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On dual shearlet frames

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Abstract. In This paper, we give a necessary condition for function in L^2 with its dual to generate a dual shearlet tight frame with respect to admissibility.

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1. Introduction

We begin by recalling some notations and denitions[1, 2, 4]. For $j, k \in \mathbb{Z}$, let

$$A_{a_0^j} = \begin{bmatrix} {a_0}^j & 0 \\ 0 & {a_0}^{\frac{j}{2}} \end{bmatrix} \quad , \qquad S_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

where $A_{a_0^j}$ and S_k are called *parabolic scaling matrices* and *shearing matrix*, respectively.

For $\psi \in L^2(\mathbb{R}^2)$, a *discrete shearlet system* associated with ψ is defined by

$$\{\psi_{j,k,m} = a_0^{-\frac{3}{4}j}\psi(S_k A_{a_0^{-j}} \cdot -m) : \; j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \tag{1}$$

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with $a_0 > 0$.

The *discrete shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is the mapping defined by

$$f \mapsto \mathcal{SH}_\psi f(j,k,m),$$

where

$$\mathcal{SH}_\psi f(j,k,m) = \langle f, \psi_{j,k,m} \rangle, \quad (j,k,m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^2.$$

If $\psi \in L^2(\mathbb{R}^2)$ satisfies

$$c_\psi := \int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\xi)|^2}{|\xi_1|^2} d\xi < \infty, \tag{2}$$

it is called an admissible shearlet.

Throughout this paper, we assume that H is a measurable subset of $\mathbb{R}^2$ such that

$$\chi_H(x) = \chi_{S^T_{-1} A_{2^{-1}} H}(x) \quad a.e. \quad \text{and} \quad |\mathrm{H} \setminus \mathrm{H}^\circ| = 0,$$

where H° denotes the interior of H, $H \setminus H^\circ := \{x \in \mathbb{R}^2 : x \in H \text{ and } \mathrm{x} \notin \mathrm{H}^\circ\}$, and $|H \setminus H^\circ|$ denotes the Lebesgue measure of $H \setminus H^\circ$. We consider the subspace $L^2(H)^\vee$ of $L^2(\mathbb{R}^2)$ defined as

$$L^2(H)^\vee = \{f : f \in L^2(\mathbb{R}^2) : \mathrm{supp}\widehat{\mathrm{f}} \subseteq \mathrm{H}\}.$$

Also, we will use the notation of the cube

$$\Theta_a(v) := \{w \in \mathbb{R}^2 : |w_i - v_i| \leqslant a, i = 1, 2\}, \tag{3}$$

with radius a and center at $v = (v_1, v_2)$, where $w = (w_1, w_2)$.

To define a dual shearlet tight frame (DSTF) in $L^2(H)^\vee$, we need to recall a shearlet frame in $L^2(H)^\vee$.

A discrete shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ as defined in (1) is called a shearlet frame for $L^2(H)^\vee$, if there exist constants $0 < A \leqslant B < \infty$ such that for all $f \in L^2(H)^\vee$,

$$A\|f\|^2 \leqslant \sum_{j,k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^2} |\langle f, \psi_{j,k,m} \rangle|^2 \leqslant B\|f\|^2, \qquad f \in L^2(H)^\vee \tag{4}$$

A discrete shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ forms a Bessel sequence for $L^2(H)^\vee$, if only the right hand side inequality in (4) holds.

We say that ψ with $\tilde{\psi}$ generates a DSTF in $L^2(H)^\vee$ if ψ and $\tilde{\psi}$ are a Bessel sequences and for some non-zero constant B,

$$B\langle f, g \rangle = \sum_{j,k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^2} \langle f, \psi_{j,k,m} \rangle \langle \tilde{\psi}_{j,k,m}, g \rangle, \qquad f, g \in L^2(H)^\vee. \tag{5}$$

2. Main results

In this section, we discuss a necessary condition for ψ with $\tilde{\psi}$ in $L^2(H)^\vee$ to generate a DSTF via admissibility.

Proposition 2.1 If $\{\psi_{j,k,m}\}_{j,k,m}$ forms a Bessel sequence with Bessel bound B, then

$$\sum_{j,k\in\mathbb{Z}} |\widehat{\psi}(S_{-k}^T A_{2^{-j}}\xi|^2 \leqslant B \tag{6}$$

and ψ is admissible shearlet.

Proof. First, we observe, using (4), that

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle|^2 \leqslant B\|\hat{f}\|^2, \tag{7}$$

for all $f \in L^2(H)^\vee$ and for any $j,k \in \mathbb{Z}$, we have

$$\sum_{m\in\mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle|^2 = 2^{\frac{3}{2}j} \sum_{m\in\mathbb{Z}^2} |\int_{[0,2\pi]^2} \sum_{l\in\mathbb{Z}^2} \hat{f}(A_{2^j} S_k^T(w+2\pi l))\overline{\hat{\psi}}(w+2\pi l) e^{2\pi i m^T.w} dw|^2 \tag{8}$$

$$= 2^{\frac{3}{2}j} \int_{\mathbb{R}^2} |\sum_{l\in\mathbb{Z}^2} \hat{f}(A_{2^j} S_k^T(w+2\pi l))\overline{\hat{\psi}}(w+2\pi l)|^2 dw,$$

where the last equality in (8) is obtained by the Parseval equality.
Then by (7) and (8), we have

$$\sum_{j,k\in\mathbb{Z}} 2^{\frac{3}{2}j} \int_{\mathbb{R}^2} |\sum_{l\in\mathbb{Z}^2} \hat{f}(A_{2^j} S_k^T(w+2\pi l))\overline{\hat{\psi}}(w+2\pi l)|^2 dw \leqslant B\|\hat{f}\|^2, \tag{9}$$

for all $f \in L^2(H)^\vee$, consider $v \in \mathbb{R}^2$ and the function

$$\hat{f}(\xi) = \frac{1}{2\varepsilon}\chi_{\Theta_\varepsilon(v)}(\xi), \tag{10}$$

where $\varepsilon > 0$, χ_Θ denotes the characteristic function of a set Θ and $\Theta_\varepsilon(v)$ is defined by (3).
For any positive integer N and all sufficiently small $\varepsilon > 0$, in (9) we obtain

$$\sum_{k\in\mathbb{Z}} \sum_{|j|\leqslant N} 2^{\frac{3}{2}j} \int_{\Theta_{2^{-\frac{3}{2}j}\varepsilon}(S_{-k}^T A_{2^{-j}} v)} |\hat{\psi}(w)|^2 dw \leqslant B.$$

Hence, by taking $\varepsilon \to 0$ and $N \to \infty$, (6) follows. ■

By using proposition 2.1, we obtain the following result which gives a necessary condition for ψ with $\tilde{\psi}$ to generate a DSTF.

Theorem 2.2 Let ψ with $\tilde{\psi}$ in $L^2(H)^\vee$ generate a DSTF in $L^2(H)^\vee$ with bound B, then we have

$$\sum_{j,k\in\mathbb{Z}} \overline{\hat{\psi}}(S_{-k}^T A_{2^{-j}}\xi)\hat{\tilde{\psi}}(S_{-k}^T A_{2^{-j}}\xi) = B\chi_H(\xi) \quad a.e.. \tag{11}$$

In particular, ψ is admissible.

Proof. Let $H_0 := H^\circ \setminus \{0\}$. From the assumption $|H \setminus H^\circ| = 0$, to prove (11) it suffices to prove that

$$\sum_{j,k\in\mathbb{Z}} \overline{\hat{\psi}}(S_{-k}^T A_{2^{-j}}\xi)\hat{\tilde{\psi}}(S_{-k}^T A_{2^{-j}}\xi) = B \qquad a.e.\ \xi \in H_0. \tag{12}$$

By the Parseval equality and the polarization identity, setting $T := [0, 2\pi)^2$, we have the equality

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \langle f, \psi_{j,k,m}\rangle\langle\tilde{\psi}_{j,k,m}, g\rangle$$

$$= \sum_{j,k\in\mathbb{Z}} 2^{-\frac{3}{2}j} \int_T [\hat{f}(A_{2^j}S_k^T\cdot), \hat{\psi}](\eta)[\hat{\tilde{\psi}}, \hat{g}(A_{2^j}S_k^T\cdot)](\eta)d\eta, \quad f, g \in L^2(\mathbb{R}^2), \tag{13}$$

where the bracket product is defined as

$$[f, g](\eta) = \sum_{m\in\mathbb{Z}^2} f(\eta + 2\pi m)\overline{g(\eta + 2\pi m)}.$$

by definition, ψ with $\tilde{\psi}$ satisfies Equation (5). By (13), we can rewrite (5) as

$$B\langle\hat{f}, \hat{g}\rangle = \sum_{j,k\in\mathbb{Z}} 2^{-\frac{3}{2}j} \int_T [\hat{f}(A_{2^j}S_k^T\cdot), \hat{\psi}](\eta)[\hat{\tilde{\psi}}, \hat{g}(A_{2^j}S_k^T\cdot)](\eta)d\eta, \quad f, g \in L^2(H)^\vee. \tag{14}$$

For any fixed $k \in \mathbb{Z}$, we consider

$$M^j := A_{2^j} = \begin{bmatrix} 2^j & 0 \\ 0 & 2^{\frac{j}{2}} \end{bmatrix}.$$

Now, let $\hat{f}(\zeta) = \hat{g}(\zeta) = \frac{1}{\sqrt{|D_l(\xi,\gamma_l)|}}\chi_{D_l(\xi,\gamma_l)}(\zeta)$, where for $l \in \mathbb{Z}$ and $\gamma_l \in \mathbb{Z}^2$, we define

$$D_l(\xi, \gamma_l) := \{M^l[S_k^T(x + 2\pi\gamma_l)] \ : \ x \in T\}, \quad \xi \in H_0.$$

Since $\xi \neq 0$ and $\xi \in H^\circ$, we can choose $l_\xi < 0$ such that

$$M^j D_l(\xi, \gamma_l) \cap D_l(\xi, \gamma_l) = \emptyset, \quad \forall j < 0,\ l \leqslant l_\xi,\ j, l \in \mathbb{Z}. \tag{15}$$

For a detailed proof of (15), the reader is referred to [3].

It is obvious that $f, g \in L^2(H)^\vee$. Hence (14) yields

$$\begin{aligned} B &= B\langle \hat{f}, \hat{g}\rangle \\ &= \sum_{k\in\mathbb{Z}} \Big[\sum_{j\geqslant l-l_N} 2^{-\frac{3}{2}j} \int_T [\hat{f}(A_{2^j}S_k^T\cdot), \hat{\psi}](\eta)[\hat{\tilde{\psi}}, \hat{g}(A_{2^j}S_k^T\cdot)](\eta)d\eta \\ &\quad + \sum_{j<l-l_N} 2^{-\frac{3}{2}j} \int_T [\hat{f}(A_{2^j}S_k^T\cdot), \hat{\psi}](\eta)[\hat{\tilde{\psi}}, \hat{g}(A_{2^j}S_k^T\cdot)](\eta)d\eta\Big], \end{aligned} \tag{16}$$

with the integer $l_N < 0$ depending only on N. Since $\hat{f}(\zeta) = \hat{g}(\zeta) = \frac{1}{\sqrt{|D_l(\xi,\gamma_l)|}}\chi_{D_l(\xi,\gamma_l)}(\zeta)$, then for any $j \geqslant l - l_N$, we obtain

$$[\hat{f}(A_{2^j}S_k^T\cdot), \hat{\psi}](\eta)[\hat{\tilde{\psi}}, \hat{g}(A_{2^j}S_k^T\cdot)](\eta) = \frac{1}{|D_l(\xi,\gamma_l)|}[\overline{\hat{\psi}}\hat{\tilde{\psi}}, \chi_{D_l(\xi,\gamma_l)}(A_{2^j}S_k^T\cdot)](\eta). \tag{17}$$

Hence, By (15) and (17), in (16) we have

$$\begin{aligned} B &= \lim_{l\to\infty} \tfrac{1}{|D_l(\xi,\gamma_l)|} \int_{D_l(\xi,\gamma_l)} \sum_{k\in\mathbb{Z}} \sum_{j\leqslant l_N - l} \overline{\hat{\psi}}(A_{2^j}S_k^T\eta)\hat{\tilde{\psi}}(A_{2^j}S_k^T\eta)d\eta \\ &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \overline{\hat{\psi}}(A_{2^j}S_k^T\eta)\hat{\tilde{\psi}}(A_{2^j}S_k^T\eta), \end{aligned}$$

then the result follows. ■

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