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On the commuting graph of non-commutative rings of order $p^n q$

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Abstract. Let R be a non-commutative ring with unity. The commuting graph of R denoted by $\Gamma(R)$, is a graph with vertex set $R \setminus Z(R)$ and two vertices a and b are adjacent iff $ab = ba$. In this paper, we consider the commuting graph of non-commutative rings of order pq and p^2q with $Z(R) = 0$ and non-commutative rings with unity of order p^3q. It is proved that $C_R(a)$ is a commutative ring for every $0 \neq a \in R \setminus Z(R)$. Also it is shown that if $a, b \in R \setminus Z(R)$ and $ab \neq ba$, then $C_R(a) \cap C_R(b) = Z(R)$. We show that the commuting graph $\Gamma(R)$ is the disjoint union of k copies of the complete graph and so is not a connected graph.

Keywords: Commuting graph; non-commutative ring; non-connected graph; algebraic graph.

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1. Introduction

Let G be a simple graph on vertex set $V(G)$ and edge set $E(G)$. A graph is said to be *connected* if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining v and u is called the *distance* between v and u and denoted by $d(v, u)$. The maximum value of the distance function in a connected graph G is called the *diameter* of G and denoted by $diam(G)$. If G is a graph, then the *complement* of G, denoted by G^c is a graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. The *complete graph* K_n is the graph with

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n vertices in which each pair of vertices are adjacent. We show $G = tK_m$ for disjoint union of t complete graph of size m. G is *complete* $t-$*partite* graph if there is a partition $V_1 \cup V_2 \cup \ldots \cup V_t = V(G)$ of the vertex set, such that v_i and v_j are adjacent if and only if v_i and v_j are in different parts of the partition. If $|V_k| = n_k$, then G is denoted by $K_{n_1,n_2,\ldots,n_t}$.

Let R be a non-commutative ring with unity 1 and let $Z(R)$ denotes the center of R. We are assuming $1 \neq 0$. A ring with unity is a division ring if every non-zero element a has a multiplicative inverse (that is, an element x with $ax = xa = 1$). If X is either an element or a subset of the ring R, then $C_R(X)$ denotes the *centralizer* of X in R. We introduce a graph with vertex set $R \setminus Z(R)$ and join two vertices a and b if $a \neq b$ and $ab = ba$. This graph is called the *commuting graph* of R and denoted by $\Gamma(R)$.

Akbari et.al [3] determined the diameters of some induced subgraphs of $\Gamma(M_n(D))$, for a division ring D and $n \geqslant 3$. Also they showed that if F is an algebraically closed field or n is a prime number and $\Gamma(M_n(F))$ is a connected graph, then diameter of $\Gamma(M_n(F))$ is equal to 4. Akbari and Raja [4] showed that if A, N, F and T are the sets of all non-invertible, nilpotent, idempotent and involutions matrices over division ring D, respectively, then $\Gamma(A)$, $\Gamma(N)$, $\Gamma(F)$ and $\Gamma(T)$ are connected graphs. In [1], two rings with distinct cardinality and the same commuting graphs are introduced. In [2], it has been shown that for a non-commutative ring R, the graph $\Gamma(R)^c$ is Hamiltonian and $\partial(\Gamma(R)^c) \leqslant 2$. In [9], it has been shown that for a non-commutative ring R, the diameter of $\Gamma(R)^c$ is one if and only if R is the non-commutative ring on 4 elements. Also they characterized all rings where the complements of their commuting graphs are planar.

In this work, we consider the commuting graph of non-commutative rings of order pq and p^2q with $Z(R) = 0$ and non-commutative rings with unity of order p^3q. We show that for $0 \neq a \in R \setminus Z(R)$, $C_R(a)$ is a commutative ring . Also $C_R(a) \cap C_R(b) = Z(R)$ for $0 \neq a, b \in R \setminus Z(R)$ and $ab \neq ba$. The main result is that the commuting graph $\Gamma(R)$ is the disjoint union of some copies of complete graphs.

2. Commuting graph of non-commutative rings

Throughout this paper, p and q are distinct prime numbers.

Lemma 2.1 [8] Let R be a finite ring of order m with a unity. If m has a cube free factorization, then R is a commutative ring.

As our first result, we prove the following Lemma.

Lemma 2.2 Let R be a non-commutative ring and $Z(R) \neq (0)$. Then $[R : Z(R)]$ is not prime.

Proof. Let $[R : Z(R)] = t$ be prime. Then group $(R, +)/(Z(R), +)$ is a cyclic group of order t. Let $(R, +)/(Z(R), +) = \langle a + Z(R)\rangle$. Then for any two elements of $x, y \in R$, there exist integer numbers n, m such that $x + Z(R) = na + Z(R)$ and $y + Z(R) = ma + Z(R)$. So there exist elements z_1 and z_2 in $Z(R)$ such that $x = na + z_1$ and $y = ma + z_2$. It is clear that $xy = yx$. This contradicts the fact that R is non-commutative ring.

Lemma 2.3 Let R be a finite ring of order p^2 or pq and $Z(R) \neq \{0\}$. Then R is commutative ring.

Proof. On the contrary let R be a finite non-commutative ring and $Z(R) \neq (0)$. If $|R| = p^2$, then $|Z(R)| = p$. So for any $a \in R \setminus Z(R)$, $|C_R(a)| = p^2$ and $a \in Z(R)$. This is contradiction. If $|R| = pq$, then $|Z(R)| \in \{p, q\}$. This is contradiction by Lemma 2.2. Hence R is a commutative ring.

Lemma 2.4 Let R be a non-commutative ring and $a, b \in R \setminus Z(R)$ such that $C_R(a)$ and $C_R(b)$ be commutative rings. If $ab = ba$, then $C_R(a) = C_R(b)$.

Proof. Let $x \in C_R(a)$. Since $ab = ba$ and $C_R(a)$ is commutative ring, $xb = bx$. So $C_R(a) \subseteq C_R(b)$. Similarly $C_R(b) \subseteq C_R(a)$. Thus $C_R(a) = C_R(b)$.

Lemma 2.5 Let R be a non-commutative ring of order p^3 and $|Z(R)| \neq 0$, then $|Z(R)| = p$.

Proof. Since $Z(R)$ is addition subgroup of R, $|Z(R)| \in \{1, p, p^2, p^3\}$. Also, since R is a non-commutative ring and $|Z(R)| \neq 1$, then $|Z(R)| = p$ or p^2. By Lemma 2.2, $[R : Z(R)] \neq p$. So $|Z(R)| = p$.

2.1 *Orders pq and p^2q*

Lemma 2.6 Let R be a non-commutative ring of order p^nq for $n = 1, 2$ and $Z(R) = \{0\}$. Then for every $0 \neq a \in R$, $C_R(a)$ is a commutative ring.

Proof. Let $0 \neq a \in R$. If $|R| = pq$, then $|C_R(a)| = p, q$ or pq. If $|C_R(a)| = pq$, then R is a commutative ring. This is contradiction. So $|C_R(a)|$ is prime. Hence $C_R(a)$ is a commutative ring. Let $|R| = p^2q$. Since $|C_R(a)| \mid |R|$, $|C_R(a)| \in \{p, q, p^2, pq\}$. If $|C_R(a)| = p$ or q, then $C_R(a)$ is a commutative ring. Let $C_R(a)$ be a ring of order p^2 or pq. Since $a \in Z(C_R(a))$, $Z(C_R(a)) \neq (0)$. By Lemma 2.3, $C_R(a)$ is a commutative ring. This completes the proof.

Theorem 2.7 Let R be a non-commutative ring of order p^nq for $n = 1, 2$ and $Z(R) = \{0\}$. If $0 \neq a, b \in R$ and $ab \neq ba$, then $C_R(a) \cap C_R(b) = \{0\}$.

Proof. On the contrary suppose that $C_R(a) \cap C_R(b) \neq 0$. Suppose $x \in C_R(a) \cap C_R(b)$. So $xa = ax$ and $xb = bx$. By Lemmas 2.4 and 2.6, $C_R(a) = C_R(x) = C_R(b)$. Hence $ab = ba$. This is impossible. Therefore $C_R(a) \cap C_R(b) = \{0\}$.

Theorem 2.8 Let R be a non-commutative ring of order pq such that $Z(R) = \{0\}$. Then the following is hold:

(i) $\Gamma(R) = \frac{pq-1}{p-1} K_{p-1}$ if $(p-1) \mid (pq-1)$.
(ii) $\Gamma(R) = \frac{pq-1}{q-1} K_{q-1}$ if $(q-1) \mid (pq-1)$.
(iii) $\Gamma(R) = l_1 K_{p-1} \cup l_2 K_{q-1}$ where $l_1(p-1) + l_2(q-1) = pq - 1$.

Proof. Let $a, b \in R \setminus Z(R)$ and $ab \neq ba$. By Theorem 2.7, $C_R(a) \cap C_R(b) = \{0\}$. Now if $x \in C_R(a), y \in C_R(b)$ and $xy = yx$, then by Lemma 2.4, $C_R(a) = C_R(x)$, $C_R(b) = C_R(y)$ and $C_R(x) = C_R(y)$. So $C_R(a) = C_R(b)$, which is impossible. Therefore $\Gamma(R)$ is the disjoint union of the complete graphs. Since R is non-commutative ring, for $0 \neq a \in R$, $|C_R(a)| = p$ or q. If for every $0 \neq a \in R$, $|C_R(a)| = p$, then $|V(\Gamma(R))| = l(p-1)$. On the other hand $|V(\Gamma(R))| = pq - 1$. Thus $l = \frac{pq-1}{p-1}$. So $\Gamma(R) = (\frac{pq-1}{p-1})K_{(p-1)}$ if $(p-1) \mid (pq-1)$. If for every $0 \neq a \in R$, $|C_R(a)| = q$, then $\Gamma(R) = \frac{pq-1}{q-1} K_{(q-1)}$ if $(q-1) \mid (pq-1)$. Let $|C_R(a)| = p$ and $|C_R(b)| = q$ for some $a, b \in R$. Hence $\Gamma(R)$ is the disjoint union of l_1 copies of complete graph $K_{(p-1)}$ and l_2 copies of complete graph $K_{(q-1)}$ where $l_1(p-1) + l_2(q-1) = pq - 1$. This completes the proof.

Theorem 2.9 Let R be a non-commutative ring of order p^2q such that $Z(R) = \{0\}$. Then the following is hold:

(i) $\Gamma(R) = \frac{p^2q-1}{t-1}K_{t-1}$ such that $t \in \{p, q, p^2, pq\}$ and $t \mid (p^2q-1)$.

(ii) $\Gamma(R) = l_1K_{p-1} \cup l_2K_{q-1} \cup l_3K_{p^2-1} \cup l_4K_{pq-1}$ such that $\sum_{i=1}^{4} l_i = p^2q-1$.

Proof. Likewise the proof of Theorem 2.8, $\Gamma(R)$ is the disjoint union of the complete graphs. Since R is non-commutative ring, for $0 \neq a \in R$, $|C_R(a)| \in \{p, q, p^2, pq\}$. If for every $0 \neq a \in R$, $|C_R(a)| = t$ for $t \in \{p, q, p^2, pq\}$, then $|V(\Gamma(R))| = l(t-1)$. Also $|V(\Gamma(R))| = p^2q-1$. Thus $\Gamma(R) = \frac{p^2q-1}{t-1}K_{t-1}$ if $(t-1) \mid (p^2q-1)$ for $t \in \{p, q, p^2, pq\}$. Now let $|\{r \in R \setminus Z(R)\,;\, |C_R(r)| = p\}| = l_1, |\{r \in R \setminus Z(R)\,;\, |C_R(r)| = q\}| = l_2, |\{r \in R \setminus Z(R)\,;\, |C_R(r)| = p^2\}| = l_3$ and $|\{r \in R \setminus Z(R)\,;\, |C_R(r)| = pq\}| = l_4$. Then $|V(\Gamma(R))| = l_1(p-1)+l_2(q-1)+l_3(p^2-1)+l_4(pq-1)$. Thus $\Gamma(R) = l_1K_{p-1} \cup l_2K_{q-1} \cup l_3K_{p^2-1} \cup l_4K_{pq-1}$ where $\sum_{i=1}^{4} l_i = p^2q-1$. This completes the proof.

2.2 *Order p^3q*

Theorem 2.10 Let R be a non-commutative ring with a unity of order p^3q and $a \in R \setminus Z(R)$. Then $C_R(a)$ is a commutative ring.

Proof. By Lemma 2.2 and since R is non-commutative ring with unity, $|Z(R)| \in \{p, p^2, q, pq\}$.
Case 1: Let $|Z(R)| = p$. Since $C_R(a)$ is the addition subgroup of R and $a \notin Z(R)$, $|C_R(a)| \in \{p^2, p^3, pq, p^2q\}$.
Subcase i: If $|C_R(a)| = p^2, pq$ or p^2q, then by Lemma 2.1, $C_R(a)$ is a commutative ring.
Subcase ii: If $|C_R(a)| = p^3$ and $C_R(a)$ is a non-commutative ring, then by Lemma 2.5, $|Z(C_R(a)| = p$. It is clear that $Z(R) \cup (a + Z(R)) \subseteq Z(C_R(a))$. Thus $p + p \leqslant p$. This is impossible.
Case 2: Let $|Z(R)| = p^2$. Since $|Z(R)| \mid |C_R(a)|$, $|C_R(a)| \in \{p^3, p^2q\}$. If $|C_R(a)| = p^2q$, then by Lemma 2.1, $C_R(a)$ is a commutative ring. If $|C_R(a)| = p^3$ and $C_R(a)$ is a non-commutative ring, then likewise case 1, subcase ii, $2p^2 \leqslant p$. Hence $C_R(a)$ is a commutative ring.
Case 3: Let $|Z(R)| = q$. Then $C_R(a)$ is of order pq or p^2q. So this is a commutative ring.
Case 4: If $|Z(R)| = pq$, then $|C_R(a)| = p^2q$. Hence $C_R(a)$ is a commutative ring. This completes the proof.

Theorem 2.11 Let R be a non-commutative ring with a unity of order p^3q such that $|Z(R)|$ is not prime. If $a, b \in R \setminus Z(R)$ and $ab \neq ba$, then $C_R(a) \cap C_R(b) = Z(R)$.

Proof. Since $|Z(R)| \in \{p^2, pq\}$, the proof falls naturally into two parts:
Part 1: If $|Z(R)| = p^2$, then for every $x \in R \setminus Z(R)$, $|C_R(x)| \in \{p^3, p^2q\}$. Thus for $a, b \in R \setminus Z(R)$ there are three cases:
Case i: If $|C_R(a)| = |C_R(b)| = p^3$, then $|C_R(a) \cap C_R(b)| = p^2$ or p^3. Since $ab \neq ba$, $|C_R(a) \cap C_R(b)| \neq p^3$. So $C_R(a) \cap C_R(b) = Z(R)$.
Case ii: If $|C_R(a)| = |C_R(b)| = p^2q$, then $|C_R(a) \cap C_R(b)| = p^2$ or p^2q. If $|C_R(a) \cap C_R(b)| = p^2q$, then $ab = ba$. This is not true. Hence $C_R(a) \cap C_R(b) = Z(R)$.
Case iii: Let $|C_R(a)| = p^3$ and $|C_R(b)| = p^2q$. Then $|C_R(a) \cap C_R(b)| = p^2$. So $C_R(a) \cap C_R(b) = Z(R)$.
Part 2: If $|Z(R)| = pq$, then for every $x \in R \setminus Z(R)$, $|C_R(x)| = p^2q$. Since $|Z(R)| \mid |C_R(a) \cap C_R(b)|$ and $|C_R(a) \cap C_R(b)| \mid p^2q$, $|C_R(a) \cap C_R(b)| \in \{pq, p^2q\}$. If $|C_R(a) \cap C_R(b)| = p^2q$, then $ab = ba$. This is impossible. So $|C_R(a) \cap C_R(b)| = pq$. And $C_R(a) \cap C_R(b) = Z(R)$.

Theorem 2.12 Let R be a non-commutative ring with a unity of order p^3q. If $|Z(R)|$ is not prime, then the following is hold:

(i) $\Gamma(R) = (\frac{pq-1}{p-1})K_{(p^3-p^2)}$ if $(p-1) \mid (pq-1)$.
(ii) $\Gamma(R) = (\frac{pq-1}{q-1})K_{(p^2q-p^2)}$ if $(q-1) \mid (pq-1)$.
(iii) $\Gamma(R) = l_1K_{(p^3-p^2)} \cup l_2K_{(p^2q-p^2)}$ where $l_1(p-1) + l_2(q-1) = pq-1$.
(iv) $\Gamma(R) = (p+1)K_{(p^2q-pq)}$.

Proof. Since $|Z(R)| \in \{p^2, pq\}$, the proof falls naturally into two parts:

Part 1: If $|Z(R)| = p^2$, then $|C_R(a)| \in \{p^3, p^2q\}$ for every $a \in R \setminus Z(R)$. Suppose $|C_R(a)| = p^3$ for every $a \in R \setminus Z(R)$. Let $a, b \in R \setminus Z(R)$ and $ab \neq ba$. By Theorem 2.11, $C_R(a) \cap C_R(b) = Z(R)$. Now if $x \in C_R(a), y \in C_R(b)$ and $xy = yx$, then by Lemma 2.4, $C_R(a) = C_R(x)$, $C_R(b) = C_R(y)$ and $C_R(x) = C_R(y)$. So $C_R(a) = C_R(b)$, which is impossible. Therefore $\Gamma(R)$ is the disjoint union of l copies of the complete graph of size $p^3 - p^2$. So $|V(\Gamma(R))| = l(p^3 - p^2)$. On the other hand $|V(\Gamma(R))| = |R| - |Z(R)| = p^3q - p^2$. Thus $l = \frac{pq-1}{p-1}$. Hence $\Gamma(R) = (\frac{pq-1}{p-1})K_{(p^3-p^2)}$ if $(p-1) \mid (pq-1)$. Suppose $|C_R(a)| = p^2q$ for every $a \in R \setminus Z(R)$. By similar argument $\Gamma(R)$ is the disjoint union of l copies of the complete graph of size p^2q-p^2 where $l = \frac{pq-1}{q-1}$. So $\Gamma(R) = (\frac{pq-1}{q-1})K_{(p^2q-p^2)}$ if $(q-1) \mid (pq-1)$. Let $|C_R(a)| = p^3$ and $|C_R(b)| = p^2q$ for some $a, b \in R \setminus Z(R)$. Then by Theorem 2.11, $C_R(a) \cap C_R(b) = Z(R)$. It is easy to see that if $x \in C_R(a)$ and $y \in C_R(b)$, then $xy \neq yx$. Hence $\Gamma(R)$ is the disjoint union of l_1 copies of the complete graph of size $p^3 - p^2$ and l_2 copies of the complete graph of size $p^2q - p^2$. So $|V(\Gamma(R))| = l_1(p^3 - p^2) + l_2(p^2q - p^2)$. On the other hand we have $|V(\Gamma(R))| = |R| - |Z(R)| = p^3q - p^2$. Thus $p^3q - p^2 = l_1(p^3 - p^2) + l_2(p^2q - p^2)$. Therefore $\Gamma(R) = l_1K_{(p^3-p^2)} \cup l_2K_{(p^2q-p^2)}$, where l_1 and l_2 satisfy in $l_1(p-1) + l_2(q-1) = pq-1$, and this prove the Part (iii).

Part 2: If $|Z(R)| = pq$, then $|C_R(a)| = p^2q$. Likewise Part 1, $\Gamma(R)$ is the disjoint union of l copies of the complete graph of size $p^2q - pq$ where $l(p^2q - pq) = p^3q - pq$. Therefore $\Gamma(R) = (p+1)K_{(p^2q-pq)}$.

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