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Some notes on L-projections on Fourier-Stieltjes algebras

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Abstract. In this paper, we investigate the relation between L-projections and conditional expectations on subalgebras of the Fourier-Stieltjes algebra $B(G)$, and we will show that compactness of G plays an important role in this relation.

Keywords: L-projection, conditional expectation, Fourier-Stieltjes algebra, spine of Fourier-Stieltjes algebra, Locally compact group.

1. Introduction

The concept of conditional expectation is fundamental for a large part of probability theory. Let $(X, \mathcal{S}, \mu)$ be a probability space and $\mathcal{T}$ a σ-subalgebra of $\mathcal{S}$. The conditional expectation operator $E^{\mathcal{T}} : L^1(X, \mathcal{S}, \mu) \to L^1(X, \mathcal{T}, \mu)$ is determined by the relation $\int_T E^{\mathcal{T}}(f)\ d\mu = \int_T f\ d\mu$ for $T \in \mathcal{T}$ and all $f \in L^1(X, \mathcal{S}, \mu)$. Existence and uniqueness of $E^{\mathcal{T}}$ follows from the *Radon-Nikodym* theorem. In [2], Douglas gave a complete characterization of norm one projections on $L^1(X, \mathcal{S}, \mu)$ related closely to the notion of conditional expectation.

The notion of conditional expectation (or quasi-expectation in [9]) is defined for any algebra. Tomiyama in [11], proved that if A is a unital C^*-algebra and $P : A \to A$ is a norm one projection with $P(1) = 1$ and $P(A)$ is a C^*-subalgebra of A, then P is a conditional expectation. In view of this fundamental theorem, A.T.-M. Lau and R.J. Loy in [7], explored the relation between norm one projections and conditional expectations on Banach algebras related to locally compact groups.

In this paper, we investigate the relation between L-projections and conditional expectations on $B(G)$ and its certain subalgebras, for instance $A^*(G)$, and we will show that the compactness of G plays an important role in this relation.

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2. Preliminaries

Let X be a Banach space and $P : X \to X$ be a projection, i.e. P is a bounded idempotent operator, then P is called *L-projection* if $\|x\| = \|Px\| + \|(I-P)x\|$ for all $x \in X$. It is clear that if P is an L-projection then $\|P\| = 1$.
Let A be an algebra. An idempotent operator $P : A \to A$ is a conditional expectation, if $P(b_1ab_2) = b_1P(a)b_2$ for all $b_1, b_2 \in P(A)$ and $a \in A$. The following proposition is a part of [7, Proposition 2.1], and its proof is a straightforward calculation.

Proposition 2.1 *Let A be a Banach algebra and $P : A \to A$ an idempotent operator such that $P(A)$ is a subalgebra of A, then the following statements are equivalent:*

(1) P is a conditional expectation.
(2) If $b_1, b_2 \in P(A)$ and $a \in \ker P$ then $P(b_1ab_2) = 0$.

In [3], P. Eymard introduced $B(G)$ and $A(G)$, then proved that $A(G)$ is a closed ideal in $B(G)$. In [6], M. Ilie and N. Spronk introduced $A^*(G)$, the spine of Fourier-Stieltjes algebra, as a subalgebra of $B(G)$. We give a brief introduction of $A^*(G)$. Let G be a locally compact group. We will denote the topology on G and the almost periodic compactification of G by τ_G and G^{ap} respectively. Let the continuous homomorphism $\eta_{ap} : G \to G^{ap}$ be the compactification homomorphism. It is clear that $\tau_{ap} := \eta_{ap}^{-1}(\tau_{G^{ap}})$ is a group topology on G. Suppose that τ is a group topology on G such that there are locally compact group G_τ and continuous homomorphism $\eta_\tau : G \to G_\tau$ with the following three properties:

(1) $\overline{\eta_\tau(G)} = G_\tau$
(2) $\tau = \eta_\tau^{-1}(\tau_{G_\tau})$
(3) $\tau_{ap} \subseteq \tau$.

So G_τ is unique up to topological isomorphism between locally compact groups. The set of such τ is shown by $\mathcal{T}_{nq}(G)$. It is trivial that $\tau_G, \tau_{ap} \in \mathcal{T}_{nq}(G)$. If $\tau_1, \tau_2 \in \mathcal{T}_{nq}(G)$, we let $\tau_1 \vee \tau_2$ denote the smallest group topology on G which includes τ_1 and τ_2. By [6], we know that $\tau_1 \vee \tau_2 \in \mathcal{T}(G)$. Under this operation $\mathcal{T}_{nq}(G)$ is a semigroup in which all elements are idempotent. From [3], we know that $A_\tau(G) := A(G_\tau) \circ \eta_\tau$ is a closed subalgebra of $B(G)$ such that $A(G_\tau)$ is isomorphic to $A_\tau(G)$ as Banach algebras.

Theorem 2.2 *If $\tau_1, \tau_2 \in \mathcal{T}_{nq}(G)$ and $\tau_1 \neq \tau_2$, then we have*

$$A_{\tau_1}(G)A_{\tau_2}(G) \subseteq A_{\tau_1 \vee \tau_2}(G) \quad , \quad A_{\tau_1}(G) \cap A_{\tau_2}(G) = \{0\}$$

Proof . This follows from [6, Lemma 3.4 and Proposition 3.1]. ■

Definition 2.3 *We let*

$$A^*(G) = \bigoplus_{1}^{\tau \in \mathcal{T}_{nq}(G)} A_\tau(G) \quad (\textit{in the sense of Banach spaces})$$

and call this space the spine of $B(G)$, it is clear that $A^*(G)$ is a closed subalgebra of $B(G)$. We refer the reader to [6], for more details about $A^*(G)$.

3. L-projections on $B(G)$

Let G be a locally compact group. By [7, Proposition 3.8], if every positive contractive projection $P : B(G) \to B(G)$ whose range is a $*$-subalgebra, is a conditional expectation , then G is compact. Now, we prove a similar result for L-projections.

PROPOSITION 3.1 *Let G be a locally compact group. If every L-projection $P : B(G) \to B(G)$ whose range is a $*$-subalgebra, is a conditional expectation, then G is compact.*

Proof . By [8, Theorem 2.1] or [1, Theorem 3.18, Corollary 3.13], there is a unique continuous unitary representation π of G such that $B(G) = A(G) \oplus A_\pi(G)$, where

$$A_\pi(G) = \overline{span}\,\{\langle \pi(g)\xi, \eta\rangle \ ; \ \xi, \eta \in \mathcal{H}_\pi \ , \ g \in G\}$$

Furthermore this is an ℓ^1-direct sum, that is if $f \in B(G)$ then there are unique elements $f_\rho \in A(G)$ and $f_\pi \in A_\pi(G)$ such that $f = f_\rho + f_\pi$ and $\|f\| = \|f_\rho\| + \|f_\pi\|$. Define $P : B(G) \to A(G)\,; f \mapsto f_\rho$, since

$$\|f\| = \|f_\rho\| + \|f_\pi\| = \|P(f)\| + \|(I - P)(f)\|$$

P is an L-projection. By [3, Proposition 3.8], $A(G)$ is a $*$-subalgebra of $B(G)$. So P is a conditional expectation by the hypothesis. If $f \in A(G)$ and $g \in A_\pi(G)$, then $P(fgf) = 0$ by Proposition 2.1, and since $A(G)$ is an ideal in $B(G)$, then $P(fgf) = fgf$. Consequently

$$\forall f \in A(G)\ ,\ \forall g \in A_\pi(G)\ :\ f^2 g = 0 \qquad (1)$$

Let $g \in A_\pi(G)$. By (1), for each $x \in G$ and each $f \in A(G)$, we have $f(x)g(x) = 0$. But from [3, Lemma 3.2], we know that $A(G)$ separates the points of G. Thus $g = 0$ and $A_\pi(G) = \{0\}$. Therefore $B(G) = A(G)$, so G is compact. ■

We prove the preceding proposition for $A^*(G)$.

PROPOSITION 3.2 *Let G be a locally compact group. If every L-projection $P : A^*(G) \to A^*(G)$ whose range is a $*$-subalgebra, is a conditional expectation, then G is compact.*

Proof . Suppose G is not compact. Since G is not topologically isomorphic with the compact group G^{ap}, by [12, Theorem 3] we know that $A(G) \neq A_{\tau_{ap}}(G)$, and by Theorem 2.2, $A(G) \cap A_{\tau_{ap}}(G) = \{0\}$. Let $\tau_1, \tau_2 \in \mathcal{T}_{nq}(G)$ and $\tau_1 \neq \tau_{ap}$. Thus $\tau_1 \vee \tau_2 \neq \tau_{ap}$. So by Theorem 2.2, we have :

$$A_{\tau_1 \vee \tau_2}(G) \cap A_{\tau_{ap}}(G) = \{0\} \qquad , \qquad A_{\tau_1}(G) A_{\tau_2}(G) \subseteq A_{\tau_1 \vee \tau_2}(G).$$

Therefore the Banach algebra

$$A := \bigoplus_{1}^{\tau_{ap} \neq \tau \in \mathcal{T}_{nq}(G)} A_\tau(G)$$

is an ideal in $A^*(G)$. By [3, Proposition 3.8], $A_\tau(G) \cong A(G_\tau)$. So the Banach algebra A is a $*$-subalgebra of $A^*(G)$ and we have $A^*(G) = A \oplus_1 A_{\tau_{ap}}(G)$. Let $P : A^*(G) \to A$ be the canonical projection. Clearly P is an L-projection and $P(A^*(G)) = A$ is a $*$-subalgebra of $A^*(G)$. So by the hypothesis, P is a conditional expectation. Since $A(G) \subseteq A$, then A separates the points of G, and since A is an ideal in $A^*(G)$, by the same argument in the preceding proposition, we have $A_{\tau_{ap}}(G) = \{0\}$. Since $A_{\tau_{ap}}(G) \cong A(G^{ap}) = B(G^{ap})$, the constant function 1_G , is in the $A_{\tau_{ap}}(G)$ which is a contradiction. So G is compact. ■

The following theorem strengthens the conclusions of two preceding propositions.

Theorem 3.3 *Let G be a locally compact group, and A is a subalgebra of $B(G)$.*

(1) Suppose that $A(G) \subsetneq A$. If every L-projection $P : A \to A$ whose range is a $*$-subalgebra, is a conditional expectation, then G is compact and $A = B(G)$.
(2) Let A is a $*$-subalgebra and $A_{\tau_{ap}}(G) \subsetneq A$. If every L-projection $P : A \to A$ whose range is a $*$-subalgebra, is a conditional expectation, then G is compact and $A = B(G)$.
(3) Let $A_{\tau_{ap}}(G) \subsetneq A$, if every L-projection $P : A \to A$ whose range is a subalgebra, is a conditional expectation, then G is compact and $A = B(G)$.

Proof . **1**) As we discussed in the proof of Proposition 3.1, $B(G) = A(G) \oplus_1 A_\pi(G)$. Suppose that G is not compact. So $A(G) \neq B(G)$ and $A_\pi(G) \neq \{0\}$. Let $B := A \cap A_\pi(G)$. Since $A(G) \subsetneq A$, then $B \neq \{0\}$ and $A = A(G) \oplus_1 B$. The canonical projection $P : A \to A(G)$ is an L-projection with range $A(G)$. So P is a conditional expectation. Similar to the proof of Proposition 3.1, $B = \{0\}$ which is a contradiction. So G is compact and consequently $A(G) = A = B(G)$.

2) By [10], $B(G) = A_{\mathcal{PIF}}(G) \oplus_1 A_{\tau_{ap}}(G)$, where $A_{\mathcal{PIF}}(G)$ is a closed ideal in $B(G)$, (note that in [10], $A_{\tau_{ap}}(G)$ was shown by $A_{\mathcal{F}}(G)$). If G is not compact, as it was shown in the Proposition 3.2, $A(G) \cap A_{\tau_{ap}}(G) = \{0\}$ and by [10, p. 681, Remark (2)], we know that $A(G) \subseteq A_{\mathcal{PIF}}(G)$. Since $B(G)$ and $A_{\tau_{ap}}(G)$ are closed under the complex conjugation, so is $A_{\mathcal{PIF}}(G)$, i.e. $A_{\mathcal{PIF}}(G)$ is a $*$-subalgebra of $B(G)$. Let $B := A \cap A_{\mathcal{PIF}}(G)$, since A and $A_{\mathcal{PIF}}(G)$ are $*$-subalgebras of $B(G)$, then B is a $*$-subalgebra, and since $A_{\tau_{ap}}(G) \subsetneq A$, then $B \neq \{0\}$. Now, let $P : A \to B$ be the canonical projection. Since $A = B \oplus_1 A_{\tau_{ap}}(G)$, then P is an L-projection whose range is a $*$-subalgebra. So P is a conditional expectation, by the hypothesis. Since $A_{\mathcal{PIF}}(G)$ is an ideal and A is a subalgebra of $B(G)$, then B is an ideal in A. Hence we have:

$$\forall f \in B \quad , \quad \forall g \in A \quad : \quad f^2 g = fgf = P(fgf) = 0 \qquad (1)$$

Since $A_{\tau_{ap}}(G) \cong A(G^{ap}) = B(G^{ap})$, the constant function 1_G , is in $A_{\tau_{ap}}(G)$. By taking $g = 1_G$ in the relation (1), we have $f = 0$ for every $f \in B$, i.e. $B = \{0\}$, and this is a contradiction. Hence G is compact and $A_{\tau_{ap}}(G) = A = B(G)$.

3) Proof of this part is similar to the proof of part (2), but it should be noted that since A is not necessarily closed under the complex conjugation, then B is just a subalgebra.

■

Corollary 3.4 *According to the part (2) of the preceding theorem, if every L-projection $P : B_\rho(G) \to B_\rho(G)$ whose range is a $*$-subalgebra of $B_\rho(G)$, is a conditional expectation, then G is compact.*

Lemma 3.5 *Let G be an abelian locally compact group. G is compact and 0-dimentional iff $\hat{G}$ is a discrete torsion group.*

Proof . Let G be a compact 0-dimentional group. Since G is compact, $\hat{G}$ is discrete by [5, Theorem 23.17]. Let $\Phi \in \hat{G}$, by [5, Corollary 24.18], there is a compact subgroup H of $\hat{G}$ that contains Φ. Since $\hat{G}$ is discrete, then H is finite and therefore Φ is of finite order. Consequently $\hat{G}$ is a torsion group. Conversely, let $\hat{G}$ be a discrete torsion group. By [5, Theorem 23.17 , 24.8], G is compact, and since $\hat{G}$ is a torsion group, G is 0-dimentional by [5, Theorem 24.21 , 24.8]. ■

Let G be an abelian locally compact group. By *Bochner's* theorem, [4, Theorem 33.3], the $*$-Banach algebras $M(\hat{G})$ and $B(G)$, are isomorphic. Now, by the preceding lemma and [7, Theorem 3.6], we have the following corollary. See also [7, Corollary 3.12].

Corollary 3.6 *Let G be an abelian locally compact group. The following four statements are equivalent:*

(1) G is a compact 0-dimentional group.
(2) $\hat{G}$ is a discrete torsion group.
(3) Each L-projection $P : B(G) \to B(G)$ whose range is a subalgebra, is a conditional expectation.
(4) Each L-projection $P : B(G) \to B(G)$ whose range is a subalgebra and $P(1_G) = 1_G$, is a conditional expectation.

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