Psi_j f\Big\| \\ &\leqslant \sum_{j\in F} \|(\Lambda_j^* - \Gamma_j^*)\Psi_j f\| \\ &\leqslant \sum_{j\in J} \|\Lambda_j - \Gamma_j\| \|\Psi_j\| \|f\| = \lambda \Big\| \sum_{j\in F} \Lambda_j^* g_j\Big\|.\end{aligned}$$

From this the result follows by Theorem 4.1. ■

In the following we generalized a result of Krein-Milman-Rutman [4] to the situation of *ov*-basis.

Theorem 4.3 Let $\{\Lambda_j\}_{j\in J}$ be a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. If there exists a sequence $\{\varepsilon_j\}_{j\in J}$ of positive numbers, such that $\|\Lambda_j - \Gamma_j\| < \varepsilon_j$ for all $j \in J$. Then $\{\Gamma_j\}_{j\in J}$ is a *ov*-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

Proof. If $\{\Psi_j\}_{j\in J}$ is the dual *ov*-basis of $\{\Lambda_j\}_{j\in J}$. Then the result follows from Corollary 4.2, to choose ε_j small enough such that $\sum_{j\in J} \varepsilon_j \|\Psi_j\| < 1$. ■

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