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Fixed points of Ćirić and Caristi-type multivalued contractions

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Abstract. The aim of this paper is to introduce the concept of multi-valued contraction that combine a renowned Ćirić-type contraction and Caristi-type contractions in the framework of metric spaces. The existence of fixed points for such contractions equipped with some suitable hypotheses are proved and some analogues of the fixed point theorems presented herein are deduced as corollaries. Moreover, an example is given to illustrate the validity of obtained main result.

Keywords: Fixed point, Ćirić contraction, Caristi contraction, multivalued mapping, metric space.

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1. Introduction and preliminaries

The concept of multivalued contraction is introduced by Nadler [16] and the corresponding fixed point result was proposed therein. One of the initial nonlinear forms of the contraction mapping principle was given by Jaggi [14] and Dass-Gupta [12], in which the inequality is satisfied by the contraction contains rational terms. These contractions are known as rational contractions. Their considerations occupy a large area of fixed point theory. Some of the fixed point results of rational type contractions are in [1, 3, 4, 6, 10, 13]. Also, Chen [7] introduced bilateral contractions which merges two significant approaches in fixed point theory: Caristi-type and Jaggi-type contractions. An inherent property of the existing fixed point results via the bilateral contraction is that the fixed point of the concerned mapping is not necessarily unique; for example, see

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[7, Example 2]. This restriction is an indication that fixed point theorems using bilateral notions are more suitable for fixed point theory of point-to-set-valued maps. Recently, the combination of two contraction type has become more popular. Du and Karapinar [13] firstly merged Banach contraction into Caristi and established an interesting result. Also, Karapinar et al. [15] merged Ćirić-type contraction into Caristi theorem and obtained a new fixed point theorem in metric space.

In this paper, by combining a Caristi-type contractions and Ćirić-type contraction, a multivalued Caristi-Ćirić type contraction is defined and fixed point of such mapping is established in the frameworks of complete metric space. We recall some basic definitions and preliminaries that will be needed in this paper.
Let (X, d) be a metric space, $CB(X)$ be a collection of non-empty closed and bounded subset of X, and $K(X)$ be a set of non-empty compact subset of X. For $x \in X$ and $A, B \in CB(X)$, the Hausdorff metric H on $CB(X)$ induced by the metric d is given by $H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}$, where $D(a, B) = \inf_{b\in B}\{d(a, b)\}$. It is known that H is a metric on $CB(X)$ and H is called the Hausdorff metric or Pompeiu-Hausdorff metric induced by d. It is also known that $(CB(X), H)$ is a complete metric space whenever (X, d) is a complete metric space.

Definition 1.1 [11] Let (X, d) be a metric space and T be a self mapping of X. Then T is said to be a quasi-contraction if and only if there exists a number $0 \leqslant q < 1$ such that

$$d(Tx, Ty) \leqslant q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

holds for every $x, y \in X$.

Definition 1.2 [5] Let (X, d) be a metric space and T be a self mapping of X. Then T is said to be a Caristi contraction if there exists a lower semi-continuous function $\vartheta : X \longrightarrow \mathbb{R}_+$ such that $d(x, Tx) \leqslant \vartheta(x) - \vartheta(Tx)$ holds for all $x \in X$.

Definition 1.3 [2, Definition 2.2] Let (X, d) be a metric space and T be a self mapping of X. Then T is said to be a P-contractive mapping if

$$d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)|$$

for all $x, y \in X$ with $x \neq y$.

Lemma 1.4 [9] Let (X, d) be a metric space and $B \in K(X)$. The,n for every $x \in X$, there exists $y \in B$ such that $d(x, y) = D(x, B)$.

2. Main results

Motivated by the results of [8], we introduce the notion of Ćirić-Caristi type multivalued contraction and establish the corresponding fixed point theorem in the setting of metric spaces.

Definition 2.1 Let (X, d) be a metric space. A multivalued mapping $T : X \longrightarrow K(X)$ is called a Ćirić-Caristi type multivalued contraction if there is a non-increasing mapping $\vartheta : X \longrightarrow \mathbb{R}_+$ such that $D(x, Tx) > 0$ implies

$$H(Tx, Ty) \leqslant [\vartheta(x) - \vartheta(y)]M(x, y) \tag{1}$$

for all distinct $x, y \in X$ with $x \leqslant y$, where

$$M(x,y) = \max\{d(x,y), [D(x,Tx) + D(y,Ty)], [D(x,Ty) + D(y,Tx)]\}.$$

Theorem 2.2 Let (X,d) be a complete metric space and $T : X \longrightarrow K(X)$ be a multivalued mapping. Moreover, suppose that the following conditions are satisfied:

A1: T is a Ćirić-Caristi type multivalued contraction;
A2: there exists $\lambda \in (0, \frac{1}{2})$ such that

$$\lambda = \sup_{x,y\in X} \{\vartheta(x) - \vartheta(y) : d(x,y) > 0\}.$$

Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_0 \in X$ be arbitrary. Then, by Lemma 1.4, there exists $x_1 \in Tx_0$ such that $d(x_0,x_1) = D(x_0,Tx_0)$. Again, by Lemma 1.4, there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) = D(x_1,Tx_1)$. Continuing in this manner, we construct a sequence (x_n) in X such that $x_{n+1} \in Tx_n$ and $d(x_n,x_{n+1}) = D(x_n,Tx_n)$ for all $n \geqslant 0$. Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. Hence, we presume that $x_n \neq x_{n+1}$ for all n. By Lemma 1.4, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned}
d(x_1,x_2) &\leqslant H(Tx_0,Tx_1)\\
&\leqslant [\vartheta(x_0) - \vartheta(x_1)]M(x_0,x_1)\\
&= [\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0,x_1), [D(x_0,Tx_0) + D(x_1,Tx_1)],\\
&\quad [D(x_0,Tx_1) + D(x_1,Tx_0)]\}\\
&\leqslant [\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0,x_1), [d(x_0,x_1) + d(x_1,x_2)], [d(x_0,x_2) + d(x_1,x_1)]\}\\
&= [\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0,x_1), [d(x_0,x_1) + d(x_1,x_2)], d(x_0,x_2)\}\\
&\leqslant [\vartheta(x_0) - \vartheta(x_1)]\max\{d(x_0,x_1), [d(x_0,x_1) + d(x_1,x_2)], [d(x_0,x_1) + d(x_1,x_2)]\}\\
&\leqslant \lambda[d(x_0,x_1) + d(x_1,x_2)],
\end{aligned}$$

from which we have $d(x_1,x_2) \leqslant \frac{\lambda}{1-\lambda} d(x_0,x_1)$. Again, by Lemma 1.4, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned}
d(x_2,x_3) &\leqslant H(Tx_1,Tx_2)\\
&\leqslant [\vartheta(x_1) - \vartheta(x_2)]M(x_1,x_2)\\
&= [\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), [D(x_1,Tx_1) + D(x_2,Tx_2)],\\
&\quad [D(x_1,Tx_2) + D(x_2,Tx_1)]\}\\
&\leqslant [\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), [d(x_1,x_2) + d(x_2,x_3)], [d(x_1,x_3) + d(x_2,x_2)]\}\\
&= [\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), [d(x_1,x_2) + d(x_2,x_3)], d(x_1,x_3)\}\\
&\leqslant [\vartheta(x_1) - \vartheta(x_2)]\max\{d(x_1,x_2), [d(x_1,x_2) + d(x_2,x_3)], [d(x_1,x_2) + d(x_2,x_3)]\}\\
&\leqslant \lambda[d(x_1,x_2) + d(x_2,x_3)].
\end{aligned}$$

That is,

$$d(x_2,x_3) \leqslant (\frac{\lambda}{1-\lambda})d(x_1,x_2) \leqslant (\frac{\lambda}{1-\lambda})^2 d(x_0,x_1).$$

Taking $\beta = \frac{\lambda}{1-\lambda}$ and continuing in this manner inductively, we have

$$d(x_n, x_{n+1}) \leqslant \beta^n d(x_0, x_1).$$

Now, we show that the sequence (x_n) in X is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $n \leqslant m$. Then

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_{m+1}) \\ &\leqslant \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) + \cdots + \beta^{m-1} d(x_0, x_1) \\ &\leqslant (\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}) d(x_0, x_1) \\ &\leqslant (\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}) d(x_0, x_1) \longrightarrow 0 \end{aligned}$$

as $n, m \longrightarrow 0$. Hence, (x_n) in X is a Cauchy sequence. The completeness of (X, d) guarantees the existence of $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, to see that $u \in Tu$, we apply

$$\begin{aligned} D(u, Tu) &\leqslant d(u, x_n) + D(x_n, Tu) \\ &\leqslant d(u, x_n) + H(Tx_{n-1}, Tu) \\ &\leqslant d(u, x_n) + [\vartheta(x_{n-1}) - \vartheta(u)] M(x_{n-1}, u) \\ &= d(u, x_n) + \lambda \max\{d(x_{n-1}, u), [D(x_{n-1}, Tx_{n-1}) + D(u, Tu)], \\ &\qquad [D(x_{n-1}, Tu) + D(u, Tx_{n-1})]\} \\ &\leqslant d(u, x_n) + \lambda \max\{d(x_{n-1}, u), [d(x_{n-1}, x_n) + D(u, Tu)], \\ &\qquad [D(x_{n-1}, Tu) + d(u, x_n)]\}. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality gives

$$D(u, Tu) \leqslant \lambda \max\{0, D(u, Tu), D(u, Tu)\} \leqslant \lambda D(u, Tu);$$

that is, $(1 - \lambda) D(u, Tu) \leqslant 0$. This implies that $u \in Tu$. ■

In the next result, motivated by the concept of P-Contraction introduced by [2], a notion of multivalued P-Contraction type is introduced in the framework of metric space.

Definition 2.3 Let (X, d) be a metric space. A multivalued mapping $T : X \longrightarrow K(X)$ is called a multivalued P-contraction type if there is a non-increasing mapping $\vartheta : X \longrightarrow \mathbb{R}_+$ such that $D(x, Tx) > 0$ implies

$$H(Tx, Ty) \leqslant [\vartheta(x) - \vartheta(y)] K(x, y) \tag{2}$$

for all distinct $x, y \in X$ with $x \leqslant y$, where

$$K(x, y) = \max\{d(x, y), d(x, y) + |D(x, Tx) - D(y, Ty)|\}.$$

Theorem 2.4 Let (X, d) be a metric space, $T : X \longrightarrow K(X)$ be a multivalued mapping and $f : X \longrightarrow 2^{\mathbb{R}}$ be a mapping defined by $f(x) = \{d(x, Tx)\}$. Moreover, suppose that the following conditions are satisfied:

B1: T is a multivalued P-contraction type;
B2: there exist $x_0, x_1 \in X$ such that $f(x_0) \subseteq f(x_1)$;
B3: there exists $\lambda \in (0, \frac{1}{2})$ such that

$$\lambda = \sup_{x,y \in X} \{\vartheta(x) - \vartheta(y) : d(x, y) > 0\}.$$

Then there exists $u \in X$ such that $u \in Tu$.

Proof. Following the proof of Theorem 2.2, we construct a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, for $n = 1, 2, \cdots$. So, we presume that $x_n \neq x_{n+1}$ for all n. By Lemma 1.4, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leqslant H(Tx_0, Tx_1) \\ &\leqslant [\vartheta(x_0) - \vartheta(x_1)]K(x_0, x_1) \\ &= [\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), d(x_0, x_1) + |D(x_0, Tx_0) - D(x_1, Tx_1)|\} \\ &\leqslant [\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), d(x_0, x_1) + |d(x_0, x_1) - d(x_1, x_2)|\}. \end{aligned}$$

By using condition B2, we get

$$\begin{aligned} d(x_1, x_2) &\leqslant [\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), d(x_0, x_1) - d(x_0, x_1) + d(x_1, x_2)\} \\ &\leqslant [\vartheta(x_0) - \vartheta(x_1)] \max\{d(x_0, x_1), d(x_1, x_2)\} \\ &\leqslant \lambda \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

Suppose that $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$. Then, $d(x_1, x_2) \leqslant \lambda d(x_1, x_2)$. The last inequality yields $d(x_1, x_2) \leqslant 0$, a contradiction. Hence, we must have $d(x_0, x_1) > d(x_1, x_2)$. Again by Lemma 1.4, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leqslant H(Tx_1, Tx_2) \\ &\leqslant [\vartheta(x_1) - \vartheta(x_2)]K(x_1, x_2) \\ &= [\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_1, x_2) + |D(x_1, Tx_1) - D(x_2, Tx_2)|\} \\ &\leqslant [\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_1, x_2) + |d(x_1, x_2) - d(x_2, x_3)|\}. \end{aligned}$$

By using condition B2, we get

$$\begin{aligned} d(x_2, x_3) &\leqslant [\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_1, x_2) - d(x_1, x_2) + d(x_2, x_3)\} \\ &\leqslant [\vartheta(x_1) - \vartheta(x_2)] \max\{d(x_1, x_2), d(x_2, x_3)\} \\ &\leqslant \lambda d(x_1, x_2) \\ &\leqslant \lambda[\lambda d(x_0, x_1)] \\ &= \lambda^2 d(x_0, x_1). \end{aligned}$$

Continuing in this manner inductively, we have

$$d(x_n, x_{n+1}) \leqslant \lambda^n d(x_0, x_1).$$

Now, we show that (x_n) in X is a Cauchy sequence. Let $m, n \in N$ with $n \leqslant m$. Then

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_{m+1}) \\ &\leqslant \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &\leqslant (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x_0, x_1) \\ &\leqslant (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x_0, x_1) \longrightarrow 0 \end{aligned}$$

as $n, m \longrightarrow 0$. Hence, (x_n) in X is a Cauchy sequence. The completeness of (X, d) guarantees $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, we have

$$\begin{aligned} D(u, Tu) &\leqslant d(u, x_n) + D(x_n, Tu) \\ &\leqslant d(u, x_n) + H(Tx_{n-1}, Tu) \\ &\leqslant d(u, x_n) + [\vartheta(x_{n-1}) - \vartheta(u)] K(x_{n-1}, u) \\ &= d(u, x_n) + \lambda \max\{d(x_{n-1}, u), d(x_{n-1}, u) + |D(x_{n-1}, Tx_{n-1}) - D(u, Tu)|\} \\ &= d(u, x_n) + \lambda \max\{d(x_{n-1}, u), d(x_{n-1}, u) + |d(x_{n-1}, x_n) - D(u, Tu)|\} \\ &\leqslant d(u, x_n) + \lambda \max\{d(x_{n-1}, u), d(x_{n-1}, u) - d(x_{n-1}, x_n) + D(u, Tu)\}. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality gives

$$D(u, Tu) \leqslant \lambda \max\{0, D(u, Tu)\} \leqslant \lambda D(u, Tu);$$

that is, $D(u, Tu) \leqslant 0$. This implies that $u \in Tu$. ■

Corollary 2.5 [8, Theorem 4] Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a single valued mapping. Assume that there exists $\vartheta : X \longrightarrow [0, \infty)$ with $d(x, fx) > 0$ such that

$$d(fx, fy) \leqslant [\vartheta(x) - \vartheta(y)] M(x, y)$$

for all distinct $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), [d(x, fx) + d(y, fy)], [d(x, fy) + d(y, fx)]\}.$$

Then f has a fixed point.

Proof. Consider a multivalued mapping $T : X \longrightarrow K(X)$ defined by $Tx = \{fx\}$ for all $x \in X$. We see that $D(x, fx) = d(x, fx) > 0$ implies $H(fx, fy) = d(fx, fy) \leqslant [\vartheta(x) - \vartheta(y)] M(x, y)$. It follows that the assumption of Theorem 2.2 coincides with that of Corollary 2.5. Hence, there exists $u \in X$ such that $u \in Tu = \{fu\}$; that is, $u = fu$. ■

Corollary 2.6 [16] Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a multi-valued mapping. Suppose that there exists $k \in (0, 1)$ such that

$$H(Tx, Ty) \leqslant k d(x, y) \tag{3}$$

for every $x, y \in X$. Then T has a fixed point.

Proof. Take $\vartheta(x) - \vartheta(y) = k$ for all $x, y \in X$. Then, (3) becomes

$$\begin{aligned} H(Tx, Ty) &\leqslant kd(x, y) \\ &\leqslant [\vartheta(x) - \vartheta(y)] \max\{d(x, y), [d(x, Tx) + d(y, Ty)], [d(x, Ty) + d(y, Tx)]\}. \end{aligned}$$

Consequently, all the conditions of Theorem 2.2 are satisfied. Hence, T has a fixed point in X. ∎

Corollary 2.7 [13, Theorem 1] Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a self-mapping. Suppose that there exists a function $\vartheta : X \longrightarrow \mathbb{R}_+$ with ϑ bounded from below such that $d(x, fx) > 0$ implies that $d(fx, fy) \leqslant [\vartheta(x) - \vartheta(y)]d(x, y)$ for all $x, y \in X$. Then f has a fixed point.

Corollary 2.8 [15, Theorem 4] Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a self-mapping. Suppose that there exists a function $\vartheta : X \longrightarrow \mathbb{R}_+$ with $d(x, fx) > 0$ implies that

$$d(fx, fy) \leqslant [\vartheta(x) - \vartheta(y)]N(x, y)$$

for all $x, y \in X$, where

$$N(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

Then f has a fixed point.

Corollary 2.9 [8, Theorem 4] Suppose that f is self mapping on complete metric space (X, d). If there is a function $\vartheta : X \longrightarrow \mathbb{R}_+$ with $d(x, fx) > 0$ such that

$$d(fx, fy) \leqslant [\vartheta(x) - \vartheta(y)]K(x, y)$$

for all $x, y \in X$, where

$$K(x, y) = \max\{d(x, y), [d(x, fx) + d(y, fy)], [d(x, fy) + d(y, fx)]\}.$$

Then f has a fixed point.

***Example* 2.10** Let $X = \{(2, 2), (2, 3), (4, 4)\}$ be equipped with the metric $d : X \times X \longrightarrow \mathbb{R}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

It is clear that (X, d) is a complete metric space. Define an order on X by $(x_1, x_2) \leqslant (y_1, y_2)$ iff $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$. Suppose that $T : X \longrightarrow K(X)$ is a multivalued mapping defined as follows:

$$Tx = \begin{cases} \{(2, 3), (2, 2)\} & \text{if } x = (2, 2), \\ \{(4, 4)\} & \text{if } x \neq (2, 2). \end{cases}$$

Let $\vartheta : X \longrightarrow \mathbb{R}$ be given as follows $\vartheta(2, 2) = 7$, and $\vartheta(2, 3) = 4$. It is clear that ϑ is non-increasing. Now, we examine the following cases:

Case I: for $x = (2,2)$, we have

$$D((2,2),T(2,2)) = \inf\{d((2,2),y) : y \in T(2,2)\} = d((2,2),(2,3),(2,2))$$
$$= d((2,2),(2,3)) = 1.$$

Case II: for $x = (2,3)$, we have

$$D((2,3),T(2,3)) = \inf\{d((2,3),y) : y \in T(2,3)\} = d((2,3),(4,4)) = \sqrt{5}.$$

Case III: for $x = (4,4)$, we have

$$D((4,4),T(4,4)) = \inf\{d((4,4),y) : y \in T(4,4)\} = d((4,4),(4,4)) = 0.$$

Now, for $x \in X$ with $D(x,Tx) > 0$ i.e $x \in \{(2,2),(2,3)\}$, we have
$H(T(2,2),T(2,3)) = H((2,3),(4,4)) = 3$, $\vartheta(2,2) - \vartheta(2,3) = 3$ and

$$\begin{aligned} M((2,2),(2,3)) &= \max\{d((2,2),(2,3)), [D((2,2),T(2,2)) + D((2,3),T(2,3))], \\ &\quad [D((2,2),T(2,3)) + D((2,3),T(2,2))]\} \\ &= \max\{1, [d((2,2),(2,3)) + d((2,3),(4,4))], \\ &\quad [d((2,2),(4,4)) + d((2,3),(2,3))]\} \\ &= \max\{1, [1+\sqrt{5}], [\sqrt{8}]\} \\ &= 1+\sqrt{5}. \end{aligned}$$

Hence,

$$H(T(2,2),T(2,3)) = 3 \leqslant 3.(1+\sqrt{5}) = [\vartheta(2,2) - \vartheta(T(2,2))]M((2,2),(2,3)).$$

Thus, for all $x, y \in X$ with $x \neq y$, $D(x,Tx) > 0$ and $D(y,Ty) > 0$ imply $H(Tx,Ty) \leqslant [\vartheta(x) - \vartheta(y)]M(x,y)$, where

$$M(x,y) = \max\{d(x,y), [D(x,Tx) + D(y,Ty)], [D(x,Ty) + D(y,Tx)]\}.$$

It follows that all the hypotheses of Theorem 2.2 are satisfied. We see that T has a fixed point. On the other hand, we demonstrate hereunder that Theorem 2.2 properly subsumes the main ideas of Nadler [16]. For that, take $x = a$ and $y = b$. Now, for all $k \in (0,1)$, consider the following cases:
Case I: For $a = (2,2)$ and $b = (2,3)$, we see that

$$d(T(2,2),T(2,3)) = d(2,3) = 1 > kd(2,3).$$

Case II: For $a = (2,2)$ and $b = (4,4)$, we see that

$$d(T(2,2),T(4,4)) = d(2,4) = 2 > kd(2,4).$$

Case III: For $a = (4,4)$ and $b = (2,3)$, we see that

$$d(T(4,4),T(2,3)) = d(4,3) = 1 > kd(4,3).$$

This shows that Example 2.10 doesn't satisfy the conditions of multivalued fixed point theorem due to Nadler [16].

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References

[1] M. Abbas, V. C Rajić, T. Nazir, S. Radenović, Common fixed point of mappings satisfying rational inequalities in ordered complex valued generalized metric spaces, Afri. Math. 26 (2015), 17-30.
[2] I. Altun, G. Durmaz, M. Olgun, P-contractive mappings on metric spaces, J. Nonl. Func. Anal. (2018), 2018:43.
[3] M. Arshad, E. Karapinar, J. Ahmad, Some unique fixed point theorems for rational contractions in partially ordered metric spaces, J. Ineq. Appl. (2013), 2013:248.
[4] I. Cabrera, J. Harjani, K. Sadarangani, A fixed point theorem for contractions of rational type in partially ordered metric spaces, Annali. Dell. Univ. Di Ferrara. 59 (2013), 251-258.
[5] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241-251.
[6] S. Chandok, B. S. Choudhury, N. Metiya, Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions, J. Egypt. Math. Soc. 23 (1) (2015), 95-101.
[7] C. M. Chen, Gh. Heidary Joonaghany, E. Karapinar, F. Khojasteh, On bilateral contractions, Mathematics. 7 (2019), 6:538.
[8] Z. Cheng, The generalization of Ćirić and Caristi type fixed point theorem, Wuhan Univ. J. Nat. Sci. 28 (1) (2023), 011-014.
[9] B. S. Choudhury, N. Metiya, C. Bandyopadhyay, P. Maity, Fixed points of multivalued mappings satisfying hybrid rational Pata-type inequalities, The Journal of Analysis. 27 (2019), 813-828.
[10] B. S. Choudhury, N. Metiya, T. Som, C. Bandyopadhyay, Multivalued fixed point results and stability of fixed point sets in metric spaces, Facta Univ. Ser. Math. Info. 30 (2015), 501-512.
[11] L. B. Ćirić, A generalization of Banach contraction principle, Proc. Am. Math. Soc. 45 (1974), 267-273.
[12] B. K. Dass, S. Gupta, An extension of Banach contraction principle through rational expression, Indian J. Pure Appl. Math. 6 (12) (1975), 1455-1458.
[13] W. S. Du, E. Karapinar, A note on Caristi-type cyclic maps: Related results and applications, J. Fixed Point Theory Appl. (2013), 2013:344.
[14] D. S. Jaggi, Some unique fixed point theorems, Indian J. Pure Appl. Math. 8 (2) (1977), 223-230.
[15] E. Karapinar, F. Khojasteh, W. Shatanawi, Revisiting Ćirić type contraction with Caristi approach, Symmetry. 11 (2019), 6:726.
[16] S. B. Nadler, Multi-valued contraction mappings, Paci. J. Math. 30 (2) (1969), 475-488.