

Frames for compressed sensing using coherence

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Abstract. We give some new results on sparse signal recovery in the presence of noise, for weighted spaces. Traditionally, were used dictionaries that have the norm equal to 1, but, for random dictionaries this condition is rarely satisfied. Moreover, we give better estimations then the ones given recently by Cai, Wang and Xu.

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1. Introduction

Compressed sensing (also known as compressive sensing or compressive sampling) is a relatively new field of research, started by the work of E. Candés, D. Donoho, J. Romberg and T. Tao (see [5], [6], [9]). Using the theory of compressed sensing we can recover signals and images from far fewer samples or measurements than were traditionally thought necessary. In order to do this, compressed sensing is based on two principles: sparsity and incoherence. Using the concept of sparsity it is possible to exactly recover a signal c knowing that it is sparse or nearly sparse in the sense that it has only a limited number of nonzero components. On the other hand, incoherence says that unlike the signal of interest, the sampling/sensing waveforms have an extremely dense representation in a proper basis ([4]).

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Important applications of compressed sensing are in signal processing, image processing ([11]), coding and information theory ([5]), compressive radar ([1], [16]), MRI (Magnetic Resonance Imaging), where using compressed sensing techniques we can obtain benefits of image speed, reducing costs (see for example [13], [14]). Successfully, a significant number of compressive sensing recovery algorithms were discovered, for example orthogonal matching pursuit ([15]), basis pursuit ([7]), l_1 -minimization. Current directions of research are in computational biology, geophysical data analysis, astronomy, communications and much more other areas.

In this paper, we present a new result on sparse signals recovery in the presence of noise, which generalizes and completes a result of T. Cai et al., presented in the paper [2]. Our paper is in connection to frame theory; we use the synthesis operator to establish new results on mutual coherence.

Let $\mathcal{H} = \mathcal{H}_n$ be an real n -dimensional space, with the inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{F} = \{f_1, f_2, \dots, f_N\} \subset \mathcal{H}_n$. We associate the following operators:

The *analysis operator*

$$\Theta : \mathcal{H}_n \rightarrow \mathbb{R}^N,$$

which is given by

$$\Theta x := (\langle x, f_1 \rangle, \langle x, f_2 \rangle, \dots, \langle x, f_N \rangle)$$

and the *synthesis operator*

$$T : \mathbb{C}^N \rightarrow \mathcal{H}, \quad T(c_1, c_2, \dots, c_N) = \sum_{j=1}^N c_j f_j.$$

We suppose that $w_i = \|f_i\| \neq 0$, $i = \overline{1, N}$.

For $c = (c_1, c_2, \dots, c_N) \in \mathbb{C}^N$ and for $0 < p < \infty$ we denote

$$\|c\|_{p,w} := \left(\sum_{i=1}^N |c_i|^p w_i^p \right)^{1/p}, \quad \langle c, d \rangle_w = \sum_{i=1}^N c_i d_i w_i^2$$

and

$$\langle c, d \rangle = \sum_{i=1}^N c_i d_i.$$

We define by

$$\mu := \max_{i \neq j} \frac{|\langle f_i, f_j \rangle|}{\|f_i\| \|f_j\|}$$

the coherence of \mathcal{F} . Usually, it is assumed that $\|f_i\| = 1, i = \overline{1, N}$ but, for random

dictionaries is very rarely satisfied. It is known that μ satisfies the Welch's inequality [18]

$$\sqrt{\frac{N-n}{n(N-1)}} \leq \mu \leq 1.$$

We also denote by $\|c\|_0 = \#\{i : c_i \neq 0\}$ the cardinality of the support of c .

Clearly, $\|c_1 + c_2\|_0 \leq \|c_1\|_0 + \|c_2\|_0$, but $\|\cdot\|_0$ isn't homogeneous. We say that c is s -sparse if $\|c\|_0 \leq s$.

It is known that if \mathcal{F} is a frame for \mathcal{H}_n , then the equation

$$y = Tc$$

has a solution for any $y \in \mathcal{H}_n$.

We consider the equation

$$y = Tc + z$$

where z is an unknown noise term, with $\|z\|_2 \leq \varepsilon$ and the problem

$$(P_{1,w}) \quad \min_{\tilde{c} \in \mathbb{R}^N} \|\tilde{c}\|_{1,w} \quad \text{subject to } \|y - T\tilde{c}\|_{2,w} \leq \eta.$$

The solution of this problem is given in the final part of the present paper. Our result extends and gives better estimations than the one presented in reference [2]. Also, we present a result related to Orthogonal Matching Pursuit algorithm (in connection with a result given by J.A. Tropp [17]) for dictionaries with the norm not necessarily equal to 1.

2. Preliminary results

Lemma 2.1 (Basic Lemma) Let \mathcal{H}_n be an real n -dimensional Hilbert space and T the synthesis operator for \mathcal{F} . Then, for all $c, d \in \mathbb{R}^N$, we have

(i) $\text{supp } c \cap \text{supp } d = \emptyset$ implies $|\langle Tc, Td \rangle| \leq \mu \|c\|_{1,w} \|d\|_{1,w}$

(ii) $(1 + \mu) \|c\|_{2,w}^2 - \mu \|c\|_{1,w}^2 \leq \|Tc\|^2 \leq (1 - \mu) \|c\|_{2,w}^2 + \mu \|c\|_{1,w}^2$

(iii) if c is s -sparse, then we have

$$[1 - \mu(s - 1)] \|c\|_{2,w}^2 \leq \|Tc\|^2 \leq [1 + \mu(s - 1)] \|c\|_{2,w}^2$$

Proof. For all $c, d \in \mathbb{R}^N$ we have,

$$\begin{aligned} |\langle Tc, Td \rangle - \langle c, d \rangle_w| &= \left| \left\langle \sum_{i=1}^N c_i f_i, \sum_{j=1}^N d_j f_j \right\rangle - \sum_{i=1}^N c_i d_i w_i^2 \right| \\ &= \left| \sum_{i \neq j} c_i d_j \langle f_i, f_j \rangle \right| \\ &\leq \mu \sum_{i \neq j} |c_i| |d_j| \|f_i\| \|f_j\| \\ &= \mu (\|c\|_{1,w} \|d\|_{1,w} - \langle c, d \rangle_w) \end{aligned}$$

The relation in (i) follows from the above computation because $\text{supp } c \cap \text{supp } d = \emptyset$ implies $\langle c, d \rangle_w = 0$.

The relation in (ii) follows for $d = c$, and the relation in (iii) it follows from (ii) using Cauchy-Schwarz inequality. ■

The following three Propositions are extensions to weighted spaces of some well-known results.

Proposition 2.2 Let $\|c\|_0 \leq s$, $\|d\|_0 \leq s$ and $Tc = Td$. For

$$s < \frac{1}{2} \left(\frac{1}{\mu} + 1 \right)$$

it follows that $c = d$.

Proof. $0 = \|T(c-d)\|^2 \geq [1 - \mu(2s-1)] \|c-d\|_{2,w}^2$. It follows that $\|c-d\|_{2,w} = 0$ which implies that $c = d$. ■

Proposition 2.3 Let $s < 1 + \frac{1}{\mu}$. Then $\{f_1, \dots, f_s\}$ is linear independent.

Proof. If

$$\sum_{i=1}^s c_i f_i = 0,$$

then, by Lemma 2.1, it follows

$$\|c\|_{2,w}^2 = 0$$

hence $c = 0$. ■

Proposition 2.4 Let

$$\delta_s = \sup_{\substack{c \neq 0 \\ \|c\|_0 \leq s}} \frac{|||Tc|||^2 - \|c\|_{2,w}^2}{\|c\|_{2,w}^2}.$$

Then $\delta_s \leq \mu(s-1)$.

Proof. Immediately from Lemma 2.1. ■

When the elements of \mathcal{F} are all of norm 1 and $\mu < \frac{1}{2s-1}$, J.A. Tropp [17] shown that Orthogonal Matching Pursuit algorithm will recovery any s -sparse signal for measurements $y = Tc$.

We will show that, in the general case when the elements are not necessarily of the norm equal to 1, we have a more general result with a much more easier proof.

Theorem 2.5 Let $c \neq 0$, $c \in \mathbb{R}^N$ and j_0 be such that

$$\langle Tc, f'_{j_0} \rangle = \min_{1 \leq j \leq N} \langle Tc, f'_j \rangle, \text{ where } f'_j = \frac{f_j}{\|f_j\|}, \quad 1 \leq j \leq N.$$

If c is s -sparse and $\delta_s + \mu s < 1$, then $j_0 \in \text{supp } c$.

Proof. If $j_0 \notin \text{supp } c$, then we have

$$\mu \geq \frac{\|Tc\|^2}{\|c\|_{1,w}^2} \tag{1}$$

Indeed,

$$\begin{aligned} \|Tc\|^2 &= \langle Tc, \sum_{j=1}^N c_j f_j \rangle \\ &= \sum_{j=1}^N c_j \langle Tc, f_j \rangle \\ &\leq \sum_{j=1}^N |c_j| w_j \frac{|\langle Tc, f_j \rangle|}{\|f_j\|} \\ &\leq \frac{|\langle Tc, f_{j_0} \rangle|}{\|f_{j_0}\|} \|c\|_{1,w}. \end{aligned}$$

But $j_0 \notin \text{supp } c$ implies that $|\langle f_j, f_{j_0} \rangle| \leq \mu \|f_j\| \|f_{j_0}\|$, for $j \in \text{supp } c$.
Hence

$$|\langle Tc, f_{j_0} \rangle| = \left| \sum_{j \in \text{supp } c} c_j \langle f_j, f_{j_0} \rangle \right| \leq \mu \sum_{j \in \text{supp } c} |c_j| w_j \|f_{j_0}\| = \mu \|f_{j_0}\| \|c\|_{1,w}.$$

Hence

$$\|Tc\|^2 \leq \mu \|c\|_{1,w}^2.$$

Using Cauchy-Schwarz inequality, from relation (1), we have

$$\begin{aligned} \mu &\geq \frac{\|Tc\|^2}{\|c\|_{1,w}^2} \geq \frac{(1 - \delta_s) \|c\|_{2,w}^2}{\|c\|_{1,w}^2} \\ &\geq \frac{1 - \delta_s}{s}. \end{aligned}$$

Corollary 2.6 If c is s -sparse and $\mu < \frac{1}{2s-1}$, then $j_0 \in \text{supp } c$. ■

Proof. From Proposition 2.4 and Theorem 2.5. ■

3. Main results

For a vector $c \in \mathbb{R}^N$ we denote by c_s the vector c with all but the s -largest entries set to zero. Also, for $T_0 \subset \{1, 2, \dots, N\}$, we denote by T_0^c the complement of T_0 .

Theorem 3.1 Let $c, d \in \mathbb{R}^N$ such that $\|c\|_{1,w} \geq \|d\|_{1,w}$ and let $v = d - c$. If

$$\mu < \frac{1}{2s-1}$$

then we have

$$\|v\|_{2,w} \leq \frac{\sqrt{3 - \frac{1}{2s-1}}}{1 - \mu(2s-1)} \|Tv\| + \frac{2\sqrt{\mu(1+\mu)s}}{1 - \mu(2s-1)} e_0,$$

where

$$e_0 = \frac{\|c - c_s\|_{1,w}}{\sqrt{s}}.$$

Proof. We denote by T_0 the locations of the s largest coefficients of c . We have

$$\|v_{T_0^c}\|_{1,w} \leq \|v_{T_0}\|_{1,w} + 2e_0. \quad (2)$$

Indeed,

$$\begin{aligned} \|c\|_{1,w} &\geq \|d\|_{1,w} = \sum_{i=1}^N w_i |v_i + c_i| \\ &= \sum_{i \in T_0} w_i |v_i + c_i| + \sum_{i \in T_0^c} w_i |v_i + c_i| \\ &\geq \|c_{T_0}\|_{1,w} - \|v_{T_0}\|_{1,w} + \|v_{T_0^c}\|_{1,w} - \|c_{T_0^c}\|_{1,w}, \end{aligned}$$

hence

$$2\|c_{T_0^c}\|_{1,w} + \|v_{T_0}\|_{1,w} \geq \|v_{T_0^c}\|_{1,w},$$

i.e. relation (2). We have

$$\begin{aligned} \langle Tv, Tv_{T_0} \rangle &= \langle Tv_{T_0} + Tv_{T_0^c}, Tv_{T_0} \rangle \\ &= \|Tv_{T_0}\|^2 + \langle Tv_{T_0^c}, Tv_{T_0} \rangle \\ &\geq [1 - \mu(s-1)] \|v_{T_0}\|_{2,m}^2 - \mu \|v_{T_0^c}\|_{1,m} \|v_{T_0}\|_{1,w} \end{aligned}$$

But v_{T_0} is s -sparse and from Lemma 2.1 we have,

$$\|v_{T_0}\|_{1,w} \leq \sqrt{s}\|v_{T_0}\|_{2,w} \tag{3}$$

We obtain

$$\|Tv\|\|Tv_{T_0}\| \geq [1 - \mu(s - 1)]\|v_{T_0}\|_{2,w}^2 - \mu\|v_{T_0^c}\|_{1,m}\sqrt{s}\|v_{T_0}\|_{2,w}$$

But

$$\|Tv_{T_0}\| \leq \sqrt{1 + \mu(s - 1)}\|v_{T_0}\|_{2,w} \tag{4}$$

So

$$\sqrt{1 + \mu(s - 1)}\|Tv\| \geq [1 - \mu(s - 1)]\|v_{T_0}\|_{2,w} - \mu\sqrt{s}\|v_{T_0^c}\|_{1,w}$$

But

$$\|v_{T_0^c}\|_{1,w} \leq \|v_{T_0}\|_{1,w} + 2\sqrt{se_0}. \tag{5}$$

Using relations (3) and (5), we obtain

$$\|v_{T_0^c}\|_{1,w} \leq \sqrt{s}[\|v_{T_0}\|_{2,w} + 2e_0] \tag{6}$$

It follows that

$$\begin{aligned} \sqrt{1 + \mu(s - 1)}\|Tv\| &\geq [1 - \mu(s - 1)]\|v_{T_0}\|_{2,w} - \mu\sqrt{s}[\sqrt{s}(\|v_{T_0}\|_{2,w} + 2e_0)] \\ &\geq [1 - \mu(s - 1)]\|v_{T_0}\|_{2,w} - \mu s\|v_{T_0}\|_{2,w} - 2\mu se_0 \end{aligned}$$

So we obtain the following inequality

$$\|v_{T_0}\|_{2,w} \leq \frac{\sqrt{1 + \mu(s - 1)}}{1 - \mu(2s - 1)}\|Tv\| + \frac{2\mu se_0}{1 - \mu(2s - 1)} \tag{7}$$

On the other hand, by the basic lemma (Lemma 2.1), we have

$$\|Tv\|^2 \geq (1 + \mu)\|v\|_{2,w}^2 - \mu\|v\|_{1,w}^2$$

and by using the equation (3) and the equation (5) we obtain

$$\begin{aligned} \|v\|_{1,w} &= \|v_{T_0}\|_{1,w} + \|v_{T_0^c}\|_{1,w} \\ &\leq 2\|v_{T_0}\|_{1,w} + 2\sqrt{se_0} \\ &\leq 2\sqrt{s}\|v_{T_0}\|_{2,w} + 2\sqrt{se_0}. \end{aligned}$$

so

$$\|v\|_{1,w} \leq 2\sqrt{s}\|v_{T_0}\|_{2,w} + 2\sqrt{se_0}. \tag{8}$$

and then

$$\|Tv\|^2 \geq (1 + \mu)\|v\|_{2,w}^2 - 4\mu s(\|v_{T_0}\|_{2,w} + e_0)^2$$

which is equivalent to

$$(1 + \mu)\|v\|_{2,w}^2 \leq \|Tv\|^2 + 4\mu s(\|v_{T_0}\|_{2,w} + e_0)^2 \quad (9)$$

Combining the equation (7) and the equation (9), we have

$$\begin{aligned} (1 + \mu)\|v\|_{2,w}^2 &\leq \|Tv\|^2 + 4\mu s \left(\frac{\sqrt{1 + \mu(s-1)}}{1 - \mu(2s-1)} \|Tv\| + \frac{2\mu s e_0}{1 - \mu(2s-1)} + e_0 \right)^2 \\ &= \|Tv\|^2 + 4\mu s \left(\frac{\sqrt{1 + \mu(s-1)}}{1 - \mu(2s-1)} \|Tv\| + \frac{(1 + \mu)e_0}{1 - \mu(2s-1)} \right)^2. \end{aligned}$$

Hence

$$(1 + \mu)\|v\|_{2,w}^2 \leq \|Tv\|^2 + \left(2\sqrt{\mu s} \frac{\sqrt{1 + \mu(s-1)}}{1 - \mu(2s-1)} \|Tv\| + 2\sqrt{\mu s} \frac{(1 + \mu)e_0}{1 - \mu(2s-1)} \right)^2.$$

We use the following inequality

$$\alpha^2 + (m\alpha + \beta)^2 \leq (\sqrt{1 + m^2}\alpha + \beta)^2$$

where

$$\begin{aligned} \alpha &= \|Tv\| \\ m &= 2\sqrt{\mu s} \frac{\sqrt{1 + \mu(s-1)}}{1 - \mu(2s-1)} \\ \beta &= 2\sqrt{\mu s} \frac{(1 + \mu)e_0}{1 - \mu(2s-1)} \end{aligned}$$

With this notations, we have

$$(\sqrt{1 + \mu})^2 \|v\|_{2,w}^2 \leq (\sqrt{1 + m^2}\alpha + \beta)^2$$

so

$$\|v\|_{2,w}^2 \leq \frac{1}{\sqrt{1 + \mu}} (\sqrt{1 + m^2}\alpha + \beta)^2$$

But

$$\sqrt{1 + m^2} = \frac{\sqrt{(8s^2 - 8s + 1)\mu^2 + 2\mu + 1}}{1 - \mu(2s-1)}$$

So we obtain

$$\|v\|_{2,w} \leq \frac{1}{\sqrt{1+\mu}} \left(\frac{\sqrt{(8s^2 - 8s + 1)\mu^2 + 2\mu + 1}}{1 - \mu(2s - 1)} \|Tv\| + \frac{2\sqrt{\mu s}(1 + \mu)e_0}{1 - \mu(2s - 1)} \right)$$

And so

$$\|v\|_{2,w} \leq \frac{\|Tv\|}{1 - \mu(2s - 1)} \sqrt{\frac{(8s^2 - 8s + 1)\mu^2 + 2\mu + 1}{1 + \mu}} + \frac{2\sqrt{\mu s}\sqrt{1 + \mu}e_0}{1 - \mu(2s - 1)}.$$

We denote

$$F(\mu) = \frac{(8s^2 - 8s + 1)\mu^2 + 2\mu + 1}{1 + \mu}$$

Since

$$F'(\mu) = \frac{(8s^2 - 8s + 1)\mu^2 + 2(8s^2 - 8s + 1)\mu + 1}{(1 + \mu)^2} \geq 0.$$

and $\mu \leq \frac{1}{2s-1}$ it follows that

$$F(\mu) \leq F\left(\frac{1}{2s-1}\right) = \frac{6s-4}{2s-1}.$$

Finally, we obtain that

$$\|v\|_{2,w} \leq \frac{\sqrt{3 - \frac{1}{2s-1}}}{1 - \mu(2s - 1)} \|Tv\| + \frac{2\sqrt{\mu s}(1 + \mu)}{1 - \mu(2s - 1)} e_0.$$

■

Theorem 3.2 Assume that

$$\mu < \frac{1}{2s - 1}$$

and $\|z\|_2 \leq \varepsilon$. Then the solution c^* of $(P_{1,w})$ obeys

$$\|c^* - c\|_{2,w} \leq \frac{\sqrt{3 - \frac{1}{2s-1}}}{1 - \mu(2s - 1)} (\eta + \varepsilon) + \frac{2\sqrt{\mu s}(1 + \mu)}{1 - \mu(2s - 1)} e_0.$$

Proof. This result it follows from Theorem 3.1 since

$$\begin{aligned} \|Tv\|_2 &= \|Tc^* - y - (Tc - y)\|_2 \\ &\leq \|Tc^* - y\|_2 + \|Tc - y\|_2 \\ &\leq \varepsilon + \eta. \end{aligned}$$

■

In [2] the authors obtained, for c s -sparse and $\|f_i\| = 1, i = \overline{1, N}$, the following estimation

$$\|c^* - c\|_{2,w} \leq \frac{\sqrt{3(1+\mu)}}{1 - (2s-1)\mu}(\eta + \varepsilon).$$

We notice that our estimation, given in the above Theorem is better. The authors of the paper presented in reference [2] indicate that there exist a relation like the one given in the above Theorem, but without the specification of the constants.

References

- [1] R. Baraniuk, P. Steeghs, Compressive radar imaging, IEEE Radar Conference, Waltham, Massachusetts, April 2007.
- [2] T. Cai, L. Wang, and G. Xu, Stable Recovery of Sparse Signals and an Oracle Inequality, IEEE Trans. Inf. Theory, 56(2010) 3516–3522.
- [3] E. Candés, The restricted isometry property and its implications for compressed sensing, C. R. Acad. Sci. Paris, Ser. I 346 (2008) 589–592.
- [4] E. Candés, M. Wakin, An introduction to Compressive Sampling, IEEE Signal Processing Magazine, 25(2)(2008) 21–30.
- [5] E. Candés, T. Tao, Decoding by Linear Programming, IEEE Trans. Inform. Theory, 51(12)(2005) 4203–4215.
- [6] E. Candés, J. Romberg, T. Tao, Stable signal recovery from incomplete and inaccurate measurements, Comm. Pure Appl. Math. 59(2006) 1207–1223.
- [7] S.S. Chen, D.L. Donoho, M.A. Saunders, Atomic decomposition by basis pursuit, SIAM J. Sci. Comput., 20(1)(1998) 33–61.
- [8] O. Christensen, An Introduction to Frames and Riesz bases, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003.
- [9] D. L. Donoho, Compressed sensing, IEEE Trans. Inform. Theory, 52(4)(2006) 1289–1306.
- [10] D. L. Donoho and M. Elad, Optimally Sparse Representation in General (nonorthogonal) Dictionaries via L_1 Minimization, the Proc. Nat. Aca. Sci., 100(2003) 2197–2202.
- [11] M. Duarte, M. Davenport, D. Takhar, J. Laska, T. Sun, K. Kelly, R. Baraniuk, Single-pixel imaging via compressive sampling, IEEE Signal Processing Magazine, 25(2)(2008) 83–91.
- [12] M. Elad, Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing, Springer, 2010.
- [13] M. Lustig, D.L. Donoho, J.M. Pauly, Sparse MRI: The application of compressed sensing for rapid MR imaging, Magnetic Resonance in Medicine, 58(6)(2007) 1182–1195.
- [14] M. Lustig, D.L. Donoho, J.M. Santos, J.M. Pauly, Compressed sensing MRI, IEEE Signal Processing Magazine, 25(2)(2008) 72–82.
- [15] S.G. Mallat, Z. Zhang, Matching pursuits with time-frequency dictionaries, IEEE Trans. Signal Proc., 41(12)(1993) 3397–3415.
- [16] L. Potter, P. Schniter, J. Ziniel, Sparse reconstruction for RADAR, SPIE Algorithms for Synthetic Aperture Radar Imagery XV, 2008.
- [17] J.A. Tropp, Greed is good: Algorithmic results for sparse approximation, IEEE Trans. Inform. Theory, 50(10)(2004) 2231–2242.
- [18] L.R. Welch, Lower Bounds on the Maximum Cross Correlation of Signals, IEEE Trans. Inform. Theory, 20(1974) 397–399.