Journal of
Linear and Topological Algebra
Vol. 01, No. 02, Summer 2013, 67- 69

A note on uniquely (nil) clean ring

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Abstract.A ring R is uniquely (nil) clean in case for any $a \in R$ there exists a uniquely idempotent $e \in R$ such that $a - e$ is invertible (nilpotent). Let $C = \begin{pmatrix} A & V \\ W & B \end{pmatrix}$ be the Morita Context ring. We determine conditions under which the rings A, B are uniquely (nil) clean. Moreover we show that the center of a uniquely (nil) clean ring is uniquely (nil) clean.

Keywords: Full element, uniquely clean ring, nil clean ring

1. Introduction

We say that an element $a \in R$ is uniquely (nil) clean provided that there exists a unique idempotent $e \in R$ such that $a - e \in R$ is invertible (nilpotent). A ring R is uniquely (nil) clean in case every element in R is uniquely (nil) clean. As is well known, every uniquely nil clean ring is uniquely clean. Many authors have studied such rings, see [1, 2, 4, 6, 8].

A Morita Context $(A, B, W, V, \psi, \varphi)$ consists two rings A, B, two bimodules ${}_AV_B$, ${}_BW_A$ and a pair of bimodule homomorphisms $\psi : V \otimes_B W \longrightarrow A$, $\phi : W \otimes_A V \longrightarrow B$, such that $\psi(v \otimes w)v' = v\phi(w \otimes v')$, $\phi(w \otimes v)w' = w\psi(v \otimes w')$. We can form

$$C = \{ \begin{pmatrix} a & v \\ w & b \end{pmatrix} \mid a \in A, b \in B, v \in V, w \in W\}$$

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and define a multiplication on C as follows:

$$\begin{pmatrix} a & v \\ w & b \end{pmatrix} \begin{pmatrix} a' & v' \\ w' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(v \otimes w') & av' + vb' \\ wa' + bw' & \phi(w \otimes v') + bb' \end{pmatrix}$$

A routine check shows that, with this multiplication (and entry-wise addition), C becomes an associative ring. We call C a Morita Context ring [3]. Obviously, the class of the rings of Morita Contexts includes all 2×2 matrix rings and all formal triangular matrix rings. In recent years, many authors studied Morita Contexts from different points of view [5, 7].

In this paper in the first section we obtain the relationship of uniquely (nil) cleanness between Morita Context ring C and A, B. At last in the second section, we investigate if the center of a uniquely (nil) clean rings are uniquely (nil) clean? Throughout, all rings are associative rings with identity. $Z(R)$ will denote, the center of R.

1.1 *Morita Context ring*

The following results are useful tools needed in the proof of main results.

THEOREM 1.1 *(see [8, Theorem 2.2] and [6, Corollary 3.3.7]) Every factor ring of uniquely (nil) clean ring is again uniquely (nil) clean.*

LEMMA 1.2 *Every idempotent in a uniquely clean ring is central.*

Proof Let $e^2 = e \in R$. If $r \in R$, then $e + (er - ere)$ is an idempotent. Hence $1 + (er - ere)$ is a unit, so the fact that $[e + (er - ere)] + 1 = e + [1 + (er - ere)]$ implies that $e + (er - ere) = e$ because R is uniquely clean. It follows that $er = ere$, and similarly $re = ere$. ■

LEMMA 1.3 *Every idempotent in uniquely nil clean ring is central.*

Proof Let $e \in R$ be an idempotent and let r be any element of R. Notice that the element $e + er(1 - e)$ can be written as $e + (er(1 - e))$ or as $(e + er(1 - e)) + 0$ as the sum of an idempotent and a nilpotent. Since R is uniquely nil clean, this shows that $e = e + er(1 - e)$, implying that $er(1 - e) = 0$. It can likewise be shown that $(1 - e)re = 0$, so e is central. ■

THEOREM 1.4 *Let* $C = \begin{pmatrix} A & V \\ W & B \end{pmatrix}$ *be the Morita Context with* $\varphi, \psi = 0$. *If* C *is a uniquely (nil) clean ring then* A, B *are uniquely (nil) clean rings.*

Proof Let $I = \begin{pmatrix} 0 & V \\ W & B \end{pmatrix}$, $J = \begin{pmatrix} A & V \\ W & 0 \end{pmatrix}$. One can check that I, J are ideals of C and $C/I \simeq A$, $C/J \simeq B$. The uniquely (nil) cleanness of A, B follows from Theorem 2.1. ■

The following example shows that the converse of Theorem 2.4 is not true.

Example 1.5 Let $C = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. One can check that $\mathbb{Z}_2$ is uniquely (nil) clean. Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a noncentral idempotent in C, then C is not uniquely (nil) clean, by Lemma 2.2 and Lemma 2.3.

Corollary 1.6 *Let R, S be two rings, and M be an (R, S)-bimodule. Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the formal triangular matrix ring. If E is a uniquely (nil) clean ring then R and S are uniquely (nil) clean rings.*

Proof Formal triangular matrix rings are special cases of the Morita Context rings with zero morphisms, therefore the result follows by Theorem 2.1. ■

2. The center of uniquely (nil) clean rings

It is interesting to know if the center of a ring shares the same property with the ring. We don't know if the center of a clean ring is necessarily clean? But we have:

Theorem 2.1 *The center of a uniquely (nil) clean ring is uniquely (nil) clean.*

Proof Let R be a uniquely (nil) clean ring and $x \in Z(R)$. Then there exists a unique idempotent $e \in R$ such that $x - e \in R$ is invertible (nilpotent). Since $e \in Z(R)$ by Lemma 2.2 and Lemma 2.3, then $x - e \in Z(R)$. Thus x is uniquely (nil) clean in $Z(R)$. ■

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