Journal of Linear and Topological Algebra
Vol. 12*, No.* 03*,* 2023*,* 211- 223
DOR: 20.1001.1.22520201.2023.12.03.6.8
DOI: 10.30495/JLTA.2023.707862

Basis extension and construction of tight frames

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Received 7 August 2023; Revised 30 October 2023, Accepted 30 October 2023.
Communicated by Amin Mahmoodi

Abstract. The notion of compression has received enormous attention in recent years because of its necessity in terms of the computational cost and other applicable features. But many times the notion expansion appears to be quite useful. Tight frames are quite useful in signal reconstruction, signal and image de-noising, compressed sensing because of the availability of a simple, explicit reconstruction formula. So in this paper, we discuss the extension of a basis by including some very sparse (at most two nonzero components) vectors so that the new frame becomes a tight frame. We do the basis extension in finite dimensional Hilbert spaces (both real and complex) to construct tight frames. We formulate constructive algorithms to do the aforementioned task. The algorithms guarantee us to produce tight frames with very less computational cost, and the new tight frames compensate for multiple erasures. The algorithms also do not disturb the vectors in the given basis. We also present one application of the aforementioned concept.

Keywords: Frames, tight frames, basis extension.

2010 AMS Subject Classification: 42C15, 46B15.

1. Introduction

Frames are more flexible substitutes for bases as they allow redundancy. The notion of frames was initiated by Duffin and Schaeffer [13] in 1952 while studying nonharmonic Fourier series. With the emergence of the wavelet era Daubechies, Grossmann and Meyer [12] reintroduced and developed the theory of frames in 1986. The flexible structure of frames drew the attention of many engineers, mathematicians and physicists because of its wide application in various well known fields like signal processing [18], coding and communications [23], image processing [5], sampling [15, 16], numerical analysis, filter theory [3]. Recently, it is emerged as an important tool in compressive sensing, quantum information processing, data analysis, coding theory and in several other areas.

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

Tight frames are widely applicable because of the availability of a simple and explicit reconstruction formula. Special types of tight frames are applied to solve problems in communications [21–24]. A physical interpretation of tight frames has been given in Casazza et al. [6]. Gobel et al. [20] constructed tight frames for the space of real valued functions defined on a graph of finite set of points, and they applied them for denoising a function f with given noisy observations. Cotfas and Gazeau [9] applied the notion of tight frames in crystal and quasi-crystal physics. They described Honeycomb lattice and diamond structure in terms of tight frames. Construction of equiangular tight frames have been discussed in [1, 2, 19]. Construction of k-angle tight frames have been studied by Datta and Oldroyd [10]. For some recent work on tight frames and their applications one may refer [8, 11, 14].

Feng et al. [17] addressed the problem of constructing finite tight frames with prescribed norm for each vector in the frame. They have employed the Householder transformations for this purpose. The computation of eigenvalues required for this method is itself a tedious job and the computational cost is very high for large matrices. Casazza and Leon [7] discussed the construction of tight frames with a given positive, self adjoint, invertible operator, and with a given set of lengths of frame vectors to be constructed. In this case also the conditions are too restrictive, and the computation of eigenvalues requires a large amount of computation. Cahill et al. [4] established a method of constructing finite frames with a given spectrum of frame operator and prescribed set of lengths of frame vectors. Their method is also very restrictive and imposes high computational costs. There are two standard methods of extending a frame to a tight frame.
(1) Using the eigenvalues and eigenvectors of a given frame operator: Given a frame $\{f_i\}_{i=1}^m$ for $\mathbb{R}^n$. Let $\{e_i\}_{i=1}^n$ be the eigenvectors for the frame operator with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$. Construct the vectors $g_i = \sqrt{\lambda_1 - \lambda_i} e_i$ for $i = 2, 3, .., n$. Then the collection $\{f_i\}_{i=1}^m \cup \{g_i\}_{i=2}^n$ forms a tight frame.
(2) Extending vectors to equal norm orthogonal bases: Given a frame $\{f_i\}_{i=1}^m$ for $\mathbb{R}^n$, for each $i = 1, 2, ..., m$ add vectors $\{f_{ij}\}_{j=2}^n$ so that $\{f_i\} \cup \{f_{ij}\}_{j=2}^n$ is an equal norm orthogonal set. The construction of the vectors $\{f_{ij}\}_{j=2}^n$ is again done by using eigenvalues and eigenvectors. Now the collection $\{f_1\} \cup \{f_{1j}\}_{j=2}^n \cup \{f_2\} \cup \{f_{2j}\}_{j=2}^n \cup .... \cup \{f_n\} \cup \{f_{nj}\}_{j=2}^n$ forms a tight frame.

Both the above methods require the computation of eigenvalues and eigenvectors. The computation of eigenvalues and eigenvectors are tedious job. To overcome the issue of computing eigenvalues and eigenvectors, we propose new simple algorithms to construct tight frames in finite dimensional real and complex Hilbert spaces. Given any basis we extend it by adding some sparse vectors into it to form a tight frame without disturbing the given basis elements. Also our algorithm will provide the tight frame bound automatically without much computation. Although it seems little inconvenient adding extra vectors into a basis, but as a result it gives a tight frame which is easier to handle. Also the additional vectors are very sparse with at most two nonzero components. The new tight frame also compensates for erasures. Initially, we start with the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, then generalize the algorithm for $\mathbb{R}^n$. We also construct tight frames in complex domains $\mathbb{C}^2$ and $\mathbb{C}^3$, and then generalize our algorithm to $\mathbb{C}^n$. As an application, we discuss the effect of the process on the numerical range of the frame operator.

2. Basis expansion and construction of tight frames

We begin the section by recalling the basic concepts of frame and tight frame. Let H be a finite dimensional Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$.

Definition 2.1 A sequence of elements $\{\phi\}_{i\in I} \subset H$ is called a frame for H if there exist real positive constants A, B such that

$$A\|x\|^2 \leqslant \sum_{i\in I} |\langle x, \phi_i\rangle|^2 \leqslant B\|x\|^2, \quad \forall x \in H. \tag{1}$$

If $A = B$, then (1) becomes $\sum_{i\in I} |\langle x, \phi_i\rangle|^2 = A\|x\|^2$ for all $x \in H$. In that case, the frame $\{\phi\}_{i\in I}$ is called a tight frame. If $A = B = 1$, then (1) becomes $\sum_{i\in I} |\langle x, \phi_i\rangle|^2 = \|x\|^2$ for all $x \in H$. In that case the frame $\{\phi\}_{i\in I}$ is called a Parseval frame.

Let $\{\phi\}_{i\in I} \subset H$ be a frame for H. The map $L : \ H \to l^2$ defined by $Lx = \{\langle x, \phi_i\rangle\}_{i\in I}$ is called the analysis or pre-frame operator. L is bounded and linear. Its adjoint $L^* : \ l^2 \to H$, given by $L^*\big(\{c_i\}_{i\in I}\big) = \sum_{i\in I} c_i\phi_i$ is called the analysis operator. The operator $S = L^*L : H \to H$, defined by $Sx = \sum_{i\in I}\langle x, \phi_i\rangle\phi_i$ is called the frame operator. The operator S is bounded, linear, positive, self adjoint and invertible. When the frame $\{\phi\}_{i\in I}$ is a tight frame with bound A, then the frame operator $S = AI$, where I is the identity operator. In this case, we have a nice explicit reconstruction formula $x = \frac{1}{A}\sum_{i\in I}\langle x, \phi_i\rangle\phi_i$. That is why tight frames are more useful in applications.

2.1 *Construction of tight frames in* $\mathbb{R}^2$

Consider a set of vectors $M = \{(1, 0), (1, 1)\}$ in $\mathbb{R}^2$. It is a minimal spanning set for $\mathbb{R}^2$. It is also a frame which is not tight. Now consider the set $M' = \{(1, 0), (1, 1), (-1, 1), (0, 1)\}$. This is a tight frame with bound 3. Here we have extended the set M to make a tight frame without disturbing the given vectors. So naturally the following questions arise:
1. Is it always possible to extend a minimal spanning set into a tight frame?
2. How to construct the tight frames?
3. How many extra vectors are to be added into the given system?
4. What will be the bound for the newly formed tight frame?
5. Can we formulate an algorithm to perform the above mentioned task?
6. Can we extend the algorithm to $\mathbb{R}^3$ and generalize it to $\mathbb{R}^n$?
7. Can we construct such an algorithm for $\mathbb{C}^2$, $\mathbb{C}^3$ and generalize it to $\mathbb{C}^n$?

The answers to Q.1, Q.5, Q.6 and Q.7 are affirmative. The following algorithm will address Q.1, Q.2, Q.3 and Q.4.

Algorithm 1
Input basis $F = \{(a, b), (c, d)\}$
If F is a tight frame then stop. Print the frame F is tight. Else Let $F = F'$
Include vectors $(-a, b)$ and $(-c, d)$ into F'
Compute $M = \max\{2(a^2 + c^2), 2(b^2 + d^2)\}$
Include vectors $\sqrt{M - 2(a^2 + c^2)}(1, 0)$ and $\sqrt{M - 2(b^2 + d^2)}(0, 1)$ into F'
Remove the zero vector from F'
The new set of vectors F' forms a tight frame.
The bound for the tight frame F' is M.

The above algorithm guarantees us to produce a tight frame from a given basis in $\mathbb{R}^2$.

The maximum number of extra vectors to be added is 3.

Geometrical interpretation of this construction:
Given basis $F = \{(a, b), (c, d)\}$. If we extend it by using above algorithm then we obtain the tight frame

$$F' = \{(a, b), (c, d), (-a, b), (-c, d), (\sqrt{M - M_1}, 0) \text{ or } (0, \sqrt{M - M_2})\},$$

where $M_1 = 2(a^2 + c^2)$, $M_2 = 2(b^2 + d^2)$ and $M = \max\{M_1, M_2\}$. Note that $(-a, b)$ and $(-c, d)$ are the reflections of the given vectors (a, b) and (c, d), respectively, about the y-axis. $(\sqrt{M - M_1}, 0)$ and $(0, \sqrt{M - M_2})$ are the adjustment vectors, where $(\sqrt{M - M_1}, 0)$ lies on $x - axis$, whereas $(0, \sqrt{M - M_2})$ lies on $y-$axis.

***Example* 2.2**

Input basis	Output tight frame	Bound
$\{(2, 1), (1, -1)\}$	$\{(2, 1), (1, -1), (-2, 1), (-1, -1), \sqrt{6}(0, 1)\}$	10
$\left\{\left(\frac{1}{2}, \frac{1}{3}\right), (1, -2)\right\}$	$\left\{\left(\frac{1}{2}, \frac{1}{3}\right), (1, -2), \left(-\frac{1}{2}, \frac{1}{3}\right), (-1, -2), \sqrt{\frac{103}{18}}(1, 0)\right\}$	$\frac{\sqrt{74}}{3}$
$\{(1, 1), (1, -1)\}$	$\{(1, 1), (1, -1)\}$	2

Proposition 2.3 If $F = \{(a, b), (c, d)\}$ is a given basis in $\mathbb{R}^2$ with the property that $a^2 + c^2 = b^2 + d^2$. If we construct a tight frame F' by using the Algorithm *1*, then there will be a maximum of 2 additional vectors.

Proof. The proof is straightforward from the Algorithm *1*. As $a^2 + c^2 = b^2 + d^2$, and $M = \max\{2(a^2+c^2), 2(b^2+d^2)\}$. Hence $\sqrt{M - 2(a^2 + c^2)} = 0$ and $\sqrt{M - 2(b^2 + d^2)} = 0$. As a result, we obtain two zero vectors, which have no contribution in the construction tight frame, and we remove them from F'. Hence there will be a maximum of 2 additional vectors in F'. ■

2.2 *Construction of tight frames in* $\mathbb{R}^3$

The algorithm in $\mathbb{R}^2$ with a little modification can be generalized to $\mathbb{R}^3$. Given a basis $\{(a, b, c), (d, e, f), (g, h, i)\}$ in $\mathbb{R}^3$. We extend the basis to construct a tight frame.

Algorithm 2
Input basis $F = \{(a, b, c), (d, e, f), (g, h, i)\}$
If F is a tight frame then stop. Print the frame F is tight. Else Let $F = F'$
Include vectors
$\{(-a, b, 0), (a, 0, -c), (0, -b, c), (-d, e, 0), (d, 0, -f), (0, -e, f), (-g, h, 0),$
$(g, 0, -i), (0, -h, i)\}$ into F'.
Compute $M = \max\{3(a^2 + d^2 + g^2), 3(b^2 + e^2 + h^2), 3(c^2 + f^2 + i^2)\}$
Include vectors $\sqrt{M - 3(a^2 + d^2 + g^2)}(1, 0, 0)$, $\sqrt{M - 3(b^2 + e^2 + h^2)}(0, 1, 0)$
and $\sqrt{M - 3(c^2 + f^2 + i^2)}(0, 0, 1)$ into F'
Remove the zero vector from F'
The new set of vectors F' forms a tight frame.
The bound for the tight frame F' is M.

The above algorithm guarantees us to produce a tight frame from a given basis in $\mathbb{R}^3$.

The maximum number of extra vectors to be added is 11.

***Example* 2.4**

Input basis	Output tight frame	Bound
$\{(1,0,0),(1,1,1),(1,1,0)\}$	$\{(1,0,0),(1,1,1),(1,1,0),(-1,0,0),(1,0,0),$ $(-1,1,0),(1,0,-1),(0,-1,1),(-1,1,0),(1,0,0),$ $(0,-1,0),\sqrt{3}(0,1,0),\sqrt{6}(0,0,1)\}$	9
$\{(\frac{1}{2},\frac{1}{3},1),(1,-2,5),(2,3,4)\}$	$\{(\frac{1}{2},\frac{1}{3},1),(1,-2,5),(2,3,4),(-\frac{1}{2},\frac{1}{3},0),(\frac{1}{2},0,-1),$ $(0,-\frac{1}{3},1),(-1,-2,0),(1,0,-5),(0,2,5),(-2,3,0),$ $(2,0,-4),(0,-3,4),\frac{\sqrt{441}}{2}(1,0,0),\sqrt{\frac{260}{3}}(0,1,0)\}$	126
$\{(1,2,1),(2,-1,1),(1,1,2)\}$	$\{(1,2,1),(2,-1,1),(1,1,2),(-1,2,0),$ $(1,0,-1),(0,-2,1),(-2,-1,0),(2,0,-1),$ $(0,1,1),(-1,1,0),(1,0,-2),(0,-1,2)\}$	18

In the third row of the above table, one can observe that the number of extra vectors added in the basis to form a tight frame is 9. But in general by the algorithm there should be 11 additional vectors. This happened because of the additional property of the given basis, $1^2+2^2+1^2 = 2^2+(-1)^2+1^2 = 1^2+1^2+2^2$. That is the sum of the squares of the first components of each vector is the same as the sum of the squares of the second components of each vector is the same as the sum of the squares of the third components of each vector. Hence we have the following proposition:

Proposition 2.5 If $\{(a,b,c),(d,e,f),(g,h,i)\}$ is a given basis in $\mathbb{R}^3$ with the property that $a^2+d^2+g^2 = b^2+e^2+h^2 = c^2+f^2+i^2$. If we construct a tight frame by using the Algorithm *2*, then there will be a maximum of 9 additional vectors.

Proof. The proof is straightforward from the Algorithm *2*. As $a^2+d^2+g^2 = b^2+e^2+h^2 = c^2+f^2+i^2$, and $M = \max\{3(a^2+d^2+g^2), 3(b^2+e^2+h^2), 3(c^2+f^2+i^2)\}$. Hence $\sqrt{M-3(a^2+d^2+g^2)} = 0$ and $\sqrt{M-3(b^2+e^2+h^2)} = 0$. As a result, we have two additional zero vectors, which have no contribution in the construction tight frame, and we remove them. Hence there will be a maximum of 9 additional vectors. ∎

Although there may be smaller tight frames containing a given basis, our algorithm guarantees us to provide a tight frame for any given basis. Also the algorithm is very simple to implement and requires very less number of computations.

2.3 *Construction of tight frames in* $\mathbb{R}^n$

Now, we generalize our algorithm for $\mathbb{R}^n$. Given a basis

$$\{X_1 = (x_{11}, x_{12}, ..., x_{1n}), X_2 = (x_{21}, x_{22}, ..., x_{2n}), ..., X_n = (x_{n1}, x_{n2}, ..., x_{nn})\}$$

in $\mathbb{R}^n$. We extend it to form a tight frame.

Algorithm 3

Input basis $F = \{X_1 = (x_{11}, x_{12}, ..., x_{1n}), X_2 = (x_{21}, x_{22}, ..., x_{2n}), ...,$
$X_n = (x_{n1}, x_{n2}, ..., x_{nn})\}$
If F is a tight frame then stop. Print the frame F is tight.
Else,
let $F = F'$.

Include vectors

$$\begin{aligned}
&\{(-x_{11}, x_{12}, 0, ..., 0), (x_{11}, 0, -x_{13}, 0, ..., 0), ..., (x_{11}, 0, .., 0, -x_{1n}),\\
&(0, -x_{12}, x_{13}, 0, .., 0), (0, x_{12}, 0, -x_{14}, 0, .., 0), ..., (0, x_{12}, 0, .., 0, -x_{1n}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots, (0, 0, ..., -x_{1n-1}, x_{1n}),\\
&(-x_{21}, x_{22}, 0, ..., 0), (x_{21}, 0, -x_{23}, 0, ..., 0), ..., (x_{21}, 0, .., 0, -x_{2n}),\\
&(0, -x_{22}, x_{23}, 0, .., 0), (0, x_{22}, 0, -x_{24}, 0, .., 0), ..., (0, x_{22}, 0, .., 0, -x_{2n}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots, (0, 0, ..., -x_{2n-1}, x_{2n}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&\{(-x_{n1}, x_{n2}, 0, ..., 0), (x_{n1}, 0, -x_{n3}, 0, ..., 0), ..., (x_{n1}, 0, .., 0, -x_{nn}),\\
&(0, -x_{n2}, x_{n3}, 0, .., 0), (0, x_{n2}, 0, -x_{n4}, 0, .., 0), ..., (0, x_{n2}, 0, .., 0, -x_{nn}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots, (0, 0, ..., -x_{nn-1}, x_{nn})\} \text{ into } F'.
\end{aligned}$$

Compute

$$M = \max\left\{n(x_{11}^2 + x_{21}^2 + ... + x_{n1}^2), n(x_{12}^2 + x_{22}^2 + .. + x_{n2}^2), ..., n(x_{1n}^2 + x_{2n}^2 + .. + x_{nn}^2)\right\}.$$

Compute

$$\begin{aligned}
M_1 &= \sqrt{M - n(x_{11}^2 + x_{21}^2 + ... + x_{n1}^2)},\\
M_2 &= \sqrt{M - n(x_{12}^2 + x_{22}^2 + ... + x_{n2}^2)},\\
&\dots\dots\dots\dots\dots\dots,\\
M_n &= \sqrt{M - n(x_{1n}^2 + x_{2n}^2 + ... + x_{nn}^2)}.
\end{aligned}$$

Include vectors $M_1(1, 0, ..., 0), M_2(0, 1, 0, ..., 0), ..., M_n(0, 0, ..., 1)$ into F'.
Remove the zero vectors from F'.
The new set of vectors F' forms a tight frame.
The bound for the tight frame F' is M.

Given any basis in $\mathbb{R}^n$, our algorithm extends it and provides us a tight frame without much computation. In this process more number of vectors are added into the given system. So it seems unwanted, but after the process we get a tight frame with very less number of computations. Tight frames are easy to handle and more applicable in nature. Also the new tight frame compensates for erasures. Therefore, it gives justice to the process of adding some more vectors into the given system.

Theorem 2.6 Let

$$F = \left\{(x_{11}, x_{12}, ..., x_{1n}), (x_{21}, x_{22}, ..., x_{2n}), ..., (x_{n1}, x_{n2}, ..., x_{nn})\right\}$$

be a given basis in $\mathbb{R}^n$. Suppose we extend it by using Algorithm *3* to form a tight frame F'. Then $\max|F'| = \frac{1}{2}(n^3 - n^2 + 4n - 2)$, where $\max|F'|$ indicates maximum number of nonzero vectors in F'. Moreover, if

$$x_{11}^2 + x_{21}^2 + ... + x_{n1}^2 = x_{12}^2 + x_{22}^2 + ... + x_{n2}^2 = ... = x_{1n}^2 + x_{2n}^2 + ... + x_{nn}^2,$$

then $\max|F'| = \frac{1}{2}(n^3 - n^2 + 2n)$.

Proof. We can estimate this by using simple counting. Initially, we include the vectors from the given basis, that gives rise to n vectors in F'. Then in the second step of the algorithm we include $n\binom{n}{2}$ number of vectors into F'. In the final step of the algorithm we include vectors

$$M_1(1,0,...,0), M_2(0,1,0,...,0), ..., M_n(0,0,...,1)$$

into F', and these are another n vectors. At least one of these vectors is a zero vector and we remove the zero vectors. That gives at most $n-1$ nonzero vectors to be included in F'. Therefore, we have

$$\begin{aligned}\max|F'| &= n + n\binom{n}{2} + n - 1\\ &= 2n - 1 + n\frac{n!}{2!(n-2)!} = 2n - 1 + \frac{n^2(n-1)}{2} = \frac{1}{2}(n^3 - n^2 + 4n - 2).\end{aligned}$$

In the second part of the theorem, if

$$x_{11}^2 + x_{21}^2 + ... + x_{n1}^2 = x_{12}^2 + x_{22}^2 + ... + x_{n2}^2 = ... = x_{1n}^2 + x_{2n}^2 + ... + x_{nn}^2,$$

then $M_1 = M_2 = ... = M_n = 0$. So, the vectors

$$M_1(1,0,...,0), M_2(0,1,0,...,0), ..., M_n(0,0,...,1)$$

are all zero vectors, and we remove them all. Therefore,

$$\max|F'| = n + n\binom{n}{2} = \frac{n^3 - n^2 + 2n}{2}.$$

■

3. Construction of tight frames in $\mathbb{C}^2$

Given a basis $\mathcal{B}$ in $\mathbb{C}^2$, we extend it to form a tight frame. We apply similar a procedure as discussed in the real case. We treat an element in $\mathbb{C}^2$ as an element in $\mathbb{R}^4$. Specifically, we represent an element $(a,b) \in \mathbb{C}^2$ as $(a,b) = (a_1 + ia_2, b_1 + ib_2) = (a_1, a_2, b_1, b_2)$. We construct similar algorithms as in the real case but in this case the number of basis elements is 2. Also in this case we have to add less number of vectors as compared to $\mathbb{R}^4$.

Suppose we are given a basis $\mathcal{B} = \{(a,b),(c,d)\} = \{(a_1+ia_2, b_1+ib_2),(c_1+ic_2, d_1+id_2)\}$ of $\mathbb{C}^2$. We represent the basis as $\mathcal{B} = \{(a_1,a_2,b_1,b_2),(c_1,c_2,d_1,d_2)\}$. We construct the following algorithm to extend $\mathcal{B}$ to form a tight frame for $\mathbb{C}^2$.

Algorithm 1
Input basis $\mathcal{B} = \{(a,b),(c,d)\} = \{(a_1+ia_2, b_1+ib_2),(c_1+ic_2, d_1+id_2)\}$
If $\mathcal{B}$ is a tight frame then stop.
Print the frame $\mathcal{B}$ is tight.
Else
Let $\mathcal{B}' = \{(a_1,a_2,b_1,b_2),(c_1,c_2,d_1,d_2)\}$
Include vectors
$\{(-a_1,0,b_1,0),(-a_1,0,0,b_2),(0,-a_2,b_1,0),(0,-a_2,0,b_2),$
$(-c_1,0,d_1,0),(-c_1,0,0,d_2),(0,-c_2,d_1,0),(0,-c_2,0,d_2)\}$ into $\mathcal{B}'$.
Compute $M = \max\{3(a_1^2+a_2^2+c_1^2+c_2^2), 3(b_1^2+b_2^2+d_1^2+d_2^2)\}$
Denote $M_1 = 3(a_1^2+a_2^2+c_1^2+c_2^2)$ and $M_2 = 3(b_1^2+b_2^2+d_1^2+d_2^2)$.
Include vectors $\sqrt{M-M_1}(1,0,0,0)$, $\sqrt{M-M_2}(0,0,1,0)$ into $\mathcal{B}'$.
The new set of vectors
$\mathcal{B}' = \big\{(a_1+ia_2, b_1+ib_2),(c_1+ic_2, d_1+id_2),(-a_1,b_1),(-a_1,ib_2),(-ia_2,b_1),(-ia_2,ib_2),$
$(-c_1,d_1),(-c_1,id_2),(-ic_2,d_1),(-ic_2,id_2),(\sqrt{M-M_1},0),(0,\sqrt{M-M_2})\big\}$ forms a tight frame.
Remove the zero vector from $\mathcal{B}'$
The bound for the tight frame $\mathcal{B}'$ is M.

***Example* 3.1**

Input basis	Output tight frame	Bound
$\big\{(1+i,1-i),(3+i,2i)\big\}$	$\big\{(1+i,1-i),(3+i,2i),(-1,1),(-1,-i),$ $(-i,1),(-i,-i),(-3,0),(-3,2i),$ $(-i,0),(-i,2i),(0,\sqrt{18})\big\}$	36
$\Big\{\big(\frac{1}{2}+i,-\frac{1}{3}+i\big),$ $(-1+2i,2+i)\Big\}$	$\Big\{\big(\frac{1}{2}+i,-\frac{1}{3}+i\big),\big(-\frac{1}{2},-\frac{1}{3}\big),\big(-\frac{1}{2},i\big),$ $\big(-i,-\frac{1}{3}\big),\ \big(-i,i\big)\ (-1+2i,2+i),(1,2),$ $(1,i),\ (-2i,2),(-2i,i),(0,\sqrt{\frac{5}{12}})\ \Big\}$	$\frac{75}{4}$

3.1 *Construction of tight frames in* $\mathbb{C}^3$

Suppose we are given a basis $\mathcal{B} = \{(a,b,c),(d,e,f),(g,h,k)\}$
$= \{(a_1+ia_2, b_1+ib_2, c_1+ic_2),(d_1+id_2, e_1+ie_2, f_1+if_2),(g_1+ig_2, h_1+ih_2, k_1+ik_2)\}$ of $\mathbb{C}^3$. We represent the basis as $\mathcal{B} = \{(a_1,a_2,b_1,b_2,c_1,c_2),(d_1,d_2,e_1,e_2,f_1,f_2),$
$(g_1,g_2,h_1,h_2,k_1,k_2)\}$. We construct the following algorithm to extend $\mathcal{B}$ to form a tight frame for $\mathbb{C}^3$.

Algorithm 2
Input basis $\mathcal{B} = \{(a,b,c),(d,e,f),(g,h,k)\}$
$= \{(a_1+ia_2, b_1+ib_2, c_1+ic_2),(d_1+id_2, e_1+ie_2, f_1+if_2),(g_1+ig_2, h_1+ih_2, k_1+ik_2)\}$
If $\mathcal{B}$ is a tight frame then stop.
Print the frame $\mathcal{B}$ is tight.
Else
Let $\mathcal{B}' = \{(a_1,a_2,b_1,b_2,c_1,c_2),(d_1,d_2,e_1,e_2,f_1,f_2),(g_1,g_2,h_1,h_2,k_1,k_2)\}$

Include vectors
$\{(-a_1,0,b_1,0,0,0),(-a_1,0,0,b_2,0,0),(-a_1,0,0,0,c_1,0),(-a_1,0,0,0,0,c_2),$
$(0,-a_2,b_1,0,0,0),(0,-a_2,0,b_2,0,0),(0,-a_2,0,0,c_1,0),(0,-a_2,0,0,0,c_2),$
$(0,0,-b_1,0,c_1,0),(0,0,-b_1,0,0,c_2),(0,0,0,-b_2,c_1,0),(0,0,0,-b_2,0,c_2)\}$ into $\mathcal{B}'$.
Include vectors
$\{(-d_1,0,e_1,0,0,0),(-d_1,0,0,e_2,0,0),(-d_1,0,0,0,f_1,0),(-d_1,0,0,0,0,f_2),$
$(0,-d_2,e_1,0,0,0),(0,-d_2,0,e_2,0,0),(0,-d_2,0,0,f_1,0),(0,-d_2,0,0,0,f_2),$
$(0,0,-e_1,0,f_1,0),(0,0,-e_1,0,0,f_2),(0,0,0,-e_2,f_1,0),(0,0,0,-e_2,0,f_2)\}$ into $\mathcal{B}'$.
Include vectors
$\{(-g_1,0,h_1,0,0,0),(-g_1,0,0,h_2,0,0),(-g_1,0,0,0,k_1,0),(-g_1,0,0,0,0,k_2),$
$(0,-g_2,h_1,0,0,0),(0,-g_2,0,h_2,0,0),(0,-g_2,0,0,k_1,0),(0,-g_2,0,0,0,k_2),$
$(0,0,-h_1,0,k_1,0),(0,0,-h_1,0,0,k_2),(0,0,0,-h_2,k_1,0),(0,0,0,-h_2,0,k_2)\}$ into $\mathcal{B}'$.
Compute
$M=\max\{5(a_1^2+a_2^2+d_1^2+d_2^2+g_1^2+g_2^2),5(b_1^2+b_2^2+e_1^2+e_2^2+h_1^2+h_2^2),$
$5(c_1^2+c_2^2+f_1^2+f_2^2+k_1^2+k_2^2)\}$
Denote $M_1=5(a_1^2+a_2^2+d_1^2+d_2^2+g_1^2+g_2^2)$, $M_2=5(b_1^2+b_2^2+e_1^2+e_2^2+h_1^2+h_2^2)$
and $M_3=5(c_1^2+c_2^2+f_1^2+f_2^2+k_1^2+k_2^2)$.
Include vectors $\sqrt{M-M_1}(1,0,0)$, $\sqrt{M-M_2}(0,1,0)$ and $\sqrt{M-M_3}(0,0,1)$ into $\mathcal{B}'$.
The new set of vectors
$\mathcal{B}'=\{(a_1+ia_2,b_1+ib_2,c_1+ic_2),(-a_1,b_1,0),(-a_1,ib_2,0),(-a_1,0,c_1),(-a_1,0,ic_2),$
$(-ia_2,b_1,0),(-ia_2,ib_2,0),(-ia_2,0,c_1),(-ia_2,0,ic_2),(0,-b_1,c_1),(0,-b_1,ic_2),$
$(0,-ib_2,c_1),(0,-ib_2,ic_2),(d_1+id_2,e_1+ie_2,f_1+if_2),(-d_1,e_1,0),(-d_1,ie_2,0),$
$(-d_1,0,f_1),(-d_1,0,if_2),(-id_2,e_1,0),(-id_2,ie_2,0),(-id_2,0,f_1),(-id_2,0,if_2),$
$(0,-e_1,f_1),(0,-e_1,if_2),(0,-ie_2,f_1),(0,-ie_2,if_2),(g_1+ig_2,h_1+ih_2,k_1+ik_2),$
$(-g_1,h_1,0),(-g_1,ih_2,0),(-g_1,0,k_1),(-g_1,0,ik_2),(-ig_2,h_1,0),(-ig_2,ih_2,0),$
$(-ig_2,0,k_1),(-ig_2,0,ik_2),(0,-h_1,k_1),(0,-h_1,ik_2),(0,-ih_2,k_1),(0,-ih_2,ik_2),$
$(\sqrt{M-M_1},0,0),(0,\sqrt{M-M_2},0),(0,0,\sqrt{M-M_3})\}$ forms a tight frame.
Remove the zero vector from $\mathcal{B}'$
The bound for the tight frame $\mathcal{B}'$ is M.

***Example* 3.2**

Input basis	Output tight frame	Bound
$\{(1+i,1-i,i),$ $(3+i,2i,1),\ (1,i,-i)\}$	$\{(1+i,1-i,i),(-1,1,0),(-1,-i,0),$ $(-1,0,0),(-1,0,i),(-i,1,0),(-i,-i,0),$ $(-i,0,0),(-i,0,i),(0,-1,0)(0,-1,i),$ $(0,i,0),(0,i,i),(3+i,2i,1),(-3,0,0)$ $(-3,2i,0),(-3,0,1),(-3,0,0),(-i,0,0)$ $(-i,2i,0),(-i,0,1),(-i,0,0),(0,0,1),$ $(0,-2i,1),(0,-2i,0),(1,i,-i),(-1,0,0),$ $(-1,i,0),(-1,0,0),(-1,0,-i),(0,i,0),$ $(0,0,-i),(0,0,-i),(0,-i,0),(0,-i,-i)$ $(0,\sqrt{30},0),(0,0,\sqrt{50})\ \}$	65

3.2 *Construction of tight frames in* $\mathbb{C}^n$

Now, we generalize our algorithm for $\mathbb{C}^n$. Given a basis
$\{(x_{11}+iy_{11},x_{12}+iy_{12},..,x_{1n}+iy_{1n}),(x_{21}+iy_{21},x_{22}+iy_{22},...,x_{2n}+iy_{2n}),....,(x_{n1}+iy_{n1},x_{n2}+iy_{n2},...,x_{nn}+iy_{nn})\}$ in $\mathbb{C}^n$. We extend it to form a tight frame.

Algorithm 3
Input basis $\mathcal{B} = \Big\{(x_{11}+iy_{11}, x_{12}+iy_{12}, ..., x_{1n}+iy_{1n}), (x_{21}+iy_{21}, x_{22}+iy_{22}, ..., x_{2n}+iy_{2n}), ..., (x_{n1}+iy_{n1}, x_{n2}+iy_{n2}, ..., x_{nn}+iy_{nn})\Big\}$.
If $\mathcal{B}$ is a tight frame then stop. Print the frame $\mathcal{B}$ is tight.
Else,
let $\mathcal{B}' = \Big\{(x_{11}, y_{11}, x_{12}, y_{12}, ..., x_{1n}, y_{1n}), (x_{21}, y_{21}, x_{22}, y_{22}, ..., x_{2n}, y_{2n}), ..., (x_{n1}, y_{n1}, x_{n2}, y_{n2}, ..., x_{nn}, y_{nn})\Big\}$.
Include vectors

$$\{(-x_{11}, 0, x_{12}, 0, ..), (-x_{11}, 0, 0, y_{12}, 0, ..), (-x_{11}, 0, 0, 0, x_{13}, 0, ..), .., (-x_{11}, 0, .., 0, y_{1n}),$$
$$(0, -y_{11}, x_{12}, 0, ..), (0, -y_{11}, 0, y_{12}, 0, ..), (0, -y_{11}, 0, 0, x_{13}, 0, ..), .., (0, -y_{11}, 0, .., 0, y_{1n}),$$
$$(0, 0, -x_{12}, 0, x_{13}, 0, ...), (0, 0, -x_{12}, 0, 0, y_{13}, 0, ...), ..., (0, 0, -x_{12}, 0, 0, ..., y_{1n}),$$
$$..............................., (0, 0, ..., -y_{1n-1}, x_{1n}, 0), (0, 0, ..., -y_{1n-1}, 0, y_{1n})\} \text{ into } \mathcal{B}'.$$

Include vectors

$$\{(-x_{21}, 0, x_{22}, 0, ..), (-x_{21}, 0, 0, y_{22}, 0, ..), (-x_{21}, 0, 0, 0, x_{23}, 0, ..), .., (-x_{21}, 0, .., 0, y_{2n}),$$
$$(0, -y_{21}, x_{22}, 0, ..), (0, -y_{21}, 0, y_{22}, 0, ..), (0, -y_{21}, 0, 0, x_{23}, 0, ..), .., (0, -y_{21}, 0, .., 0, y_{2n}),$$
$$(0, 0, -x_{22}, 0, x_{23}, 0, ..), (0, 0, -x_{22}, 0, 0, y_{23}, 0, ..), ..., (0, 0, -x_{22}, 0, 0, ..., y_{2n}),$$
$$..., (0, 0, ..., -y_{2n-1}, x_{2n}, 0), (0, 0, ..., -y_{2n-1}, 0, y_{2n}),$$
$$..,$$
$$..,$$
$$(-x_{n1}, 0, x_{n2}, 0, ..), (-x_{n1}, 0, 0, y_{n2}, 0, ..), (-x_{n1}, 0, 0, 0, x_{n3}, 0, ..), .., (-x_{n1}, 0, .., 0, y_{nn}),$$
$$(0, -y_{n1}, x_{n2}, 0, .), (0, -y_{n1}, 0, y_{n2}, 0, .), (0, -y_{n1}, 0, 0, x_{n3}, 0, .), .., (0, -y_{n1}, 0, .., 0, y_{nn}),$$
$$(0, 0, -x_{n2}, 0, x_{n3}, 0, ..), (0, 0, -x_{n2}, 0, 0, y_{n3}, 0, ..), .., (0, 0, -x_{n2}, 0, 0, .., y_{nn}),$$
$$................................, (0, 0, ..., -y_{nn-1}, x_{nn}, 0), (0, 0, ..., -y_{nn-1}, 0, y_{nn})\Big\} \text{ into } \mathcal{B}'.$$

Compute
$M = \max\Big\{(2n-1)(x_{11}^2 + y_{11}^2 + x_{21}^2 + y_{21}^2 + ... + x_{n1}^2 + y_{n1}^2), (2n-1)(x_{12}^2 + y_{12}^2 + x_{22}^2 + y_{22}^2 + ... + x_{n2}^2 + y_{n2}^2), ..., (2n-1)(x_{1n}^2 + y_{1n}^2 + x_{2n}^2 + y_{2n}^2 + ... + x_{nn}^2 + y_{nn}^2)\Big\}$.
Denote
$M_1 = (2n-1)(x_{11}^2 + y_{11}^2 + x_{21}^2 + y_{21}^2 + ... + x_{n1}^2 + y_{n1}^2)$,
$M_2 = (2n-1)(x_{12}^2 + y_{12}^2 + x_{22}^2 + y_{22}^2 + ... + x_{n2}^2 + y_{n2}^2)$,
....,
$M_n = (2n-1)(x_{1n}^2 + y_{1n}^2 + x_{2n}^2 + y_{2n}^2 + ... + x_{nn}^2 + y_{nn}^2)$.
Include vectors $\sqrt{M-M_1}(1,0,0,..)$, $\sqrt{M-M_2}(0,1,0,..)$,...., $\sqrt{M-M_n}(0,0,..,1)$ into $\mathcal{B}'$.

The new set of vectors
$\mathcal{B}' = \Big\{(x_{11}+iy_{11}, x_{12}+iy_{12}, .., x_{1n}+iy_{1n}), (x_{21}+iy_{21}, x_{22}+iy_{22}, .., x_{2n}+iy_{2n}), .., (x_{n1}+iy_{n1}, x_{n2}+iy_{n2}, .., x_{nn}+iy_{nn}),$
$(-x_{11}, x_{12}, 0, 0, ..), (-x_{11}, iy_{12}, 0, 0, ..), (-x_{11}, 0, x_{13}, 0, ..), (-x_{11}, 0, iy_{13}), ..,$
$(-x_{11}, 0, 0, .., iy_{1n}), (-iy_{11}, x_{12}, 0, ..), (-iy_{11}, iy_{12}, 0, ..), (-iy_{11}, 0, x_{13}, 0, ..), ..,$

$(-iy_{11}, 0, .., 0, iy_{1n}), (0, -x_{12}, x_{13}, 0, ..), (0, -x_{12}, 0, iy_{13}, 0, ..), .., (0, -x_{12}, 0, .., 0, iy_{1n}),$
$..., (0, 0, ..., -iy_{1n-1}, x_{1n}), (0, 0, ..., -iy_{1n-1}, iy_{1n}),$
$(-x_{21}, x_{22}, 0, ...), (-x_{21}, iy_{22}, 0, ...), (-x_{21}, 0, x_{23}, 0, ...), ..., (-x_{21}, 0, .., 0, iy_{2n}),$
$(-iy_{21}, x_{22}, 0, ..), (-iy_{21}, iy_{22}, 0, ..), (-iy_{21}, 0, x_{23}, 0, ..), ..., (-iy_{21}, 0, .., 0, iy_{2n}),$
$(0, -x_{22}, x_{23}, 0, ...), (0, -x_{22}, iy_{23}, 0, ...), ..., (0, -x_{22}, 0, 0, ..., iy_{2n}),$
$..., (0, 0, ..., -iy_{2n-1}, x_{2n}), (0, 0, ..., -iy_{2n-1}, iy_{2n}),$
$...,$
$...,$
$(-x_{n1}, x_{n2}, 0, ...), (-x_{n1}, iy_{n2}, 0, ...), (-x_{n1}, 0, x_{n3}, 0, ...), ..., (-x_{n1}, 0, .., 0, iy_{nn}),$
$(-iy_{n1}, x_{n2}, 0, ..), (-iy_{n1}, iy_{n2}, 0, ..), (-iy_{n1}, 0, x_{n3}, 0, ..), ..., (-iy_{n1}, 0, .., 0, iy_{nn}),$
$(0, -x_{n2}, x_{n3}, 0, ...), (0, -x_{n2}, iy_{n3}, 0, ...), ..., (0, -x_{n2}, 0, 0, ..., iy_{nn}),$
$.., (0, 0, ..., -iy_{nn-1}, x_{nn}), (0, 0, ..., -iy_{nn-1}, iy_{nn}),$
$(\sqrt{M-M_1}, 0, 0, ...), (0, \sqrt{M-M_2}, 0, ..), ..., (0, 0, ..., \sqrt{M-M_n})\Big\}$ forms a tight frame.
Remove the zero vector from $\mathcal{B}'$.
The bound for the tight frame $\mathcal{B}'$ is M.

Given any basis in $\mathbb{C}^n$, our algorithm extends it and provides us a tight frame without much computation. In this process more number of vectors are added into the given system. So it seems unwanted. But, tight frames are easy to handle and more applicable in nature. Also the new tight frame compensates for erasures. Moreover, the additional vectors are very sparse with at most two nonzero components. Therefore, it gives justice to the process of adding some more vectors into the given system.

Theorem 3.3 Let $\mathcal{B} = \Big\{(x_{11} + iy_{11}, x_{12} + iy_{12}, ..., x_{1n} + iy_{1n}), (x_{21} + iy_{21}, x_{22} + iy_{22}, ..., x_{2n} + iy_{2n}), ..., (x_{n1} + iy_{n1}, x_{n2} + iy_{n2}, ..., x_{nn} + iy_{nn})\Big\}$ be a given basis in $\mathbb{C}^n$. Suppose we extend it by using Algorithm *3* to form a tight frame $\mathcal{B}'$. Then $\max|\mathcal{B}'| = (2n^3 - 2n^2 + 2n - 1)$, where $\max|\mathcal{B}'|$ indicates maximum number of nonzero vectors in $\mathcal{B}'$. Moreover, if

$$\begin{aligned} x_{11}^2 + y_{11}^2 + x_{21}^2 + y_{21}^2 + ... + x_{n1}^2 + y_{n1}^2 &= x_{12}^2 + y_{12}^2 + x_{22}^2 + y_{22}^2 + ... + x_{n2}^2 + y_{n2}^2 \\ &= ..... = x_{1n}^2 + y_{1n}^2 + x_{2n}^2 + y_{2n}^2 + ... + x_{nn}^2 + y_{nn}^2, \end{aligned}$$

then $\max|\mathcal{B}'| = (2n^3 - 2n^2 + n)$.

Proof. We can estimate this by using simple counting. Initially, we include the vectors from the given basis, that gives rise to n vectors in $\mathcal{B}'$. Then in the second step of the algorithm we include $n\Big\{\binom{2n}{2} - n\Big\}$ number of vectors into $\mathcal{B}'$. In the final step of the algorithm we include the vectors

$$\sqrt{M-M_1}(1, 0, ..., 0), \sqrt{M-M_2}(0, 1, 0, ..., 0), ..., \sqrt{M-M_n}(0, 0, ..., 1)$$

into $\mathcal{B}'$, and these are another n vectors. At least one of these vectors is a zero vector and we remove the zero vectors. That gives at most $n-1$ nonzero vectors to be included in $\mathcal{B}'$. Therefore, we have

$$\begin{aligned} \max|\mathcal{B}'| &= n + n\Big\{\binom{2n}{2} - n\Big\} + n - 1 \\ &= 2n - 1 + n\Big\{\frac{(2n)!}{2!(2n-2)!} - n\Big\} = 2n^3 - 2n^2 + 2n - 1. \end{aligned}$$

In the second part of the theorem, if

$$\begin{aligned} x_{11}^2 + y_{11}^2 + x_{21}^2 + y_{21}^2 + ... + x_{n1}^2 + y_{n1}^2 &= x_{12}^2 + y_{12}^2 + x_{22}^2 + y_{22}^2 + ... + x_{n2}^2 + y_{n2}^2 \\ &= ..... = x_{1n}^2 + y_{1n}^2 + x_{2n}^2 + y_{2n}^2 + ... + x_{nn}^2 + y_{nn}^2, \end{aligned}$$

then $M_1 = M_2 = ... = M_n = M$. So, the vectors

$$\sqrt{M - M_1}(1, 0, ..., 0), \sqrt{M - M_2}(0, 1, 0, ..., 0), ..., \sqrt{M - M_n}(0, 0, ..., 1)$$

are all zero vectors, and we remove them all. Therefore,

$$\max |\mathcal{B}'| = n + n\Big\{ \binom{2n}{2} - n \Big\} = 2n^3 - 2n^2 + n.$$

■

As direct consequences of the above theorem, we have the following corollaries.

Corollary 3.4 Given a basis $\mathcal{B}$ in $\mathbb{C}^2$, if we extend it to form a tight frame $\mathcal{B}'$ by using Algorithm *1*, then the maximum number of nonzero elements in $\mathcal{B}'$ is 11.

Corollary 3.5 Given a basis $\mathcal{B}$ in $\mathbb{C}^3$, if we extend it to form a tight frame $\mathcal{B}'$ by using Algorithm *2*, then the maximum number of nonzero elements in $\mathcal{B}'$ is 41.

4. Application: Reduction of numerical range of frame operator

Let T be a bounded linear operator in a Hilbert space H. Then the numerical range of T denoted by $N_R(T)$ is defined as $N_R(T) = \{\langle Tx, x\rangle : \ x \in H, \ \|x\| = 1\}$.

Theorem 4.1 The numerical range of frame operator is either a singleton set or a closed bounded set lying between the frame bounds. Moreover, it is a convex set.

Proof. Let $\{\phi_i\}_{i\in I}$ be a non-tight frame for H with lower and upper frame bounds A and B, respectively. The frame operator for $\{\phi_i\}_{i\in I}$ is defined as $Sx = \sum_{i\in I}\langle x, \phi_i\rangle\phi_i$. Also we have $\langle Sx, x\rangle = \sum_{i\in I} |\langle x, \phi_i\rangle|^2$ for all $x \in H$. Therefore, $A\|x\|^2 \leqslant \langle Sx, x\rangle \leqslant B\|x\|^2$ for all $x \in H$. As a result we have

$$N_R(S) = \{\langle Sx, x\rangle : \ x \in H, \ \|x\| = 1\} \subseteq [A, B].$$

If $\{\phi_i\}_{i\in I}$ is a tight frame with bound A, then $\langle Sx, x\rangle = \sum_{i\in I} |\langle x, \phi_i\rangle|^2 = A\|x\|^2$ for all $x \in H$. Therefore, $N_R(S) = \{A\}$, a singleton set. In particular, if $\{\phi_i\}_{i\in I}$ is a Parseval frame then $N_R(S) = \{1\}$. Moreover, since S is a bounded linear operator, by Toeplitz Housdorff theorem, $N_R(S)$ is a convex set. In addition, if H is finite dimensional then $N_R(S)$ is compact. ■

Let $\{\phi_i\}_{i\in I}$ be a non-tight frame for H with lower and upper frame bounds A and B, respectively. In this case $N_R(S) \subseteq [A, B]$. If we extend $\{\phi_i\}_{i\in I}$ to a tight frame with bound M, then the numerical range of the frame operator is reduced to a singleton set $\{M\}$. The singleton set $\{M\}$ may not belong to $[A, B]$.

5. Conclusion

We have developed simple algorithms to extend any basis to form a tight frame. The algorithms are simple to implement, they do not require the computation of eigenvalues and eigenvectors. Although our algorithms require more number of vectors to be added into the basis in comparison to the other existing methods, but by this we can avoid the computation of eigenvalues and eigenvectors. Eigenvalues and eigenvectors are not easy to compute for higher dimensional cases. So the methods developed in this paper can be more useful in applications. Also the tight frames constructed by our methods compensates for multiple erasures.

Acknowledgments

The author is thankful to the anonymous reviewer for the valuable comments which improved the presentation of the paper.

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