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Signature submanifolds for some equivalence problems

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Abstract. This article concerned on the study of signature submanifolds for curves under Lie group actions SE(2), SA(2) and for surfaces under SE(3). Signature submanifold is a regular submanifold which its coordinate components are differential invariants of an associated manifold under Lie group action, and therefore signature submanifold is a key for solving equivalence problems.

Keywords: Signature submanifold, eqiuvalence problem, moving frame, differential invariant.

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1. Introduction

We start this section with a motivate example; given γ and $\tilde{\gamma}$ to be curves on $\mathbb{R}^2$, are γ and $\tilde{\gamma}$ equivalence under Euclidean group action (rotations and translations) to each other? i.e. is there an element of $g \in \mathrm{SE}(2)$ such that $\gamma = g.\tilde{\gamma}$. If we can find some $g \in \mathrm{SE}(2)$ such that $\gamma = g.\tilde{\gamma}$ therefore γ and $\tilde{\gamma}$ are equivalence to each other under SE(2). Clearly, finding some g is not easy, even manifolds and groups are simple.

Let try to answer this question; when two curves are not equivalence? For example we know curves which have different arc lengths are not equivalence under SE(2) because this action preserve fixed curve's arc lenghts; so arc lenght is a usefull quantity for check equivalent. In fact the idea of solving equivalence problems come from here; by finding quantities that don't change under actions and by comparitive these, one can solve equivalence problems. These quantities of prolonged manifolds that fixed under

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group actions called differential invariants, and related signature submanifold is a regular submanifold which its coordinate components are these differential invariants. So finding signature submanifold is a key for solving equivalence problems.

We want to obtain signature submanifold by equivariant moving frame method, according to Akivis, the method of moving frames originates in work of the Estonian mathematician Martin Bartels, a teacher of both Gauss and Lobachevsky. The field is most closely associated with Elie Cartan, who forged earlier contributions by Darboux, Frenet, Serret, and Cotton into a powerful tool for analyzing the geometric properties of submanifolds and their invariants under the action of transformation groups, [5].

In the 1970s, several researchers, began the process of developing a firm theoretical foundation for the method. The final crucial step, is to define a moving frame simply as an equivariant map from the manifold back to the transformation group. All classical moving frames can be reinterpreted in this manner. Moreover, the equivariant approach is completely algorithmic, and applies to very general group actions. in recently years equivariant moving frame method has been many promotions, can be found details in[1] and [5].

In [4] P.J.Olver explain how can obtain structure of algebras of differential invariants for finite-dimensional Lie groups with equivariant moving frame method and in [5] he describe solving equivalence problem with this method. In [6] E.L.Mansfeld has obtained algebras of differential invariants with out calculate Murrer - Cartan forms and only with correction matrix, **K** and whole of computations has briefed in matrix computations.

2. Equivalence and Signatures

Definition 2.1 [5] Given a group action of G on M, two submanifolds $S, \bar{S} \subset M$ are said to be equivalent if $\bar{S} = g \cdot S$ for some $g \in G$.

Suppose we have constructed an n^{th} order moving frame $\rho : J^n \longrightarrow G$ defined on an open subset of jet space. A submanifold S is called regular if its $j_n S$ lies in the domain of definition of the moving frame. For any $k \geqslant n$, we use $J^k = I^k | j_k S$, where $I^k = \left(\ldots, H^i, \ldots, I^\alpha_J, \ldots\right), \#J \leqslant k$, to denote the k^{th} order restricted differential invariants, [5].

Definition 2.2 [5] The k^{th} order signature $\mathcal{S}^k = \mathcal{S}^k\left(S\right)$ is the set parametrized by the restricted differential invariants $J^k : j_k S \longrightarrow \mathbb{R}^{n_k}$, where $n_k = p + q\binom{p+k}{k}$ such that p is dimension of independent variables and q is dimension of dependent variables.

The submanifold S is called fully regular if J^k has constant rank $0 \leqslant t_k \leqslant p = \dim S$ for all $k \geqslant n$. In this case, $\mathcal{S}^k$ forms a submanifold of dimension t_k perhaps with self-intersections. In the fully regular case,

$$t_n \leqslant t_{n+1} \leqslant \ldots \leqslant t_s = t_{s+1} = \ldots = t \leqslant p, \tag{1}$$

where t is the differential invariant rank and s the differential invariant order of S, [5].

Theorem 2.3 [5] Two fully regular p-dimensional submanifolds $S, \bar{S} \subset M$ are (locally) equivalent if and only if they have the same differential invariant order s and their signature manifolds of order $s+1$ are identical: $\mathcal{S}^{s+1}\left(S\right) = \mathcal{S}^{s+1}\left(\bar{S}\right)$.

3. Algorithm for Finding Signature Submanifolds

3.1 *Equivariant Moving Frame*

We start by explanation the equivariant moving frame construction. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

Definition 3.1 [5] A moving frame is a smooth, G-equivariant map $\rho : M \longrightarrow G$. There are two principal types of equivariance:

$$\rho\left(g \cdot z\right) = \begin{cases} g \cdot \rho\left(z\right) & \text{left moving frame} \\ \rho\left(z\right) \cdot \left(g\right)^{-1} & \text{right moving frame} \end{cases}$$

Theorem 3.2 [5] Moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

For existance moving frame, G must be act free and regular on M. Acting freely meaning that isotropy subgroup $G_x = \{g \in G \mid g \cdot x = x\} = \{e\}$ or equivalently, the orbits all have the same dimension, r, as G itself. Regularity requires that, in addition, the orbits form a regular foliation. In practice these conditions satisfy respectively by prolongation of group action sufficiently large and by choice of a appropriate cross section.

Theorem 3.3 [5] Let G act freely and regularly on M, and let $\mathcal{K} \subset M$ be a cross-section. Given $z \in M$, let $g = (z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in \mathcal{K}$. Then $\rho : M \longrightarrow G$ is a right moving frame.

In aplication, we use the following procedure. We define the cross section $\mathcal{K}$ as a set of equations $\psi_k(z) = 0, k = 1, ..., r$. The number of equations,r, equals the dimension of the group. In order to $\mathcal{K}$ obtain the group element that takes z to $\mathcal{K}$,we solve the so called normalisation equations, $\psi_k(z) = 0, k = 1, ..., r$.
The frame $\rho(z)$ therefore satisfies $\psi_i(\rho(z) * z) = 0, i = 1, ..., r$. If the solution is unique on the domain U, then ρ is a right frame, one that satisfies $\rho(g * z) = \rho(z) \cdot g^{-1}$. One chooses the normalisation equations to minimise the computations as much as possible for the application at hand, [6].

3.2 *Differential Invariant, Invariant Differential operator and K matrix*

Definition 3.4 [6] A differential invariant for a group action G is a smooth function like this: $I : J^n \longrightarrow \mathbb{R}$ such that

$$\forall g \in G : \qquad I\left(g \cdot z\right) = I\left(z\right)$$

Theorem 3.5 [6] If ρ is a right frame, then the quantity $I(z) = \rho(z) \cdot z$ is an invariant of the group action.

3.3 *Invariant Differential Operator*

Definition 3.6 [6] We define total differential operator:

$$\begin{aligned} \mathrm{D}_i &= \frac{D}{Dx_i} \\ &= \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_{k} u_{ki}^{\alpha} \frac{\partial}{\partial u_k^{\alpha}} \end{aligned}$$

that is an operator on the jet space of a manifold.

Definition 3.7 [6] Also we define:

$$\begin{aligned} \tilde{\mathrm{D}}_i &= \frac{D}{D\tilde{x}_i} \\ &= \sum_{k=1}^{p} \left(\tilde{D}x\right)_{ik} \mathrm{D}_k \end{aligned}$$

such that:

$$\left(\tilde{D}x\right) = \left((D\tilde{x})^{-1}\right)_{ik} \quad \text{where} \quad D\tilde{x} = \begin{pmatrix} \frac{\partial \tilde{x}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{x}_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}_p}{\partial x_1} & \cdots & \frac{\partial \tilde{x}_p}{\partial x_p} \end{pmatrix}$$

Definition 3.8 [6] The map $z \longrightarrow I(z) = \rho(z) \cdot z$ is called the invariantisation map.

Definition 3.9 [6] For any prolonged action in $(x_i, u^{\alpha}, u_k^{\alpha})$-space, the specific components of $I(z)$, the invariantised jet coordinates, are denoted

$$J_i = I(x_i), \qquad I^{\alpha} = I(u^{\alpha}), \qquad I_k^{\alpha} = I(u_k^{\alpha}).$$

Definition 3.10 [6] A set of distinguished invariant operators is defined by evaluating the transformed total differential operators on the frame. They are $\mathcal{D}_j = \tilde{D}_j \mid_{g=\rho(z)}$ where the $\tilde{D}_j$ are given in 3.7.

3.4 K *Matrix*

It is of great computational importance in the applications that the matrix **K** can be calculated without explicit knowledge of the frame. All that is required are the normalisation equations $\{\psi_\lambda(z) = 0, \lambda = 1, \cdots, r\}$ and the infinitesimals. Suppose the n variables actually occurring in the $\psi_\lambda(z)$ are $\zeta_1, \cdots, \zeta_n$; typically m of these will be independent variables and $n - m$ of them will be dependent variables and their derivatives, [6].

Definition 3.11 [6] Define T to be the invariant $p \times n$ total derivative matrix

$$T_{ij} = I\left(\frac{D}{Dx_i}\zeta_j\right),$$

Also, let Φ denote the $r \times n$ matrix of infinitesimals with invariantised arguments,

$$\Phi_{ij} = \left(\frac{\partial \left(g \cdot \zeta_j \right)}{\partial g_i} \right) \Big|_{g=e} (I) \,,$$

Furthermore, define J to be the $n \times r$ transpose of the Jacobian matrix of the left hand sides of the normalisation equations $\psi_1, \ldots, \psi_r$ with invariantised arguments, that is

$$J_{ij} = \frac{\partial \psi_j \left(I \right)}{\partial I \left(\zeta_i \right)}.$$

Theorem 3.12 [6] The correction matrix $\mathbf{K}$, which provides the error terms in the process of invariant differentiation is given by

$$\mathbf{K} = -TJ \left(\Phi J \right)^{-1}. \tag{2}$$

where T, J and Φ are defined above.

Definition 3.13 [6] We denote by $\mathcal{I}^0$ the set of zeroth invariants,

$$\mathcal{I}^0 = \{ I\left(x_j\right) = J_i, I\left(u^\alpha\right) = I^\alpha \mid j = 1, \ldots, p, \alpha = 1, \ldots, q \}.$$

Theorem 3.14 [6] Suppose the normalisation equations $\psi_k = 0, \;\; k = 1 \ldots r$, yield a frame for a regular free action on some open set of the prolonged space with coordinates $(x_j, u^\alpha, u^\alpha_k)$. Then the components of the correction matrix $\mathbf{K}$, given in Theorem 3.12, together with $\mathcal{I}^0$, given in Definition3.13, form a generating set of differential invariants.

4. Signature Curve for Curves under SE(2) and SA(2) Actions

4.1 *Signature Curve for Curves under* SE(2)

The following question was answered in this section is: when two curves in the plane equivalence to each other under Euclidean group action? For answering the question we must obtain related signature submanifold of a curve under Euclidean group actions; so we must compute $\mathbf{K}$ matrix and by using this we can attain related signature submanifold.

As we consider Euclidean group act in a curve such that

$$\begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix},$$

Infinitesimal generator of this group is:

$$v_a = \partial x \qquad\qquad v_b = \partial u, \qquad\qquad v_\theta = -u\partial x + x\partial,$$

So infinitesimals table is leads to:

	x	u	u_x	u_{x^2}	u_{x^3}	u_{x^4}	$\cdots$
a	1	0	0	0	0	0	$\cdots$
b	0	1	0	0	0	0	$\cdots$
θ	$-u$	x	$1+u_x^2$	$3u_xu_{x^2}$	$3u_{x^2}^2+4u_xu_{x^3}$	$10u_{x^2}u_{x^3}+5u_xu_{x^4}$	$\cdots$

By considering normalization equations we have:

$$\tilde{x} = 0, \qquad \tilde{u} = 0, \qquad \tilde{u}_x = 0,$$
$$\psi_1(x, u, u_x) = x, \qquad \psi_2(x, u, u_x) = u, \qquad \psi_3(x, u, u_x) = u_x,$$

and

$$\zeta_1 = x, \qquad \zeta_2 = u, \qquad \zeta_3 = u_x,$$
$$J^x = 0, \qquad I^u = 0, \qquad I_1^u = 0,$$

so we find **K** matrix:

$$\Phi = \begin{array}{c c} & \begin{array}{ccc} x & u & u_x \end{array} \\ \begin{array}{c} a \\ b \\ \theta \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},$$

$$J = \begin{array}{c c} & \begin{array}{ccc} \psi_1(I) & \psi_2(I) & \psi_3(I) \end{array} \\ \begin{array}{c} J^x \\ I^u \\ I_1^u \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},$$

$$T = \begin{array}{c c} & \begin{array}{ccc} x & u & u_x \end{array} \\ x & \begin{pmatrix} 1 & 0 & I_{11}^u \end{pmatrix} \end{array},$$

$$-\mathbf{K} = \begin{array}{c c} & \begin{array}{ccc} a & b & \theta \end{array} \\ x & \begin{pmatrix} 1 & 0 & I_{11}^u \end{pmatrix} \end{array}.$$

So according to theorem 3.14 differential invariant generator for algebra of the action is:

$$I_{11}^u = \kappa = \frac{u_{x^2}}{(1 + u_x^2)^{3/2}},$$

and also differential invariant operator is:

$$\mathcal{D}_x = \frac{D_x}{\sqrt{1 + u_x^2}}.$$

Now we want abtain signature submanifold of a $C \subset M = \mathbb{R}^2$, consider the curve is a one dimentional submanifold, so sequence of t_k's has two forms:

1- When κ has a constant value:

$$t_0 = 0 = t_1 = 0 = \cdots \Longrightarrow s = 0,$$

2- When κ is a function of x:

$$t_0 = 1 = t_1 = 1 = \cdots \Longrightarrow s = 0.$$

Therefore in both cases signature submanifold of a curve is: $\mathcal{S}^1(C) = \mathcal{S}\{(\kappa, \kappa_x = \mathcal{D}_x \kappa)\}$ hence two curves are equivalence to each other if and only if they have the same curvature.

4.2 *Signature Curve for Curves under* SA(2)

As we consider Equiee Affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ act in a curve such that

$$\tilde{x} = \alpha x + \beta u + a, \qquad \tilde{u} = \gamma x + \delta u + b, \qquad \alpha\delta - \gamma\beta = 1,$$

By considering normalization equations we have:

$$\tilde{x} = 0, \quad \tilde{u} = 0, \quad \tilde{u}_x = 0, \quad \tilde{u}_{xx} = 1, \quad \tilde{u}_{xxx} = 0,$$

Similar to pre_ subsection we can obtain **K** matrix

$$-\mathbf{K} = \begin{array}{c} \\ x \end{array} \begin{array}{c} \begin{array}{ccccc} \beta & \gamma & \delta & a & b \end{array} \\ \left(\begin{array}{ccccc} \frac{-I^u_{1111}}{3} & 1 & 0 & 1 & -1 \end{array} \right) \end{array}.$$

So according to theorem 3.14 differential invariant generator for algebra of the action is:

$$\kappa = I^u_{1111} = I\left(u_{x^4}\right) = \frac{3u_{x^2}u_{x^3} - 5u^2_{x^3}}{3u^{8/3}_{x^3}},$$

$$\mathcal{D}_x = u^{1/3}_{xx} D_x.$$

Now we want abtain signature submanifold of a $C \subset M = \mathbb{R}^2$, consider the curve is a one dimentional submanifold, so sequence of t_k's has two forms:

1- When κ has a constant value:

$$t_0 = 0 = t_1 = 0 = \cdots \Longrightarrow s = 0,$$

2- When κ is a function of x:

$$t_0 = 1 = t_1 = 1 = \cdots \Longrightarrow s = 0.$$

Therefore in both cases signature submanifold of a curve is: $\mathcal{S}^1(C) = \mathcal{S}\{(\kappa, \kappa_x = \mathcal{D}_x\kappa)\}$ hence two curves are equivalence to each other if and only if they have the same curvature.

5. Signature Submanifolds for Surface under SE(3) Action

Now we proceed Euclidean transformation group act on surfaces in three dimentional space, which is a six dimentional Lie group, consist of composition of rotations and transitions that has many applications; We try to abtain generators of algebra of this group. Consider the usual action on SE (3); if R_p, R_q and R_t are rotation matrices around

x axis, y axis and u axis respectively; therefore $R_{pqt} = R_p R_q R_t$, or

$$\begin{pmatrix} \cos p \cos q & \cos p \sin q \sin t - \sin p \cos t & \cos p \sin q \cos t + \sin p \sin t \\ \sin p \cos q & \sin p \cos q \sin t + \cos p \cos t & \sin p \sin q \cos t - \cos p \sin t \\ -\sin q & \cos q \sin t & \cos q \cos t \end{pmatrix},$$

then SE (3) acts on surface $u = u(x, y)$ in $\mathbb{R}^3$ space suchlike:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{u} \end{pmatrix} = R_{pqt} \begin{pmatrix} x \\ y \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and infinitesimal vector fields are:

$$v_a = \partial_x, \qquad v_b = \partial_y, \qquad v_c = \partial_y,$$
$$v_p = -y\partial_x + x\partial_y, \qquad v_q = u\partial_x - x\partial_u, \qquad v_t = -u\partial_y + y\partial_u,$$

Therefore infinitesimals table leads to:

	x	y	u	u_x	u_y	u_{x^2}	$\cdots$
a	1	0	0	0	0	0	$\cdots$
b	0	1	0	0	0	0	$\cdots$
c	0	0	1	0	0	0	$\cdots$
p	$-y$	x	0	$-u_y$	u_x	$-2u_{xy}$	$\cdots$
q	u	0	$-x$	$-1-u_x^2$	$-u_x u_y$	$-3u_x u_{x^2}$	$\cdots$
t	0	$-u$	y	$u_x u_y$	$1+u_y^2$	$u_y u_{x^2} + 2u_x u_{xy}$	$\cdots$

	u_{xy}	u_{y^2}	u_{xy^2}	$\cdots$
a	0	0	0	$\cdots$
b	0	0	0	$\cdots$
c	0	0	0	$\cdots$
p	$u_{x^2} - u_{y^2}$	$2u_{xy}$	$2u_{x^2y} - u_{y^3}$	$\cdots$
q	$-u_y u_{x^2} - 2u_x u_{xy}$	$-2u_y u_{xy} - u_x u_{y^2}$	$-u_{x^2}u_{y^2} - 2(u_{xy}^2 + u_y u_{x^2y} + u_x u_{xy^2})$	$\cdots$
t	$2u_y u_{xy} + u_x u_{y^2}$	$3u_y u_{y^2}$	$3u_{y^2}u_{xy} + 3u_y u_{x^2y} + u_x u_{y^3}$	$\cdots$

By considering normalization equations we have:

$$\tilde{x} = \tilde{y} = \tilde{u} = \tilde{u}_x = \tilde{u}_y = \tilde{u}_{xy} = 0,$$

$$\big(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6\big)\,(x, y, u, u_x, u_y, u_{xy}) = \big(x, y, u, u_x, u_y, u_{xy}\big),$$

$$\zeta_1 = x, \qquad \zeta_2 = y, \qquad \zeta_3 = u, \qquad \zeta_4 = u_x, \qquad \zeta_5 = u_y, \qquad \zeta_6 = u_{xy},$$

$$J^x = 0, \quad J^y = 0, \quad I^u = 0, \quad I^u_1 = 0, \quad I^u_2 = 0, \quad I^u_{12} = 0,$$

$$\Phi = \begin{array}{c} \\ a \\ b \\ c \\ p \\ q \\ t \end{array} \begin{array}{c} \begin{array}{cccccc} x & y & u & u_x & u_y & u_{xy} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^u_{11} - I^u_{22} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array},$$

$$J = \begin{array}{c} \\ J^x \\ J^y \\ I^u \\ I^u_1 \\ I^u_2 \\ I^u_{12} \end{array} \begin{array}{c} \begin{array}{cccccc} \psi_1(I) & \psi_2(I) & \psi_3(I) & \psi_4(I) & \psi_5(I) & \psi_6(I) \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

$$T = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} x & u & u_x & u_{x^2} & u_{x^3} & u_{x^4} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & I^u_{11} & 0 & I^u_{112} \\ 0 & 1 & 0 & 0 & I^u_{22} & I^u_{122} \end{pmatrix} \end{array},$$

So we find **K** matrix:

$$-\mathbf{K} = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & p & q & t \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \dfrac{I^u_{112}}{I^u_{22} - I^u_{11}} & -I^u_{11} & 0 \\ 0 & 1 & 0 & \dfrac{I^u_{122}}{I^u_{22} - I^u_{11}} & 0 & I^u_{22} \end{pmatrix} \end{array}.$$

According to theorem 3.14 differential invariant generators for algebra of the action are:

$$\kappa_1 = I^u_{11}, \qquad \kappa_2 = I^u_{22},$$

1- When κ_1 and κ_2 have constant values:

$$t_0 = 0 = t_1 = 0 = \cdots \Longrightarrow s = 0,$$

2- When anyone dependes to another:

$$t_0 = 1 = t_1 = 1 = \cdots \Longrightarrow s = 0,$$

$$t_0 = 1 = t_1 = 1 < t_2 = 2 = \cdots \Longrightarrow s = 2,$$

$$t_0 = 1 < t_1 = 2 = \cdots \Longrightarrow s = 1,$$

3- When two curvatures are inedpendent to each other:

$$t_0 = 2 = t_1 = 2 = \cdots \Longrightarrow s = 0.$$

For each cases we obtain signature submanifolds:

When $s = 0$ we have

$$\mathcal{S} = \{(\kappa_1, \kappa_2, \mathcal{D}_j \kappa_i) \qquad i = 1, 2, \quad j = x, y\},$$

for $s = 1$

$$\mathcal{S}^2 = \mathcal{S} \cup \{(\mathcal{D}_{j_1} \mathcal{D}_{j_2} \kappa_i) \qquad i = 1, 2, \quad j_1, j_2 = x, y\},$$

and for $s = 2$

$$\mathcal{S}^3 = \mathcal{S}^2 \cup \{(\mathcal{D}_{j_1} \mathcal{D}_{j_2} \mathcal{D}_{j_3} \kappa_i) \qquad i = 1, 2, \quad j_1, j_2, j_3 = x, y\}.$$

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