

Numerical Solution of Heun Equation Via Linear Stochastic Differential Equation

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Abstract. In this paper, we intend to solve special kind of ordinary differential equations which is called Heun equations, by converting to a corresponding stochastic differential equation(S.D.E.). So, we construct a stochastic linear equation system from this equation which its solution is based on computing fundamental matrix of this system and then, this S.D.E. is solved by numerically methods. Moreover, its asymptotic stability and statistical concepts like expectation and variance of solutions are discussed. Finally, the attained solutions of these S.D.E.s compared with exact solution of corresponding differential equations.

Keywords: Heun equation; Wiener process; Stochastic differential equation; Linear equations system

1. Introduction to Heun equation

Most of the known theoretical issues in physics are being involved in analyzing and solving an enormous number of differential equations (O.D.E. or P.D.E.). The classification of differential equations is done according to their singularity structure around their singular points [10], [13]. In physics, one of the applied typical equations are the following hypergeometric Heun equation:

$$z(1-z)\frac{d^2w}{dz^2} + (c - (l+a+b)z)\frac{dw}{dz} - abw = 0. \quad (1)$$

As we know, this equation has three regular singular points, at zero, one and infinity. Jacobi, Legendre, Bessel, Laguerre and Hermite equations are special cases of this second order equation and can be changed to this type.

In mathematics, the local Heun equation was introduced by M. Karl and L. Heun[11]. The local Heun function is called a Heun function (HF), if it is also regular at $z = 1$, and is

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called a Heun polynomial (HP), if it is regular at all three finite singular points $z = 0, 1, a$. Heun's equation is a second-order linear ordinary differential equation of the form

$$\frac{d^2 w}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right] \frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w = 0. \quad (2)$$

The condition $\epsilon = \alpha + \beta - \gamma - \delta + 1$ is necessary to ensure regularity of the point at ∞ . The complex number q is named the accessory parameter. Heun's equation has four regular singular points: $z = 0, 1, a$ and $z = \infty$ with exponents $(0, 1 - \gamma)$, $(0, 1 - \delta)$, $(0, 1 - \epsilon)$, and (a, β) . For more information regarding Heun equation, their various solutions and its applications to theoretical physics issues, you can refer to [7-9, 14]. In this paper, we intend to solve the following stochastic Heun equation:

$$\begin{cases} y'' + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1} \right) y' + \left(\frac{\mu}{x} + \frac{\nu}{x-1} \right) y = \xi, \\ y(0) = y_0, \quad y'(0) = y_1. \end{cases}$$

such that $\alpha, \beta, \gamma, \mu, \nu$ and ξ , could be coefficients of Gaussian random variables which is named Wiener process or Brownian motion.

In general case, these equations have power series solutions with simple relations between continuous coefficients and can be generally represented in terms of simple integral transforms. In the case of nonlinear problem, we often utilize one form of the *Painleve* equation which is known as a linear second order differential equations [12]. As an applied method, because of having stochastic behavior of more physical phenomenons, we consider S.D.E. form of these equations and start to solve and analyze them.

This paper is organized as follow. In section 2, we consider the stochastic linear equation system of Heun equation which has been mentioned in various books like [4] and [3]. Afterwards, by construction fundamental matrix for this system we solve it numerically by *Rung - Kutta* method. In section 3, we consider two second order S.D.E. examples of Heun equation and solve them by mentioned method form section 2 and stochastic numerical simulation like predictor-corrector *Euler Maruyama* (E.M.) and *Milstein*' method [2]. Also, we find expectation, variance and figure of these equations answers. In section 4, the conclusion of this paper has been said and finally, in section 5, *Mathlab* code of these examples has been brought.

2. Making Stochastic Differential Equation System

In general Case, consider the second order liner S.D.E.

$$y'' = (A(t) + \alpha(t)\xi_1)y' + (B(t) + \beta(t)\xi_2)y + (C(t) + \gamma(t)\xi_3).$$

which variables $\xi_i (i = 1, 2, 3)$, are Gaussian random variables and they are named "white noise" who considered as derivation of Wiener process respect to time (*i.e.* $\frac{dW_i(t)}{dt} = \xi_i$).

The Wiener processes corresponding to following definition have some properties.

DEFINITION 2.1 A m -dimensional vector $W(t)$ of stochastic processes $W_i(t)$, ($i = 1, 2, \dots, m$) is named Wiener process or Brownian motion if;

- (a) $W(0) = 0$ a.s. (almost sure with probability one),
- (b) $W(t) - W(s)$ is normal distribution (i.e. $W \sim N(0, t - s)$), for all $0 \leq s \leq t$
- (c) for times $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are independent increments.

From this definition, it could be concluded that

$$E[W_i(t)] = 0, E[W_i^2(t)] = t \text{ for } i = 1, 2, \dots, m. \tag{3}$$

Now, the above S.D.E. could be written as following Linear system:

$$y_1' = y_2, y_2' = (A(t) + \alpha(t)\xi_1)y_2 + (B(t) + \beta(t)\xi_2)y_1 + (C(t) + \gamma(t)\xi_3) \tag{4}$$

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' &= \left(\begin{pmatrix} 0 & 1 \\ B(t) & A(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ C(t) \end{pmatrix} \right) \\ &+ \left(\begin{pmatrix} 0 & 0 \\ \beta(t)y_1 & \alpha(t)y_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \lambda\gamma(t) & (1 - \lambda)\gamma(t) \end{pmatrix} \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \end{aligned} \tag{5}$$

In this case, because of linear combination of normal random variables is again a normal random variable with mean and variance equal to corresponding linear their combination respectively, so we have put ξ_3 as a convex Combination of ξ_1, ξ_2 . Thus, this equation matrix form is made as follows:

$$dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW. \tag{6}$$

such that $\xi_i = \frac{dW_i}{dt}$ ($i = 1, 2, 3$), and W_i is Wiener process.

Now, we intend to address the constriction of this problem solutions. At First, we prove the existence and uniqueness theorem for this linear S.D.E.s system. A complete proof of this fundamental theorem in stochastic differential equations could be found in text books like [6] and [5].

THEOREM 2.2 Suppose that $D(t).y + C(t) : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$ and $F(t).y + E(t) : \mathbb{R}^n \times [0, T] \rightarrow \mathcal{M}^{m \times n}$ are continuous and satisfy in the following properties ($\mathcal{M}^{m \times n}$, is the set of $m \times n$ -dimensional matrixes in \mathbb{R}^n):

$$(1) \|D(t).(y(t) - \hat{y}(t))\| \leq L.\|y(t) - \hat{y}(t)\|, \text{ (Lipshitz inequality)} \tag{7}$$

$$\|F(t).(y(t) - \hat{y}(t))\| \leq L.\|y(t) - \hat{y}(t)\|. \tag{8}$$

$$(2) \|D(t).y + C(t)\| \leq L \cdot |1 + \|y\||, \quad (9)$$

$$\|F(t).y + E(t)\| \leq L \cdot |1 + \|y\||. \quad (10)$$

for all $0 \leq t \leq T$, and $y, \hat{y}, x \in \mathbb{R}^n$. For some suitable $L \in \mathbb{R}$ and let $y_0 \in \mathbb{R}^n$ is a random variable such that: $E[y_0^2] < \infty$. Hence, There exist a unique solution $y \in L_n^2(0, T)$ of S.D.E.:

$$\begin{cases} dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW, \\ y(0) = y_0. \quad (0 \leq t \leq T) \end{cases} \quad (11)$$

Where W , is a m -dimensional Brownian motion.

Remark 1 According to [6] and [1], if we have the inequality:

$$\text{Sup}_{0 \leq t \leq T} \{ \|C(t)\|, \|E(t)\|, \|D(t)\|, \|F(t)\| \} < \infty. \quad (12)$$

then $D(t).y + C(t)$ and $F(t).y + E(t)$ satisfy the hypotheses which have been posed in existence and uniqueness above theorem for linear S.D.E. provided $E[y_0^2] < \infty$. In special Case, if C, D, E , and F , have continuous elements in $[0, T]$, they get their finite maximum values in this interval.

On account of existence and uniqueness solution of linear S.D.E., for instance in narrow sense, the linear S.D.E. and its explicit solution is :

$$\begin{cases} dy = (D(t).y + C(t))dt + E(t).dW, \\ y(0) = y_0 \end{cases} \quad (13)$$

$$y(t) = \Phi(t) \left(y_0 + \int_0^t \Phi(s)^{-1} (C(s)ds + E(s)dW) \right), \quad (14)$$

where $\Phi(0)$ is the fundamental matrix of the following O.D.E. system [1]:

$$\frac{d\Phi}{dt} = D(t).\Phi, \quad \Phi(0) = I.$$

In other words, we have:

$$\begin{pmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\Phi}_{21} & \dot{\Phi}_{22} \end{pmatrix} = \begin{pmatrix} \Phi_{21} & \Phi_{22} \\ B(t)\Phi_{11} + A(t)\Phi_{21} & B(t)\Phi_{21} + A(t)\Phi_{22} \end{pmatrix}$$

and $\Phi_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

Consequently it turns out two second-order equations by different initial Conditions:

$$\begin{cases} \ddot{\Phi}_{11} = B(t)\Phi_{11} + A(t)\dot{\Phi}_{11} \\ \Phi_{11}(0) = 1, \dot{\Phi}_{11}(0) = 0 \end{cases}, \quad \begin{cases} \ddot{\Phi}_{12} = B(t)\Phi_{12} + A(t)\dot{\Phi}_{12} \\ \Phi_{12}(0) = 0, \dot{\Phi}_{12}(0) = 1 \end{cases} \quad (15)$$

so the explicit solution attains so

$$\begin{pmatrix} y_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \dot{\Phi}_{11} & \dot{\Phi}_{12} \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} + \int_0^t \frac{1}{\det\Phi} \begin{pmatrix} \dot{\Phi}_{12} & -\Phi_{12} \\ -\dot{\Phi}_{11} & \Phi_{11} \end{pmatrix} \begin{pmatrix} 0 \\ C(s) \end{pmatrix} ds + \begin{pmatrix} 0 & 0 \\ \lambda\gamma d(t) & (1-\lambda)\gamma(t) \end{pmatrix} \begin{pmatrix} dW_2 \\ dW_1 \end{pmatrix}.$$

Therefore we should have this equality for equation solution:

$$\begin{aligned} y_1 = y &= \Phi_{11}y(0) + \Phi_{12}y'(0) + \Phi_{11} \int_0^t \frac{1}{\det\Phi} (-\Phi_{12})(C(s)ds \\ &+ \lambda\gamma(s)dW_2 + (1-\lambda)\gamma(s)dW_1) \\ &+ \Phi_{12} \int_0^t \frac{1}{\det\Phi} (\Phi_{11})(C(s)ds + \lambda\gamma(s)dW_2 + (1-\lambda)\gamma(s)dW_1). \end{aligned} \quad (16)$$

The equation (15) are second-order Linear O.D.E. we could solve them by various methods like series solution respect to nonsingular point or *Frobenius* series respect to regular points. Also, we could apply *sinc* method to solve directly this equation or convert it to a Linear system equation and solve it by 4th-order *Rung - kutta* method.

afterwards, we decide to Compute from equality (16) that it could be done by numerical methods like E.M. predictor-corrector *E.M.* and *milstein*. Also in matrix form which is convenient for *Matlab* software, we could get the following recursive procedure.

$$\begin{aligned} y(t) &= \Phi(t) \left(y_0 + \int_0^t \Phi(s)^{-1} (C(s)ds + E(s)dW_s) \right) \\ \begin{cases} \Phi^{-1}(t_{i+1}) \cdot y(t_{i+1}) = y_0 + \int_0^{t_{i+1}} \Phi(s)^{-1} (C(s)ds + E(s)dW_s) \\ \Phi^{-1}(t_i) y(t_i) = y_0 + \int_0^{t_i} \Phi(s)^{-1} (C(s)ds + E(s)dW_s) \end{cases} \end{aligned}$$

Consequently, we get:

$$y(t_{i+1}) = y(i+1) = \Phi(t_{i+1}) \left(\Phi^{-1}(t_i) y_i + \int_{t_i}^{t_{i+1}} \Phi(s)^{-1} (C(s)ds + E(s)dW_s) \right)$$

$$y(i+1) = \Phi(t_{i+1})\Phi(t_i)^{-1}(y_i + C(t_i)\Delta t_i + E(t_i)\Delta W_i) \quad (17)$$

such that;

$$\delta W_i = W(t_{i+1}) - W(t_i) \cong \sqrt{\delta t_i} \xi_i \quad (\xi_i \sim N(0, 1))$$

The last approximation has been concluded from independent increment property Of Wiener process (for all $t, s \in [0, T]$; $W(t) - W(s) = W(t-s) \sim N(0, t-s)$.)

Of course, $W(t)$ could be computed by an infinite series of *Haar* function with standard Gaussian have been written based on this approximation. Although, it could be done according relation (12). Finally, this issue should be said that for almost each W , the random trajectories of S.D.E.

$$\begin{cases} dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW, \\ y(0) = y_0 + \xi. \end{cases}$$

Converge uniformly on interval $[0, T]$ as $\begin{cases} \xi \rightarrow 0, \\ \epsilon = F(t).y + E(t) \rightarrow 0 \end{cases}$, will be caused

the trajectories of deterministic O.D.E. $\begin{cases} \dot{y} = D(t).y + C(t) \\ y(0) = y_0. \end{cases}$

In general case, the following theorem indicates this asymptotic stability for linear stochastic systems against corresponding ordinary equation system is satisfactory. The up coming theorem was proved in [5].

THEOREM 2.3 (Dependence on Parameters)

Suppose for $k = 1, 2, \dots$ that $D^k(t) + C^k(t)$ and $F^k(t)y + E^k(t)$ satisfy the hypothesis of existence and uniqueness theorem, with the same constant L which said as a real bond in theorem. Moreover

$$\lim_{k \rightarrow \infty} E(\|y_0^k - y_0\|) = 0,$$

and for each $M > 0$, such that $\|y\| \leq M$,

$$\lim_{k \rightarrow \infty} \max_{0 \leq t \leq T} (\|D^k(t) - D(t)\| + \|C^k(t) - C(t)\| + \|F^k(t) - F(t)\| + \|E^k(t) - E(t)\|) = 0.$$

Finally, suppose that $y^k(0)$ solves:

$$\begin{cases} dy^k = (D^k(t) + C^k(t))dt + (F^k(t)y + E^k(t))dW \\ y^k(0) = y_0^k \end{cases}$$

Then the sequence $\{y^k(t)\}_{k=1}^\infty$, in interval $[0, T]$ intends almost sure to $y(t)$. That is

$$\lim_{k \rightarrow \infty} E \left(\max_{0 \leq t \leq T} \|y^k(t) - y(t)\|^2 \right) = 0,$$

where y is the unique solution of

$$\begin{cases} dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW, \\ y(0) = y_0. \end{cases}$$

Remark 2 According to asymptotic stability and solution of linear S.D.E. we could solve any nonhomogeneous linear O.D.E. by its correspond S.D.E. In addition, the analytic solution and least square error of O.D.E. Could be found in based on expectation and variance of S.D.E. solution

3. Examples

In this section, we discuss two second order stochastic differential equation which are especial cases of Heun equation. About each one of them, we find expectation, variance and by computing a great number of their stochastic solutions, according to Strong Law of Large Numbers theorem in statistics, we could draw a good approximate for the solution of corresponding O.D.E.

Example 3.1 Consider the following S.D.E.

$$\begin{cases} y'' = 2y' - y + t.\xi, \\ y(0) = y'(0) = 1. \end{cases} \tag{18}$$

by making equivalent S.D.E.system, we have

$$\dot{y} = D(t)ydt + E(t)dW.$$

that; $D(t) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $E(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$, $S = \frac{dW}{dt}$ is white noise and W , is Gaussian random variable ($W \sim N(0, t)$) which is named Wiener process.

$$\begin{cases} \dot{\Phi} = D(t).\Phi \\ \Phi(0) = I \end{cases} \rightarrow \begin{cases} \ddot{\Phi}_{11} = 2\dot{\Phi}_{11} - \Phi_{11} \\ \Phi_{11}(0) = 1, \dot{\Phi}_{11} = 0 \end{cases}, \begin{cases} \ddot{\Phi}_{12} = 2\dot{\Phi}_{12} - \Phi_{12} \\ \Phi_{12}(0) = 0, \dot{\Phi}_{12}(0) = 1 \end{cases}$$

with direct Computation we have:

$$\Phi_{11} = e^t - te^t, \quad \Phi_{12} = te^t$$

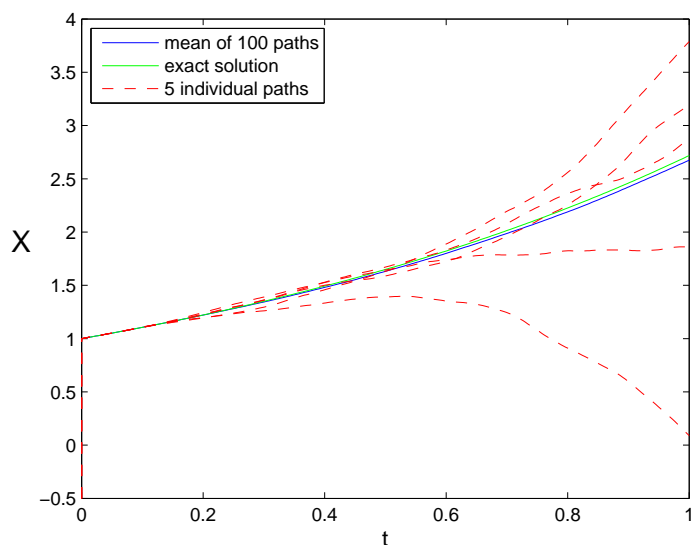


Figure 1. The exact solution with mean of $M = 100$ numerical solution by E.M. Method with $N = 2^8$ subinterval of $[0, 1]$ and also 5 individual numerical solution has been shown. In this case the least square error is 0.0435.

Thus, S.D.E. solution will be as following:

$$\begin{aligned}
 y &= \Phi \cdot \left(y_0 + \int_0^t \Phi^{-1}(s) \cdot \begin{pmatrix} 0 \\ s \end{pmatrix} \cdot dW(s) \right) \\
 &= e^t \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_0^t e^{-s} \begin{pmatrix} s+1 & -s \\ s & 1-s \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} sW(s) \right)
 \end{aligned}$$

Therefore, final solution of S.D.E. (i.e. $y = y_1$) is:

$$y_1 = y = e^t + (1-t) \cdot e^t \cdot \int_0^t -s^2 \cdot e^{-s} dW + t \cdot e^t \cdot \int_0^t ((-s) \cdot s \cdot e^{-s} dW.$$

Because of expectation of $it\hat{o}$ integral in zero, thus we will have the following results

$$E[y] = 0, \quad var(y) = E[y^2] - E^2[y] = t(1-t)e^{2t} \cdot E \left[\int_0^t -s^2 \cdot e^{-s} dW \cdot \int_0^t s(1-s)e^{-s} dW \right]$$

and with attention to property of $it\hat{o}$ internal products

$$E \left[\int_0^t F(x, t) dW \cdot \int_0^t G(x, t) dW \right] = E \left[\int_0^t F(x, t) \cdot G(x, t) dt \right],$$

we have

$$Var(y) = t(1 - t)e^{2t} \cdot \int_0^t s^3(s - 1)e^{-2s} dt.$$

Also, with asymptotic stability of equations, when $t \rightarrow 0$, the exact solution of O.D.E. system: $\begin{cases} y'' = 2y' - y, \\ y(0) = y'(0) = 1. \end{cases}$ is $y = E[y] = e^t$.

Example 3.2 Consider the following differential equation with noise

$$\begin{cases} (1 - x^2)y'' - 2xy' + n(n + 1)y = f(x)\xi_1 + g(x), \\ y(0) = y_0, y'(0) = y_1. \end{cases}$$

the equivalent system equation is $dy = (D(t)y + C(t))dt + E(t)dW_t$, such that:

$$D(t) = \begin{pmatrix} 0 & 1 \\ \frac{-n(n+1)}{1-t^2} & \frac{2t}{1-t^2} \end{pmatrix}, C(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, E(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, y = \begin{pmatrix} y \\ y' \end{pmatrix} \text{ and } \xi_t = \frac{dW}{dt}.$$

According to linear S.D.E. solution, the answer is

$$y = \Phi(t) \left(\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \int_0^t \Phi^{-1}(s)(C(s)ds + E(s)dW_s) \right), \tag{19}$$

$$\dot{\Phi}(t) = D(t) \cdot \Phi(t), \Phi(0) = I. \tag{20}$$

Consequently, two second order equations get from this system equation.

$$\begin{cases} \Phi''_{1i} = \frac{2t}{1-t^2} \Phi'_{1i} - \frac{n(n+1)}{1-t^2} \Phi_{1i}, (i = 1, 2) \\ \Phi_{1i}(0) = \delta_{1i}, \Phi'_{1i}(0) = \delta_{2i}. \end{cases} \tag{21}$$

We could solve these O.D.E.s by different methods like series solution, since functions. Also, this equation system could be solved by numerical methods of equation system like *Rung - kutta* from 4th order. Thus, recursive relation for (19) is as follows:

$$y_{i+1} = \Phi_{i+1} \cdot \Phi_i^{-1} (y_i + C(t_i)\Delta t_i + E(t_i)\Delta W_i.)$$

The equations (21) are $n - th$ order *legendre* equations. For instance, in case $n = 2$:

$$\begin{cases} \Phi''_{11} = \frac{2t}{1-t^2} \Phi'_{11} - \frac{6}{1-t^2} \Phi_{11} \\ \Phi_{11}(0) = 1, \Phi'_{11}(0) = 0 \end{cases} \rightarrow \Phi_{11}(t) = -3t^2 + 1 \tag{22}$$

$$\begin{cases} \Phi''_{12} = \frac{2t}{1-t^2} \Phi'_{12} - \frac{6}{1-t^2} \Phi_{12} \\ \Phi_{12}(0) = 0, \Phi'_{12}(0) = 1 \end{cases} \rightarrow \Phi_{12}(t) = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + 0(t^7) \tag{23}$$

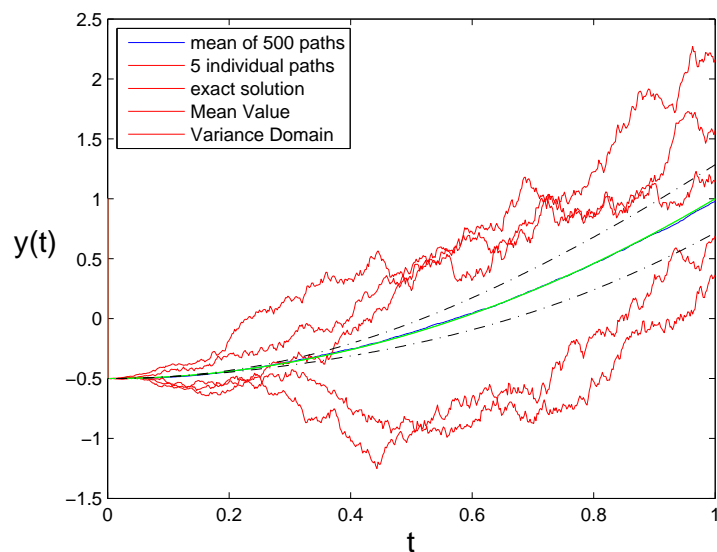


Figure 2. The exact solution with mean of $M = 100$ numerical solution by E.M. Method with $N = 2^9$ subinterval of $[0, 1]$ and also 5 individual numerical solution has been shown. In this case the least square error is 0.0234.

If we consider the following S.D.E:

$$\begin{cases} (1-t^2)y'' - 2ty' + 6y = 4t + \xi \\ y(0) = -\frac{1}{2}, y'(0) = 0. \end{cases}$$

it could be verified by *Laplace* transform method that $y = y_h + y_p = \frac{1}{2}(3t^2 - 1) + t$ is the exact solution of corresponding O.D.E., without white noise. On the other hand, S.D.E. solution is as follow:

$$y = \begin{pmatrix} y \\ y' \end{pmatrix} = \Phi \cdot \left(\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \int_0^t \Phi^{-1}(s) \cdot \left[\begin{pmatrix} 0 \\ 4s \end{pmatrix} ds + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_s \right] \right)$$

In this equality, for instance if $g(t) = 0$, we get $y = -\frac{1}{2}(-3t^2 + 1) + (t - \frac{2}{3}t^3 + \frac{1}{5}t^5) \cdot W(t)$.

Finally, $E[y] = \frac{-1}{2}(-3t^2 + 1)$ and $var(y) = t^2 \left(1 - \frac{2}{3}t^2 + \frac{1}{5}t^4 \right)$. We can see this situation in corresponding graph.

We have indicated the maximum absolute errors in numerical solution of this example. In the corresponding table for different numbers of N , the values $\|LE_{EM}^N(h)\|$ and $\|LE_M^N(h)\|$, are least squares errors for E.M. and *Milstein* methods

Table 1. Generated errors based on Numerical methods E.M. and *Milstein*

| N | $\ E_{EM}^N(h)\ ^a$ | $\ E_M^N(h)\ ^b$ |
|----------|-----------------------|-----------------------|
| 2^7 | 5.3×10^{-2} | 3.29×10^{-2} |
| 2^8 | 2.40×10^{-2} | 2.19×10^{-2} |
| 2^9 | 2.32×10^{-2} | 1.5×10^{-2} |
| 2^{10} | 2.15×10^{-2} | 4.32×10^{-3} |

^aleast squares error for E.M. method.^bleast squares error for *Milstein* method

4. Conclusion

According to this paper, we indicated that it could be analyzed stability and solved linear S.D.E. or even linear O.D.E. from different orders by converting them to linear system equation and producing some independent noise in various coefficient. By this method we could also solve the linear differential equations even with nonhomogeneous part by stochastic differentials system.

5. Appendix: *Matlab* Codes

Matlab Code of first example has been brought in the following. For more details and any other question about *Matlab* codes of the above examples, you could have a connection with authors.

```
clear all
clc
W(1)=0;
n=8; M=2; K=100;
N = 2^n;
randn('state',100);
a=randn(1,2*N+1);
Xtrue = ones(M,N+1);
Xzero = ones(M,N+1); % problem parameters
T = 1; dt = T/N;
t=[0:dt:T];
Xtrue(1,:) =exp(t); %+([0:dt:T])
R = 1; Dt = R*dt; L = N/R; % L EM steps of size Dt = R*dt
%Xem = zeros(M,N+1); % preallocate for efficiency
Xtemp = zeros(M,N+1);
Xtemp(:,1)=1;
E=zeros(M,N+1);
Z=zeros(K,N+1)
E(2,:) =4*t;
for i=1:K
```

```

dW = sqrt(dt)*randn(1,N); % Brownian increments
W = cumsum(dW); % discretized Brownian path
for j = 1:L
    Winc = sum(dW(R*(j-1)+1:R*j));
    Xtemp(:,j+1) = phiexp(t,j+1,n)*inv(phiexp(t,j,n))*...
    (Xtemp(:,j)+E(:,j)*Winc);
end
Z(i,:) = Xtemp(1,:);
end
meanZ = mean(Z);
plot([0:dt:T],[meanZ(1,:)],'b'), hold on
plot([0:dt:T],[Xtrue(1,:)],'g'), hold on
plot([0,t],[-0.5*ones(5,1),Z(1:5,:)],'r--'),
hold off % 5 individual paths
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'Horizontal','right')
emerr = norm((meanZ(1, :)-Xtrue(1, :)), 'inf') % emerr = 0.435%
Also the source of function "phiexp" which has been utilized in this code is as follow:
function F=phi(t,j,n);
N=2^n;
T = 1; dt = T/N;
t=[0:dt:T];
F(1,1)=(1-t(j))*exp(t(j));
F(1,2)=t(j)*exp(t(j));
F(2,1)=-t(j)*exp(t(j));
F(2,2)=(1+t(j))*exp(t(j));

```

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