Journal of
Linear and Topological Algebra
Vol. 02*, No.* 02*,* 2013, 67- 70

Commuting Π-regular rings

Sh. Sahebi[a,*] and M. Azadi[a]

[a] *Department of Mathematics, Faculty of Science, Islamic Azad University, Central Tehran Branch, PO. Code 14168-94351, Tehran, Iran*

Abstract. R is called commuting regular ring (resp. semigroup) if for each $x, y \in R$ there exists $a \in R$ such that $xy = yxayx$. In this paper, we introduce the concept of commuting π-regular rings (resp. semigroups) and study various properties of them.

© 2013 IAUCTB. All rights reserved.

Keywords: Regular, Commuting π-regular.

2010 AMS Subject Classification: 16E50, 16D70, 16U99.

1. Introduction

Let R be a ring (resp. semigroup). R is called Von Neumann regular ring (resp. semigroup) if for each $x \in R$ there exists $a \in R$ such that $xax = x$. Following [2], R is called π-regular ring if for any $x \in R$ there exist a positive integer n and $a \in R$ such that $x^n a x^n = x^n$. Following [6,1], R is called a commuting regular ring (resp. semigroup) if for each $x, y \in R$ there exists $a \in R$ such that $xy = yxayx$.

In recent years some authors have studied the commuting regular rings(resp. semigroups)[1, 3, 5]. We extend commuting regular rings (resp. semigroups) and introduce the concept of commuting π-regular rings (resp. semigroups) as following:

Definition 1.1 R is called a commuting π-regular ring (resp. semigroup) if for each $x, y \in R$ there exist a positive integer n and $a \in R$ such that $(xy)^n = (yx)^n a (yx)^n$.

*Corresponding author.
E-mail addresses: sahebi@iauctb.ac.ir (Sh. Sahebi), meh.azadi@iauctb.ac.ir (M. Azadi).
This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). Authors want to thank authority of IAUCTB for their support to complete this research.

Print ISSN: 2252-0201
Online ISSN: 2345-5934
© 2013 IAUCTB. All rights reserved.
http://jlta.iauctb.ac.ir

Since for each $x, y \in R$ we have $(xy)^n = (yx)^n((yx)^{-n}(xy)^n(yx)^{-n})(yx)^n$, then division rings (resp. groups) are commuting π-regular. Moreover, nil rings (resp. nil semigroups) are commuting π-regular. Because, for each $x, y \in R$ there exist positive integers n_1 and n_2 such that $(xy)^{n_1} = 0 = (yx)^{n_2}$ and so, $(xy)^{n_1} = (yx)^{n_1}(yx)^{n_2}(yx)^{n_1}$. In this paper we investigate verious properties of commuting π-regular rings (resp. semigroups). For a ring R, we use the notation $C(R)$ for the center of R.

2. Basic properties of commuting π-regular rings

In this section, we get some basic properties of commuting π-regular rings. Clearly, a commuting π- regular ring is π-regular (put $x = y$), but the converse is not true. As the following Remark, $M_2(\mathbb{Z}_2)$ is not commuting π-regular however it is π-regular.

Remark 1 *The ring of $n \times n$ matrices over a commuting π-regular ring is not necessarily commuting π-regular. For example $\mathbb{Z}_2$ is commuting π-regular ring but $M_2(\mathbb{Z}_2)$ is not. Indeed, let* $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_2)$ *then* $xy = (xy)^n = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $yx = (yx)^n = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ *for each $n \in \mathbb{N}$. If there exists* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_2)$ *such that* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ *then $1 = 0$ which is a contradiction.*

Remark 2 *The concepts of π- regular and commuting π- regular rings are the same in commutative case.*

Proposition 2.1 Every homomorphic image of a commuting π-regular ring is commuting π-regular.

Proof. Let R, S be rings and $f : R \to S$ be a ring epimorphism. Suppose that R is commuting π-regular and let $v, w \in S$. Since f is an epimorphism, there exist $x, y \in R$ such that $f(x) = v, f(y) = w$. Then since R is commuting π-regular, there exist $a \in R$ and a positive integer n such that $(xy)^n = (yx)^n a(yx)^n$. It follows that

$$(vw)^n = (f(xy)^n) = f((yx)^n a(yx)^n) = (wv)^n f(a)(wv)^n,$$

this completes the proof. ■

Corollary 2.2 Let R be a commuting π-regular ring. If I is an ideal of R, then R/I is commuting π-regular.

Proposition 2.3 Let R be a commutative ring with identity. If R is a commuting π-regular ring, then every prime ideal of R is maximal.

Proof. Let P be a prime ideal of R, then R/P is commuting π-regular by corollary 2.2. If $P \neq a + P = \overline{a} \in R/P$ then there exist a positive integer n and $\overline{b} \in R/P$ such that $\overline{a}^{2n} = \overline{a}^{2n}\overline{b}\overline{a}^{2n}$ and so $\overline{a}^{2n}(\overline{1} - \overline{b}\overline{a}^{2n}) = 0$. Therefore $\overline{b}\overline{a}^{2n} = \overline{1}$ and the proof is complete. ■

Although, the subring of a commuting π-regular ring is not necessarily commuting π-regular (for example, $\mathbb{Z}$ as a subring of $\mathbb{Q}$ is not commuting π-regular). But we have the following:

Proposition 2.4 The center $C(R)$ of every commuting π-regular ring R is again commuting π-regular.

Proof. Let $x, y \in C(R)$. Then there exist $a \in R$ and a positive integer n such that

$$(xy)^n = (yx)^n a(yx)^n = (yx)^{2n} a = a(yx)^{2n}$$

and so

$$(xy)^n a = (yx)^{2n} a^2 = a^2 (yx)^{2n}.$$

Let $z = (yx)^n a^2$ then

$$(yx)^n z(yx)^n = (yx)^{2n} a^2 (yx)^n = (yx)^n a(yx)^n = (xy)^n.$$

Now it is enough to show that $z \in C(R)$. First note that $(yx)^n a \in C(R)$, because for any $r \in R$ we have

$$(yx)^n ar = ar(yx)^n = ar(yx)^n a(yx)^n = a(yx)^{2n} ra = (yx)^n ra = r(yx)^n a$$

and so

$$zr = ((yx)^n a)ar = ar(yx)^n a = (yx)^n ara = r(yx)^n a^2 = rz$$

and the proof is complete. ■

The following shows that the corner of a commuting π-regular ring R (i.e. eRe for some idempotent e of R) is also commuting π-regular.

Proposition 2.5 Let R be a commuting π-regular ring. Then for any $e^2 = e$, eRe is commuting π-regular.

Proof. Let $x, y \in eRe$. Since R is commuting π-regular we have $(xy)^n = (yx)^n a(yx)^n$ for some $a \in R$ and a positive integer n. Note that $(yx)^n = (yx)^n e = e(yx)^n$. Thus $(yx)^n = (yx)^n eae(yx)^n$ and it follows that eRe is commuting π-regula. ■

3. Commuting π-regular semigroups

Recall the following definition from [4]:

Definition 3.1 Let S be a semigroup. A relation E on the set S is called compatible if:

$$(\forall s, t, a \in S)[(s, t) \in E, (s', t') \in E] \Rightarrow (ss', tt') \in E.$$

A compatible equivalence relation is called congruence.

Let ρ be a congruence on a semigroup S and S/ρ be the set of ρ-classes, whose elements are the subsets $x\rho$, then we can define a binary operation on the quotient set S/ρ, in a natural way as follows:

$$(a\rho)(b\rho) = (ab)\rho$$

It is easy to check that S/ρ with the above operation is a semigroup.

Proposition 3.2 Let ρ be a congruence on a commuting π-regular semigroup S. Then S/ρ is commuting π-regular.

Proof. Let $x, y \in S$, so there exist $c \in S$ and a positive integer n such that $(xy)^n = (yx)^n c(yx)^n$ and therefore

$$((x\rho)(y\rho))^n = (xy)^n\rho = ((yx)^n c(yx)^n)\rho = ((y\rho)(x\rho))^n c\rho((y\rho)(x\rho))^n$$

Thus S is commuting π-regular semigroup. ■

Definition 3.3 Let S be a semigroup. The left map $\lambda : S \to S$ is called a left translation of S if $s(\lambda t) = (\lambda s)t$, for all $s, t \in S$. The right map $\rho : S \to S$ is called a right translation of S if $(st)\rho = s(t\rho)$, for all $s, t \in S$. A left translation λ and a right traslation ρ are said to be linked if $s(\lambda t) = (s\rho)t$ for all $s, t \in S$.
The set of all linked pairs (λ, ρ) of left and right tranlation is called the translation hull of S and will be denoted by $\Omega(S)$. $\Omega(S)$ is a semigroup under the obvoius multiplication $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$ where $\lambda\lambda'$ denote the composition of the left maps λ and λ', while $\rho\rho'$ denotes the composition of the right maps ρ and ρ'.

Proposition 3.4 Let S be a commuting π-regular semigroup. For every $a \in S$ define $\lambda_a s = as$ and $s\rho_a = sa$. Then (λ_a, ρ_a) is a linked pair in $\Omega(S)$ and the set of every (λ_a, ρ_a), where $a \in S$, with multiplication of link tranlations is a commuting π-regular semigroup.

Proof. It is easy to verify that, for all $a, b \in S$, $(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab})$. Therefore the set of (λ_a, ρ_a)'s, is a semigroup. on the other hand for $a, b \in S$ there exist $t \in S$ and a positive integer n such that $(ab)^n = (ba)^n t(ba)^n$ and so

$$((\lambda_a, \rho_a)(\lambda_b, \rho_b))^n = ((\lambda_{(ab)^n}, \rho_{ab)^n}))^n = (\lambda_{(ba)^n t(ba)^n}, \rho_{(ba)^n t(ba)^n})$$

$$= ((\lambda_b, \rho_b)(\lambda_a, \rho_a))^n(\lambda_t, \rho_t)((\lambda_b, \rho_b)(\lambda_a, \rho_a))^n.$$

■

References

[1] M. Azadi, H. Doostie, L. Pourfaraj, *Certain rings and semigroups examining the regularity property,* Journal of mathematics, statistics and allied fields., 29(2008), 1:1-6.
[2] J. W. Fisher, R. L. Snider, *On the Von Neumann regularity of rings with regular prime factor rings*, Pacific J. Math., 54(1974),1: 135-144.
[3] H. Doostie, L. Pourfaraj, *On the minimal of commuting regular rings and semigroups*, Intarnal, J. Appl. Math. 19(2006), 2: 201-216.
[4] J. M. Howie, *Fundamentals of semigroup Theory*, Clarendon Press. Oxford, New York (1995).
[5] Sh. A. Safari Sabet, *Commutativity conditions for rings with unity,* Internal. J. Appl. Math. 15(2004), 1: 9-15.
[6] A. H. Yamini, Sh. A. Safari Sabet, *Commuting regular rings*, Internal. J. Appl. Math. 14(2003) 4: 357-364.