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A note on unique solvability of the absolute value equation

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Abstract. It is proved that applying sufficient regularity conditions to the interval matrix $[A - |B|, A + |B|]$, we can create a new unique solvability condition for the absolute value equation $Ax + B|x| = b$, since regularity of interval matrices implies unique solvability of their corresponding absolute value equation. This condition is formulated in terms of positive definiteness of a certain point matrix. Special case $B = -I$ is verified too as an application.

Keywords: Absolute Value Equation, Interval matrix, Positive definiteness, Regularity, Unique solvability.

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1. Introduction

The absolute value equation (AVE)

$$Ax + B|x| = b, \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, is a new topic which seems to be a useful tool in optimization since the general NP-hard linear complementarity problem (LCP) which subsumes many mathematical programming problems can be formulated as an AVE ,[4, 5] and [6]. Up to now several authors interested to study this topic, cf. e.g. Mangasarian [4], Mangasarian and Meyer [5], Prokopyev [6], Rohn [8],[10], [11], [12], [13] and Rohn et al [15].

As the main part of this paper we want to verify conditions for unique solvability of AVE.

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In particular Prokopyev proved that checking unique solvability of the AVE is NP-hard [6], Magasarian and Meyer presented a unique solvability condition for the special case of AVE ($B = -I$), where all singular values are exceed 1 [5], Rohn generalizes this condition as

$$\sigma_{\max}(|B|) < \sigma_{\min}(A) \tag{2}$$

[13]. It is worth mentioning to say that Rohn also investigated another sufficient condition [11] as

$$\rho(|A^{-1}||B|) < 1. \tag{3}$$

We want to present a new sufficient unique solvability condition which generalizes (2) and not more general than (3).
Our plan is as follows, first we present some results about interval matrices and their regularity sufficient conditions, next we present a theorem which defined a direct relation between regularity of interval matrices and unique solvability of their corresponding AVE. In section 3 we proof three sufficient unique solvability conditions with a unity method, of course the third one is a new sufficient condition which obtained via employing sufficient regularity conditions for interval matrices. In section 4 special case of AVE(1) which $B = -I$, is verified as an application.

2. Basic results and notations

2.1 *Interval matrices*

Given two matrices A_c and Δ, $\Delta \geqslant 0$, the set of matrices

$$\mathbf{A} = \{A : |A - A_c| \leqslant \Delta\} \tag{4}$$

is called a (square) interval matrix with midpoint matrix A_c and radius matrix Δ.

Definition 2.1 A square interval matrix $\mathbf{A}$ is called regular, if each $A \in \mathbf{A}$ is nonsingular, and said to be singular otherwise (i.e., if it contains a singular matrix).

The problem of checking regularity of interval matrices is known to be NP-hard and was studied by several authors [1, 3, 7, 9, 14, 16].

2.2 *Sufficient regularity conditions*

Theorem 2.2 Each of three conditions below implies regularity of $\mathbf{A}$

i) $\rho(|A_c^{-1}|\Delta) < 1$.
ii) $\|\Delta\|_2 < \sigma_{\min}(A_c)$.
iii) the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is positive definite.

The condition (i) is due to Beeck [1], (ii) is due to Rump [16] and (iii) is due to Farhadsefat, et al [3].
Of course there is another condition for checking regularity, which was investigated by Rex-Rohn, but since (iii) is a generalization of that condition and more applicable, we

don't mention it and refer the interested reader to [7].
In (i) ρ denotes the spectral radius, in (ii) $\sigma_{\min}$ is the minimum singular value, and

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)} \tag{5}$$

where $\lambda_{\max}$, $\sigma_{\max}$ denote the maximum eigenvalue and maximum singular value respectively; I denotes the identity matrix of the respective size [2].

Following we present a theorem which implies a nice relation between regularity of interval matrices and unique solvability of AVE(1). The interested reader refered to [8, 10] for the proof.

Theorem 2.3 If the interval matrix $[A-|B|, A+|B|]$ is regular then for each right-hand side b the equation

$$Ax + B|x| = b$$

has a unique solution.

By employing Theorem 2.3 and applying results in Theorem 2.2 to the interval matrix $[A - |B|, A + |B|]$, we are able to proof all previous unique solvability conditions for AVE, with a unity method, whilst we can present a new condition too as follows.

3. Sufficient conditions for unique solvability of AVE

The first condition was mentioned and proved by Rohn in [11] for the special case $|Ax| - |B||x| = b$ and we prove it now for our AVE(1) with a similar proof.

Theorem 3.1 If A be nonsingular and

$$\rho(|A^{-1}||B|) < 1, \tag{6}$$

Then for each right-hand side $b \in \mathbb{R}^n$ the AVE (1) has a unique solution.

Proof. Since due to Beeck's condition (6) implies regularity of $[A - |B|, A + |B|]$, and according to Theorem 2.3 we are done. ∎

The second condition was proved again by Rohn in [13] with contradiction method.

Theorem 3.2 Let $A, B \in \mathbb{R}^{n\times n}$ satisfy

$$\sigma_{\max}(|B|) < \sigma_{\min}(A) \tag{7}$$

Then for each right-hand side $b \in \mathbb{R}^n$ the AVE (1) has a unique solution.

Proof. Due to (5), we can rewrite (7) as follows

$$\||B|\|_2 < \sigma_{\min}(A),$$

Obviously this is Rump's condition for regularity of $[A-|B|, A+|B|]$, and due to Theorem 2.3 it follows unique solvability of (1). ∎

Now it's turn to present the main part of this paper which is a new condition for unique solvability of AVE. Notice that of course, as was shown in [3], our new condition is a generalization of (7) and, using a simple counterexample, not more general than (8).

Theorem 3.3 Let $A, B \in \mathbb{R}^{n\times n}$ and the matrix

$$A^T A - \||B|\|_2^2 I \tag{8}$$

is positive definite then for each right hand-side $b \in \mathbb{R}^n$ the AVE (1) is unique solvable.

Proof. Since due to Farhadsefat's condition, positive definiteness of (8) implies regularity of $[A - |B|, A + |B|]$, so just such as what has been mentioned in Theorem 2.1 and Theorem 2.2, the AVE (1) is unique solvable. ■

4. Special case $B = -I$

If we specialize our subject to $B = -I$ we obtain the AVE, $Ax - |x| = b$ which plays an important role in optimization problems and conditions was investigated for its unique solvability by several authors. In theorem below three conditions presented which the condition (I) was investigated by Rohn, et al [15], the condition (II) was proved by Mangasarian and Meyer [5] and the condition (III) is revealed due to our new condition.

Theorem 4.1 Each of three conditions below implies unique solvability of $Ax - |x| = b$ for each right-hand side $b \in \mathbb{R}^n$.

I) $\rho(|A^{-1}|) < 1$
II) $\sigma_{\min}(A) > 1$
III) the matrix $A^T A - I$ is positive definite.

Proof. Substituting $B = -I$ in (1), so Theorems 2.1, 2.2, 2.3 imply three aforementioned conditions. ■

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