Journal of Linear and Topological Algebra
Vol. 11*, No.* 04*,* 2022*,* 279- 292
DOR: 20.1001.1.22520201.2022.11.04.7.4
DOI: 10.30495/JLTA.2022.699627

Some new results concerning e-θ-open sets

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Received 24 November 2022; Accepted 29 December 2022.

Communicated by Ghasem Soleimani Rad

Abstract. The main purpose of this paper is to study the class of e-θ-open sets and explore some of their new properties. Also, we introduce and study some weak separation axioms by utilizing e-θ-open sets. In addition, we define the notion of e-θ-kernel and slightly e-θ-R_0 spaces. Furthermore, we apply them to discuss some fundamental properties of the graph functions. We obtain not only some characterizations but also many new results.

Keywords: e-open set, e-θ-open set, e-θ-closed set, e-θ-closure.

2010 AMS Subject Classification: 54C08, 54C10, 54C05.

1. Introduction

One of the most important objects of topology is undoubtedly the notion of open sets. Numerous researchers in the field of topology is devoted to the study of various classes of open subsets of topological spaces. Recently, many new forms of this notion have been introduced and studied by many mathematicians. For instance, in 1966, Velicko introduced the notion of θ-open [13] set which is the stronger form of open sets in topology. After him, several new forms of θ-open [13] classes such as pre-θ-open [6], semi-θ-open [5], b-θ-open [10, 11], β-θ-open [2, 7], and e-θ-open [9] were defined and studied in the literature.

In this study, we continue to study the properties of the notion of e-θ-open set. We define a new class of subsets, called e-θ-D-set, via e-θ-open sets. Also, we introduce some

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.ctb.iau.ir

new separation axioms by means of e-θ-D-sets and e-θ-open sets and investigate some of their fundamental properties.

2. Preliminaries

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A in X, respectively. The family of all open subsets containing x of X is denoted by $O(X,x)$. A subset A is said to be regular open [12] (resp. regular closed [12]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The δ-interior [13] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-$int(A)$. The subset A of a space X is called δ-open [13] if $A = \delta\text{-}int(A)$, i.e., a set is δ-open if it is the union of some regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a subset A of a space X is called δ-closed [13] if $A = \delta\text{-}cl(A)$, where $\delta\text{-}cl(A) = \{x \in X | (\forall U \in O(X,x))(int(cl(U)) \cap A \neq \emptyset)\}$. The family of all δ-open (resp. δ-closed) sets in X is denoted by $\delta O(X)$ (resp. $\delta C(X)$).

A subset A of X is said to be e-open [4] (resp. b-open [1]) if $A \subseteq cl(\delta\text{-}int(A)) \cup int(\delta\text{-}cl(A))$ (resp. $A \subseteq cl(int(A)) \cup int(cl(A))$). The complement of an e-open (resp. b-open) set is called e-closed [4] (resp. b-closed [1]). A subset A of a space X is said to be e-regular [9] (resp. b-regular [10]) if it is both e-open (resp. b-open) and e-closed (resp. b-closed). The e-interior [4] of a subset A of X is the union of all e-open sets of X contained in A and is denoted by e-$int(A)$. The e-closure [4] of a subset A of X is the intersection of all e-closed sets of X containing A and is denoted by e-$cl(A)$. The family of all e-open (resp. e-closed, e-regular, b-open, b-closed, b-regular) subsets containing x of X is denoted by $eO(X,x)$ (resp. $eC(X,x), eR(X,x), BO(X,x), BC(X,x), BR(X,x)$).

A point x of X is called an e-θ-cluster [9] (resp. b-θ-cluster [10]) point of $A \subseteq X$ if $e\text{-}cl(U) \cap A = \emptyset$ (resp. $b\text{-}cl(U) \cap A = \emptyset$) for every $U \in eO(X,x)$ (resp. $U \in BO(X,x)$). The set of all e-θ-cluster (resp. b-θ-cluster) points of A is called the e-θ-closure (resp. b-θ-closure) of A and is denoted by $e\text{-}cl_\theta(A)$ (resp. $b\text{-}cl_\theta(A)$). A subset A is said to be e-θ-closed [9] (resp. b-θ-closed [10]) if and only if $A = e\text{-}cl_\theta(A)$ (resp. $A = b\text{-}cl_\theta(A)$). The complement of an e-θ-closed (resp. b-θ-closed) set is said to be e-θ-open [9] (resp. b-θ-open [10]). The family of all e-θ-closed (resp. e-θ-open, b-θ-closed, b-θ-open) subsets of X is denoted by $e\theta C(X)$ (resp. $e\theta O(X), B\theta C(X), B\theta O(X)$). The family of all e-θ-closed (resp. e-θ-open, b-θ-closed, b-θ-open) subsets containing x of X is denoted by $e\theta C(X,x)$ (resp. $e\theta O(X,x), B\theta C(X,x), B\theta O(X,x)$). Also, the family of all e-θ-open sets containing the subset F of X will be denoted by $e\theta O(X,F)$.

Theorem 2.1 [9] Let A be a subset of a topological space X. Then,
(a) $A \in eO(X)$ if and only if $e\text{-}cl(A) \in eR(X)$,
(b) $A \in eC(X)$ if and only if $e\text{-}int(A) \in eR(X)$.

Corollary 2.2 [9] Let A and $A_\alpha (\alpha \in \Lambda)$ be any subsets of a space X. Then the following properties hold:
(a) A is e-θ-open in X if and only if for each $x \in A$ there exists $U \in eR(X,x)$ such that $x \in U \subseteq A$,
(b) If A_α is e-θ-open in X for each $\alpha \in A$, then $\cup_{\alpha \in \Lambda} A_\alpha$ is e-θ-open in X.

Theorem 2.3 [9] For a subset A of a topological space X, the following properties hold:
(a) If $A \in eO(X)$, then $e\text{-}cl(A) = e\text{-}cl_\theta(A)$,
(b) $A \in eR(X)$ if and only if A is e-θ-*open* and e-θ-*closed*.

Definition 2.4 [9] A topological space X is said to be e-regular if for each $F \in eC(X)$ and each $x \notin F$, there exists disjoint e-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 2.5 [9] For a topological space X, the following properties are equivalent:
(a) X is *e-regular*;
(b) For each $U \in eO(X)$ and each $x \in U$, there exists $V \in eO(X)$ such that $x \in V \subseteq e\text{-}cl(V) \subseteq U$;
(c) For each $U \in eO(X)$ and each $x \in U$, there exists $V \in eR(X)$ such that $x \in V \subseteq U$.

Definition 2.6 [3] A function $f : X \to Y$ is said to be e-irresolute if $f^{-1}[V] \in eO(X)$ for every $V \in eO(Y)$.

Theorem 2.7 [8] Let $f : X \to Y$ be a function. Then the following properties are aquivalent:
(a) f is weakly e-irresolute;
(b) $f[e\text{-}cl(A)] \subseteq e\text{-}cl_\theta(f[A])$ for every subset A of X.

Remark 1 *[9] It can be easily shown that e-regular $\Rightarrow$ e-θ-open $\Rightarrow$ e-open.*

Theorem 2.8 [9] For any subset A of a space X, we have

$$e\text{-}cl_\theta(A) = \bigcap\{V | A \subseteq V \text{ and } V \text{ is } e\text{-}\theta\text{-}closed\}$$

$$= \bigcap\{V | A \subseteq V \text{ and } V \in eR(X)\}.$$

Remark 2 *It is easy to prove that:*
(a) *the intersection of an arbitrary collection of e-θ-closed sets is e-θ-closed.*
(b) *X and $\emptyset$ are e-θ-closed sets.*

Remark 3 *The following example shows that the union of any two e-θ-closed sets of X need not be e-θ-closed in X.*

***Example* 2.9** [9] Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. The subsets $\{a\}$ and $\{b\}$ are e-θ-closed in (X, τ) but $\{a, b\}$ is not e-θ-closed.

Lemma 2.10 [9] For a subset A of a space X, the following properties hold:
(a) If $A \in eO(X)$, then $e\text{-}cl(A)$ is *e-regular* and $e\text{-}cl(A) = e\text{-}cl_\theta(A)$,
(b) A is *e-regular* if and only if A is *e-θ-closed* and *e-θ-open*,
(c) A is e-regular if and only if $A = e\text{-}int(e\text{-}cl(A))$,
(d) A is e-regular if and only if $A = e\text{-}cl(e\text{-}int(A))$.

Lemma 2.11 [9] For any subset A of a topological space X, $e\text{-}cl_\theta(A)$ is *e-θ-closed.*

3. On *e-θ*-open sets

Definition 3.1 A subset A of a topological space X is said to be θ-complement e-open (briefly, θ-c-e-open) provided there exists a subset U of X for which $X \setminus A = e\text{-}cl_\theta(U)$. We call a set θ-complement e-closed if its complement is θ-c-e-open.

Remark 4 *It should be mentioned that by Lemma 2.11, $X \setminus A = e\text{-}cl_\theta(U)$ is e-θ-closed and A is e-θ-open. Therefore, the equivalence of θ-c-e-open and e-θ-open is obvious from Definition 3.1.*

Theorem 3.2 Let X be a topological space and $A \subseteq X$. If A is e-open, then $e\text{-}int(e\text{-}cl_\theta(A))$ is *e-θ-open.*

Proof. Let $A \in eO(X)$.

$$\left.\begin{array}{r} A \in eO(X) \Rightarrow e\text{-}cl(A) \in eC(X) \Rightarrow X \setminus e\text{-}cl(A) \in eO(X) \\ \text{Lemma 2.10} \end{array}\right\} \Rightarrow$$

$\Rightarrow e\text{-}cl(X \setminus e\text{-}cl(A)) = e\text{-}cl_\theta(X \setminus e\text{-}cl(A)) \ldots (1)$

$$\left.\begin{array}{r} X \setminus e\text{-}int(e\text{-}cl(A)) = e\text{-}cl(X \setminus e\text{-}cl(A)) \Rightarrow e\text{-}int(e\text{-}cl(A)) = \setminus e\text{-}cl(\setminus e\text{-}cl(A)) \\ A \in eO(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow e\text{-}int(e\text{-}cl_\theta(A)) = e\text{-}int(e\text{-}cl(A)) = X \setminus e\text{-}cl(X \setminus e\text{-}cl(A)) \ldots (2)$

$(1), (2) \Rightarrow e\text{-}int(e\text{-}cl_\theta(A)) = \setminus e\text{-}cl_\theta(\setminus e\text{-}cl(A)) \Rightarrow e\text{-}int(e\text{-}cl_\theta(A)) \in e\theta O(X)$. ■

Theorem 3.3 Let X be a topological space. Then the notion of e-θ-open is equivalent to the notion of e-regular if and only if $e\text{-}cl_\theta(A)$ is e-regular for every set $A \subseteq X$.

Proof. $(\Rightarrow)$: Let $e\theta O(X) = eR(X)$ and $A \subseteq X$.

$A \subseteq X \Rightarrow e\text{-}cl_\theta(A) = e\text{-}cl_\theta(e\text{-}cl_\theta(A)) \Rightarrow e\text{-}cl_\theta(A) \in e\theta C(X) \ldots (1)$

$$\left.\begin{array}{r} e\text{-}cl_\theta(A) \in e\theta C(X) \Rightarrow X \setminus e\text{-}cl_\theta(A) \in e\theta O(X) \\ e\theta O(X) = eR(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow X \setminus e\text{-}cl_\theta(A) \in eR(X) \subseteq e\theta C(X)$

$\Rightarrow e\text{-}cl_\theta(A) \in e\theta O(X) \ldots (2)$

$(1), (2) \Rightarrow e\text{-}cl_\theta(A) \in eR(X)$.

$(\Leftarrow)$: Let $U \in e\theta O(X)$. Our aim is to show that $U \in eR(X)$.

$$\left.\begin{array}{r} U \in e\theta O(X) \overset{\text{Remark 4}}{\Rightarrow} (\exists A \subseteq X)(X \setminus U = e\text{-}cl_\theta(A)) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow X \setminus U \in eR(X)$$

$\Rightarrow U \in eR(X)$. ■

Theorem 3.4 Let X be a topological space and $B \subseteq X$. If B is e-θ-open, then B is an union of e-regular sets.

Proof. Let $B \in e\theta O(X)$ and $x \in B$.

$$\left.\begin{array}{r} B \in e\theta O(X) \overset{\text{Remark 4}}{\Rightarrow} (\exists A \subseteq X)(B = X \setminus e\text{-}cl_\theta(A)) \\ x \in B \end{array}\right\} \Rightarrow x \notin e\text{-}cl_\theta(A)$$

$\Rightarrow (\exists W_x \in eO(X, x))(e\text{-}cl(W_x) \cap A = \emptyset)$

$\Rightarrow (\exists W_x \in eO(X, x))(e\text{-}cl(W_x) \subseteq \setminus A)$

$$\left.\begin{array}{r} \Rightarrow (W_x \in eO(X, x))(e\text{-}int(e\text{-}cl(W_x)) = (e\text{-}int_\theta(e\text{-}cl(W_x)) \subseteq e\text{-}int_\theta(\setminus A) = \setminus e\text{-}cl_\theta(A) \\ \mathcal{A} := \{e\text{-}int(e\text{-}cl(W_x)) | (\forall x \in B)(\exists W_x \in eO(X, x))(e\text{-}int(e\text{-}cl(W_x)) \subseteq \setminus e\text{-}cl_\theta(A))\} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\mathcal{A} \subseteq eR(X))(B = \bigcup \mathcal{A})$. ■

Corollary 3.5 Let X be a topological space and $B \subseteq X$. If B is e-θ-closed, then B is an the intersection of e-regular sets.

4. On e-θ-D_i and e-θ-T_i topological spaces

In this chapter, we introduce some classes of sets via the notion of e-θ-open sets. Also, the relationships between these notions and some other existing notions in the literature are investigated.

Definition 4.1 A subset A of a topological space X is called an e-θ-D-set if there exist two sets $U, V \in e\theta O(X)$ such that $U \neq X$ and $A = U \setminus V$. The family of all e-θ-D-set of X and all e-θ-D-set of X containing $x \in X$ will be denoted by $e\theta D(X)$ and $e\theta D(X, x)$, respectively.

Remark 5 *It is clear that every e-θ-open set U different from X is an e-θ-D-set. However, the converse of this implication need not be true as shown by the following example.*

Example 4.2 Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then, $e\theta O(X) = 2^X \setminus \{\{b\}, \{b, c\}, \{b, d\}\}$ and $e\theta D(X) = 2^X \setminus \{X\}$. It is obvious that the set $\{b\}$ is an e-θ-D-set but it is not e-θ-open.

Definition 4.3 A topological space X is called e-θ-D_0 if for any distinct pair of points x and y in X, there exists e-θ-D-set U of X containing x but not y or e-θ-D-set V of X containing y but not x.

Definition 4.4 A topological space X is called e-θ-D_1 if for any distinct pair of points x and y in X, there exists e-θ-D-set U in X containing x but not y and e-θ-D-set V of X containing y but not x.

Definition 4.5 A topological space X is called e-θ-D_2 if for any distinct pair of points x and y in X, there exist two e-θ-D-sets U and V of X containing x and y, respectively, such that $U \cap V = \emptyset$.

Definition 4.6 A topological space X is called e-θ-T_0 if for any distinct pair of points x and y in X, there exists an e-θ-open set U of X containing x but not y and an e-θ-open set V of X containing y but not x.

Definition 4.7 A topological space X is called e-θ-T_1 if for any distinct pair of points x and y in X, there exists an e-θ-open set U of X containing x but not y and an e-θ-open set V of X containing y but not x.

Definition 4.8 A topological space X is called e-θ-T_2 if for any distinct pair of points x and y in X, there exist two e-θ-open sets U and V of X containing x and y, respectively, such that $U \cap V = \emptyset$.

Remark 6 *From Definitions 4.1 to 4.8, we obtain the following diagram:*

$$\begin{array}{ccccc} e\text{-}\theta\text{-}T_2 & \Rightarrow & e\text{-}\theta\text{-}T_1 & \Rightarrow & e\text{-}\theta\text{-}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ e\text{-}\theta\text{-}D_2 & \Rightarrow & e\text{-}\theta\text{-}D_1 & \Rightarrow & e\text{-}\theta\text{-}D_0 \end{array}$$

Theorem 4.9 Let X be a topological space. If X is e-θ-T_0, then it is e-θ-T_2.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r} x \neq y \\ X \text{ is } e\text{-}\theta\text{-}T_0 \end{array}\right\} \Rightarrow (\exists W \in e\theta O(X, x))(\exists T \in e\theta O(X, y))(y \notin W \vee x \notin T)$$

First case: Let $W \in e\theta O(X, x)$ and $y \notin W$.

$$\left.\begin{array}{r} W \in e\theta O(X, x) \Rightarrow (\exists S \in eO(X, x))(S \subseteq e\text{-}cl(S) \subseteq W) \\ y \notin W \end{array}\right\} \overset{\text{Lemma 2.10}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow y \notin e\text{-}cl(S) \in eR(X) \subseteq e\theta C(X) \Rightarrow y \in X \setminus e\text{-}cl(S) \in e\theta O(X) \\ (U := W)(V := X \setminus e\text{-}cl(S)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in e\theta O(X, x))(V \in e\theta O(X, y))(U \cap V = \emptyset).$$ ■

Theorem 4.10 Let X be a topological space. If X is e-θ-D_0, then it is e-θ-T_0.

Proof. It suffices to prove that every e-θ-D_0 space is e-θ-T_0.
Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r} (x, y \in X)(x \neq y) \\ X \text{ is } e\text{-}\theta\text{-}D_0 \end{array}\right\} \Rightarrow (\exists A \in e\theta D(X, x))(y \notin A) \vee (\exists B \in e\theta D(X, y))(x \notin B)$$

$\Rightarrow (\exists N, M \in e\theta O(X))(M \neq X)(A = M \setminus N)(x \in A)(y \notin M \vee y = M \cap N)$

First case: Let $y \notin M$.

$$\left.\begin{array}{r} U := M \\ y \notin M \end{array}\right\} \Rightarrow (U \in e\theta O(X,x))(y \notin U)$$

Second case: Let $y \in M \cap N$.

$$\left.\begin{array}{r} (y \in M \cap N \subseteq N)(V := N) \\ (A = M \setminus N)(x \in A) \Rightarrow x \notin N \end{array}\right\} \Rightarrow (V \in e\theta O(X,y))(x \notin V).$$

■

Corollary 4.11 For any topological space X, the notions which are given in Remark *6* are equivalent.

Definition 4.12 Let X be a topological space, $N \subseteq X$ and $x \in X$. The set N is called an e-θ-neighbourhood of x in X if there exists an e-θ-open set U of X such that $x \in U \subseteq N$. The family of all e-θ-neighbourhood of a point x is denoted by $\mathcal{N}_{e\theta}(x)$.

Definition 4.13 Let X be a topological space and $x \in X$. The point x which has only X as the e-θ-neighbourhood is called a point common to all e-θ-closed sets (briefly, e-θ-cc).

Theorem 4.14 Let X be a topological space. If X is e-θ-D_1, then X has no e-θ-cc-*point*.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r} (x,y \in X)(x \neq y) \\ X \text{ is } e\text{-}\theta\text{-}D_1 \end{array}\right\} \Rightarrow (\exists A \in e\theta D(X,x))(\exists B \in e\theta D(X,y))(x \notin B)(y \notin A)$$

$\Rightarrow (\exists U, V \in e\theta O(X))(U \neq X)(x \in A = U \setminus V)$

$\Rightarrow (U \in e\theta O(X))(x \in U \subseteq U \neq X)$

$\Rightarrow X \neq U \in \mathcal{N}_{e\theta}(x)$

$\Rightarrow \mathcal{N}_{e\theta}(x) \neq \{X\}$. ■

Definition 4.15 A subset A of a topological space X is called a generalized e-θ-closed set (briefly, $ge\theta$-closed) if e-$cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is e-θ-open in X. The family of all generalized e-θ-closed set in X will be denoted by $ge\theta C(X)$.

Lemma 4.16 [9] Let A be any subset of a space X. Then, $x \in e$-$cl_\theta(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in eR(X,x)$.

Theorem 4.17 For a topological space X, the following statements hold:
(a) For each pair of points x and y in X, $x \in e$-$cl_\theta(\{y\})$ implies $y \in e$-$cl_\theta(\{x\})$;
(b) For each point x in X, the singleton $\{x\}$ is $ge\theta$-closed in X.

Proof. (a) Let $x, y \in X$ and $y \notin e$-$cl_\theta(\{x\})$.

$$\left.\begin{array}{r} y \notin e\text{-}cl_\theta(\{x\}) \Rightarrow (\exists V \in eO(X,y))(e\text{-}cl(V) \cap \{x\} = \emptyset) \\ \text{Lemma 2.10} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (e\text{-}cl(V) \in eR(X,y))(e\text{-}cl(V) \cap \{x\} = \emptyset) \\ U := e\text{-}cl(V) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U \in eO(X,y))(e\text{-}cl(U) \cap \{x\} = \emptyset)$

$\Rightarrow x \notin e$-$cl_\theta(\{y\})$.

(b) Let $x \in X$, $U \in e\theta O(X)$ and $\{x\} \subseteq U$.

$(x \in X)(\{x\} \subseteq U)(U \in e\theta O(X)) \Rightarrow U \in e\theta O(X,x)$

$$\left.\begin{array}{r} \overset{\text{Corollary 2.2}}{\Rightarrow} (\exists V \in eR(X,x))(V \subseteq U) \Rightarrow (V \in eR(X,x))(V = e\text{-}cl(V) \subseteq U) \\ eR(X) \subseteq e\theta O(X) \end{array}\right\} \Rightarrow$$

$\Rightarrow (V \in eO(X,x))(e$-$cl_\theta(\{x\}) \subseteq e$-$cl_\theta(V) = e$-$cl(V) \subseteq U)$. ■

Definition 4.18 A space X is said to be e-θ-$T_{1/2}$ if $ge\theta C(X) \subseteq e\theta C(X)$.

Theorem 4.19 For a topological space X, the followings are equivalent:
(a) X is e-θ-$T_{1/2}$;
(b) X is e-θ-T_1.

Proof. $(a) \Rightarrow (b)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{l} \left.\begin{array}{r} x,y\in X \overset{\text{Theorem 4.17}}{\Rightarrow} \{x\},\{y\}\in ge\theta C(X) \\ X \text{ is } e\text{-}\theta\text{-}T_{1/2} \Rightarrow ge\theta C(X)\subseteq e\theta C(X) \end{array}\right\} \Rightarrow \{x\},\{y\}\in e\theta C(X) \\ \hfill x\neq y \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (X\setminus\{y\}\in e\theta O(X,x))(X\setminus\{x\}\in e\theta O(X,y)) \\ (U:=X\setminus\{y\})(V:=X\setminus\{x\}) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U\in e\theta O(X,x))(V\in e\theta O(X,y))(y\notin U)(x\notin V).$$

$(b) \Rightarrow (a)$: Let $A \in ge\theta C(X)$. Suppose that $A \notin e\theta C(X)$. We will obtain a contradiction.

$$\left.\begin{array}{r} A\notin e\theta C(X) \Rightarrow A\neq e\text{-}cl_\theta(A) \Rightarrow (\exists x\in X)(x\in e\text{-}cl_\theta(A)\setminus A) \\ X \text{ is } e\text{-}\theta\text{-}T_1 \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (\forall a\in A)(\exists V_a\in e\theta O(X,a))(x\notin V_a) \\ \text{Corollary 2.2} \end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r} \Rightarrow (A\subseteq \bigcup_{a\in A} V_a)(x\notin \bigcup_{a\in A} V_a\in e\theta O(X)) \\ A\in ge\theta C(X) \end{array}\right\} \Rightarrow x\notin e\text{-}cl_\theta(A)\subseteq \bigcup_{a\in A} V_a$$

This contradicts with $x \in e\text{-}cl_\theta(A)$. ■

Definition 4.20 A topological space X is called e-T_2 [3] if for any distinct pair of points x and y in X, there exist e-open sets U and V in X containing x and y, respectively, such that $U \cap V = \emptyset$.

Theorem 4.21 For a topological space X, the followings are equivalent:
(a) X is e-θ-T_2;
(b) X is e-T_2.

Proof. $(a) \Rightarrow (b)$: Obvious.

$(b) \Rightarrow (a)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r} (x,y\in X)(x\neq y) \\ X \text{ is } e\text{-}T_2 \end{array}\right\} \Rightarrow (\exists W\in eO(X,x))(\exists T\in eO(X,y))(W\cap T=\emptyset)$$
$$\left.\begin{array}{r} \Rightarrow (e\text{-}cl(W)\in eR(X,x))(e\text{-}cl(T)\in eR(X,y))(e\text{-}cl(W)\cap e\text{-}cl(T)=\emptyset) \\ (U:=e\text{-}cl(W))(V:=e\text{-}cl(T)) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U\in e\theta O(X,x))(V\in e\theta O(X,y))(U\cap V=\emptyset).$$
■

Definition 4.22 A function $f : X \to Y$ is said to be weak e-irresolute [8] (briefly, $w.e.i.$) if for each $x \in X$ and each $V \in eO(Y, f(x))$, there exists $U \in eO(X, x)$ such that $f[U] \subseteq e\text{-}cl(V)$.

Remark 7 *[8] A function $f : X \to Y$ is weak e-irresolute if and only if $f^{-1}[V]$ is e-θ-closed (resp. e-θ-open) in X for every e-θ-closed (resp. e-θ-open) set V in Y.*

Theorem 4.23 If $f : (X, \tau) \to (Y, \sigma)$ is a weak e-irresolute surjection and A is an e-θ-D-set in Y, then the inverse image of A is an e-θ-D-set in X.

Proof. Let $A \in e\theta D(Y)$.

$$\left.\begin{array}{r} A\in e\theta D(Y) \Rightarrow (\exists U,V\in e\theta O(Y))(U\neq Y)(A=U\setminus V) \\ f \text{ is weak } e\text{-irresolute surjection} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (f^{-1}[U],f^{-1}[V]\in e\theta O(X))(f^{-1}[U]\neq f^{-1}[Y]=X)(f^{-1}[A]=f^{-1}[U]\setminus f^{-1}[V])$$
$$\Rightarrow f^{-1}[A]\in e\theta D(X).$$
■

Theorem 4.24 If Y is an e-θ-D_1 space and $f : X \to Y$ is a weak e-irresolute bijection, then X is e-θ-D_1.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y)\\ f \text{ is bijective}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(f(x), f(y) \in Y)(f(x) \neq f(y))\\ Y \text{ is } e\text{-}\theta\text{-}D_1\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in e\theta D(Y, f(x)))(\exists V \in e\theta D(Y, f(y)))(f(y) \notin U)(f(x) \notin V)\\ \text{Theorem 4.23}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (y \notin f^{-1}[U] \in e\theta D(X, x))(x \notin f^{-1}[V] \in e\theta D(X, y)).$$ ■

Theorem 4.25 For a topological space X, the followings are equivalent:
(a) X is e-θ-D_1;
(b) For each pair of distinct points $x, y \in X$, there exists a weak *e-irresolute* surjection $f : X \to Y$, where Y is an e-θ-D_1 space such that $f(x) \neq f(y)$.

Proof. $(a) \Rightarrow (b)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y)\\ (Y := X)(f := \{(x, x)|x \in X\})\end{array}\right\} \overset{\text{Hypothesis}}{\Rightarrow}$$

$$\Rightarrow (f \text{ is } w.e.i. \text{ surjection})(Y \text{ is } e\text{-}\theta\text{-}D_1)(f(x) \neq f(y)).$$

$(b) \Rightarrow (a)$: Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists f \in Y^X\ w.e.i. \text{ surjection})(Y \text{ is } e\text{-}\theta\text{-}D_1)(f(x) \neq f(y))$$

$$\left.\begin{array}{r}\Rightarrow (f \in Y^X\ w.e.i. \text{ sur.})(\exists U \in e\theta D(Y, f(x)))(\exists V \in e\theta D(Y, f(y)))(f(y) \notin U)(f(x) \notin V)\\ \text{Theorem 4.23}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (y \notin f^{-1}[U] \in e\theta D(X, x))(x \notin f^{-1}[V] \in e\theta D(X, y)).$$ ■

5. Further properties

Definition 5.1 Let A be a subset of a topological space X. The e-θ-kernel (resp. b-θ-kernel [11]) of A, denoted by $e\text{-}ker_\theta(A)$ (resp. $b\text{-}ker_\theta(A)$), is defined to be the set $\bigcap\{U|(U \in e\theta O(X))(A \subseteq U)\}$ (resp. $\bigcap\{U|(U \in B\theta O(X))(A \subseteq U)\}$).

Remark 8 *For a subset A of a topological space X, the sets of* $e\text{-}ker_\theta(A)$ *and* $b\text{-}ker_\theta(A)$ *need not be equal to each other as shown by the following example.*

***Example* 5.2** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$. Then $eR(X) = e\theta O(X) = eO(X) = 2^X$ and $BR(X) = B\theta O(X) = \{\emptyset, X\}$, $BO(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. For the subset $A = \{a, b\}$, $e\text{-}ker_\theta(A) = A \neq X = b\text{-}ker_\theta(A)$.

Definition 5.3 A space X is called slightly e-θ-R_0 space (resp. slightly b-θ-R_0 space [11]) if $\bigcap\{e\text{-}cl_\theta(\{x\})|x \in X\} = \emptyset$ (resp. $\bigcap\{b\text{-}cl_\theta(\{x\})|x \in X\} = \emptyset$).

Remark 9 *A slightly* e-θ-R_0 *space need not be a slightly* b-θ-R_0 *space as shown by the following example.*

***Example* 5.4** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$. Since $\bigcap\{e\text{-}cl_\theta(\{x\})|x \in X\} = \bigcap\{e\text{-}cl_\theta(\{a\}), e\text{-}cl_\theta(\{b\}), e\text{-}cl_\theta(\{c\}), e\text{-}cl_\theta(\{d\})\} = \emptyset$, the space X is a slightly e-θ-R_0 space. On the other hand, since $\bigcap\{b\text{-}cl_\theta(\{x\})|x \in X\} =$

$\bigcap\{b\text{-}cl_\theta(\{a\}), b\text{-}cl_\theta(\{b\}), b\text{-}cl_\theta(\{c\}), b\text{-}cl_\theta(\{d\})\} = \bigcap\{X\} = X \neq \emptyset$, the space X is not a slightly b-θ-R_0 space.

Theorem 5.5 Let A be a subset of a space X. Then, $e\text{-}ker_\theta(A) = \{x \in X | e\text{-}cl_\theta(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \notin e\text{-}ker_\theta(A)$.

$$\begin{aligned} x \notin e\text{-}ker_\theta(A) &\Rightarrow x \notin \bigcap\{U | (U \in e\theta O(X))(A \subseteq U)\} \\ &\Rightarrow (\exists U \in e\theta O(X))(A \subseteq U)(x \notin U) \\ &\Rightarrow (\setminus U \in e\theta C(X))(\{x\} \subseteq \setminus U \subseteq \setminus A) \\ &\Rightarrow (\setminus U \in e\theta C(X))(e\text{-}cl_\theta(\{x\}) \subseteq e\text{-}cl_\theta(\setminus U) = \setminus U \subseteq \setminus A) \\ &\Rightarrow e\text{-}cl_\theta(\{x\}) \cap A = \emptyset \\ &\Rightarrow x \notin \{x \in X | e\text{-}cl_\theta(\{x\}) \cap A = \emptyset\} \end{aligned}$$

Then we have

$$\{x \in X | e\text{-}cl_\theta(\{x\}) \cap A = \emptyset\} \subseteq e\text{-}ker_\theta(A) \ldots (1)$$

Now, let $x \notin \{x \in X | e\text{-}cl_\theta(\{x\}) \cap A \neq \emptyset\}$.

$$\left.\begin{array}{r} x \notin \{x \in X | e\text{-}cl_\theta(\{x\}) \cap A \neq \emptyset\} \Rightarrow e\text{-}cl_\theta(\{x\}) \cap A = \emptyset \Rightarrow A \subseteq \setminus e\text{-}cl_\theta(\{x\}) \\ U := \setminus e\text{-}cl_\theta(\{x\}) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U \in e\theta O(X))(A \subseteq U)(x \notin U)$

$\Rightarrow x \notin \bigcap\{U | (U \in e\theta O(X))(A \subseteq U)\} = e\text{-}ker_\theta(A)$

Then we have

$$e\text{-}ker_\theta(A) \subseteq \{x \in X | e\text{-}cl_\theta(\{x\}) \cap A = \emptyset\} \ldots (2)$$

$(1), (2) \Rightarrow e\text{-}ker_\theta(A) = \{x \in X | e\text{-}cl_\theta(\{x\}) \cap A = \emptyset\}$. ■

Theorem 5.6 Let X be a topological space. Then, X is slightly e-θ-R_0 if and only if $e\text{-}ker_\theta(\{x\}) \neq X$ for any $x \in X$.

Proof. $(\Rightarrow)$: Suppose that there is a point y in X such that $e\text{-}ker_\theta(\{y\}) = X$.

$e\text{-}ker_\theta(\{y\}) = \{x \in X | e\text{-}cl_\theta(\{x\}) \cap \{y\} \neq \emptyset\} = X \Rightarrow (\forall x \in X)(y \in e\text{-}cl_\theta(\{x\}))$

$$\left.\begin{array}{r} \Rightarrow y \in \bigcap\{e\text{-}cl_\theta(\{x\}) | x \in X\} \\ \text{Hypothesis} \end{array}\right\} \overset{\text{Theorem 5.5}}{\Rightarrow} y \in \bigcap\{e\text{-}cl_\theta(\{x\}) | x \in X\} = \emptyset$$

This is a contradiction.

$(\Leftarrow)$: Suppose that X is not slightly e-θ-R_0.

$$\begin{aligned} X \text{ is not slightly } e\text{-}\theta\text{-}R_0 &\Rightarrow \bigcap\{e\text{-}cl_\theta(\{x\}) | x \in X\} \neq \emptyset \\ &\Rightarrow (\exists y \in X)(y \in \bigcap\{e\text{-}cl_\theta(\{x\}) | x \in X\}) \\ &\Rightarrow (\exists y \in X)(\forall x \in X)(y \in e\text{-}cl_\theta(\{x\})) \\ &\overset{\text{Theorem 2.8}}{\Rightarrow} (\forall x \in X)(y \in \bigcap\{V | (\{x\} \subseteq V)(V \in eR(X))\}) \\ &\Rightarrow (\forall x \in X)(\forall V \in eR(X, y))(\{x\} \subseteq V) \\ &\Rightarrow (\forall V \in eR(X, y))(V = X) \\ &\Rightarrow e\text{-}cl_\theta(\{x\}) = X \end{aligned}$$

This is a contradiction. ■

Theorem 5.7 Let X and Y be two topological spaces. If X is slightly e-θ-R_0, then the product $X \times Y$ is slightly e-θ-R_0.

Proof. Let X be slightly e-θ-R_0.

$\bigcap\{e\text{-}cl_\theta(\{(x, y)\}) | (x, y) \in X \times Y\} \subseteq \bigcap\{e\text{-}cl_\theta(\{x\}) \times e\text{-}cl_\theta(\{y\}) | (x, y) \in X \times Y\}$

$= \bigcap\{e\text{-}cl_\theta(\{x\}) | x \in X\} \times \bigcap\{e\text{-}cl_\theta(\{y\}) | y \in Y\} = \emptyset$. ■

Definition 5.8 A function $f : X \to Y$ is R-continuous [11] (resp. θ-R-e-continuous, R-e-continuous) if for each $x \in X$ and each e-open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $cl(f[U]) \subseteq V$ (resp. $e\text{-}cl_\theta(f[U]) \subseteq V$, $e\text{-}cl(f[U]) \subseteq V$).

Remark 10 *We have the following diagram from Definition 5.8.*

$$\theta\text{-}R\text{-}e\text{-}continuous \longrightarrow R\text{-}e\text{-}continuous \longleftarrow R\text{-}continuous$$

A function f which is R-e-continuous need not be R-continuous as shown by the following example.

Example **5.9** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f(x) = c$. The function f is R-e-continuous but it is not R-continuous.

Question 5.10 Is there any R-e-continuous function which is not θ-R-e-continuous?

Definition 5.11 A function $f : X \to Y$ is said to be e-open [4] if $f[U]$ is e-open in Y for every open set U of X.

Theorem 5.12 Let $f : X \to Y$ be a function. If f is R-e-continuous and e-open, then f is θ-R-e-continuous.

Proof. Let $x \in X$ and $V \in eO(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in eO(Y, f(x))) \\ f \text{ is } R\text{-}e\text{-continuous}\end{array}\right\} \Rightarrow \left.\begin{array}{r}(\exists U \in O(X, x))(e\text{-}cl(f[U]) \subseteq V) \\ f \text{ is } e\text{-open}\end{array}\right\} \Rightarrow$$

$$\overset{\text{Lemma } 2.10(a)}{\Rightarrow} (\exists U \in O(X, x))(e\text{-}cl_\theta(f[U]) = e\text{-}cl(f[U]) \subseteq V).$$ ■

Definition 5.13 The graph $G(f)$ of a function $f : X \to Y$ is said to be strongly e-θ-closed if for each point $(x, y) \in (X \times Y) \setminus G(f)$, there exist subsets $U \in eO(X, x)$ and $V \in e\theta O(Y, y)$ such that $(e\text{-}cl(U) \times V) \cap G(f) = \emptyset$.

Lemma 5.14 The graph G(f) of $f : X \to Y$ is strongly e-θ-closed in $X \times Y$ if and only if for each point $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in eO(X, x)$ and $V \in e\theta O(Y, y)$ such that $f[e\text{-}cl(U)] \cap V = \emptyset$.

Proof. Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r}(x, y) \notin G(f) \\ G(f) \text{ is strongly } e\text{-}\theta\text{-closed}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in eO(X, x))(\exists V \in e\theta O(Y, y))((e\text{-}cl(U) \times V) \cap G(f) = \emptyset)$$
$$\Rightarrow (\exists U \in eO(X, x))(\exists V \in e\theta O(Y, y))(\forall x \in X)((x, f(x)) \notin e\text{-}cl(U) \times V)$$
$$\Rightarrow (\exists U \in eO(X, x))(\exists V \in e\theta O(Y, y))(f[e\text{-}cl(U)] \cap V = \emptyset).$$ ■

Definition 5.15 A space X is called to be e-T_1 [3] if for each pair of distinct points in X, there exist e-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

Theorem 5.16 Let X and Y be two topological spaces. If $f : X \to Y$ is θ-R-e-continuous weak e-irresolute and Y is e-T_1, then $G(f)$ is strongly e-θ-closed.

Proof. Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r}(x, y) \notin G(f) \Rightarrow (y, f(x) \in Y)(y \neq f(x)) \\ Y \text{ is } e\text{-}T_1\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists V \in eO(Y, f(x))(y \notin V)) \\ f \text{ is } \theta\text{-}R\text{-}e\text{-continuous}\end{array}\right\} \Rightarrow \quad \left.\begin{array}{r}(\exists U \in O(X, x))(y \notin e\text{-}cl_\theta(f[U])) \\ f \text{ is weak } e\text{-irresolute}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in eO(X, x))(\backslash e\text{-}cl_\theta(f[U]) \in e\theta O(X, y))(e\text{-}cl(U) \times (\backslash e\text{-}cl_\theta(f[U])) \cap G(f) = \emptyset). \quad \blacksquare$$

Theorem 5.17 Let $f : X \to Y$ be a weak e-irresolute function. Then, f is θ-R-e-continuous if and only if for each $x \in X$ and each *e-closed* subset F of Y with $f(x) \notin F$, there exists an open subset U of X containing x and an e-θ-open subset V of Y with $F \subseteq V$ such that $f[e\text{-}cl(U)] \cap V = \emptyset$.

Proof. $(\Rightarrow)$: Let $x \in X$, $F \in eC(Y)$ and $f(x) \notin F$.

$$\left.\begin{array}{r}(x \in X)(F \in eC(Y))(f(x) \notin F) \Rightarrow Y \setminus F \in eO(Y, f(x)) \\ f \text{ is } \theta\text{-}R\text{-}e\text{-continuous}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (\exists U \in O(X, x))(e\text{-}cl_\theta(f[U]) \subseteq Y \setminus F) \\ f \text{ is weak } e\text{-irresolute}\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\Rightarrow (U \in O(X, x))(f[e\text{-}cl(U)] \subseteq e\text{-}cl_\theta(f[U]) \subseteq Y \setminus F) \\ V := Y \setminus e\text{-}cl_\theta(f[U])\end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in O(X, x))(V \in e\theta O(Y))(F \subseteq V \subseteq Y \setminus f[e\text{-}cl(U)])$$
$$\Rightarrow (U \in O(X, x))(V \in e\theta O(Y, F))(f[e\text{-}cl(U]) \cap V = \emptyset.$$

$(\Leftarrow)$: Let $x \in X$ and $V \in eO(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in eO(Y, f(x))) \Rightarrow f(x) \notin Y \setminus V \in eC(Y) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in O(X, x))(\exists W \in e\theta O(Y, Y \setminus V))(f[e\text{-}cl(U)] \cap W = \emptyset)$$
$$\Rightarrow (U \in O(X, x))(W \in e\theta O(Y, Y \setminus V))(f[U] \subseteq f[e\text{-}cl(U)] \subseteq Y \setminus W \subseteq V)$$
$$\Rightarrow (U \in O(X, x))(e\text{-}cl_\theta(f[U]) \subseteq e\text{-}cl_\theta(Y \setminus W) = Y \setminus W \subseteq V). \quad \blacksquare$$

Corollary 5.18 Let X and Y be two topological spaces and $f : X \to Y$ be a weak e-irresolute function. Then, f is *θ-R-e-continuous* if and only if for each $x \in X$ and each *e-open* subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $e\text{-}cl_\theta(f[e\text{-}cl(U)]) \subseteq V$.

Proof. $(\Rightarrow)$: Let $x \in X$ and $V \in eO(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in eO(Y, f(x))) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow \quad \left.\begin{array}{r}(\exists U \in O(X, x))(e\text{-}cl_\theta(f[U]) \subseteq V) \\ f \text{ is weak } e\text{-irresolute}\end{array}\right\} \overset{\text{Theorem 2.7}}{\Rightarrow}$$

$$\Rightarrow (\exists U \in O(X, x))(f[e\text{-}cl(U)] \subseteq e\text{-}cl_\theta(f[U]) \subseteq V)$$
$$\Rightarrow (U \in O(X, x))(e\text{-}cl_\theta(f[e\text{-}cl(U)]) \subseteq e\text{-}cl_\theta(e\text{-}cl_\theta(f[U])) = e\text{-}cl_\theta(f[U]) \subseteq V).$$

$(\Leftarrow)$: Let $x \in X$ and $V \in eO(Y, f(x))$.

$$\left.\begin{array}{r}(x \in X)(V \in eO(Y, f(x))) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in O(X, x))(e\text{-}cl_\theta(f[U]) \subseteq e\text{-}cl_\theta(f[e\text{-}cl(U)]) \subseteq V). \quad \blacksquare$$

Definition 5.19 A topological space X is said to be e-R_1 (resp. b-R_1 [11]) if for all $x, y \in X$ with $e\text{-}cl(\{x\}) \neq e\text{-}cl(\{y\})$ (resp. $b\text{-}cl(\{x\}) \neq b\text{-}cl(\{y\})$), there exist disjoint e-open (resp. b-open) sets U and V such that $e\text{-}cl(\{x\}) \subseteq U$ (resp. $b\text{-}cl(\{x\}) \subseteq U$) and $e\text{-}cl(\{y\}) \subseteq V$ (resp. $b\text{-}cl(\{y\}) \subseteq V$).

Remark 11 *An e-R_1 space need not be a b-R_1 space as shown by the following example.*

***Example* 5.20** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then, the space X is an e-R_1 space but it is not b-R_1.

Theorem 5.21 Let X be a topological space. Then, X is e-R_1 if and only if $e\text{-}cl_\theta(\{x\}) = e\text{-}cl(\{x\})$ for all $x \in X$.

Proof. $(\Rightarrow)$: Let $x \in X$.
$x \in X \Rightarrow e\text{-}cl(\{x\}) \subseteq e\text{-}cl_\theta(\{x\}) \ldots (1)$

Now, let $y \notin e\text{-}cl(\{x\})$.

$$\left.\begin{array}{r} y \notin e\text{-}cl(\{x\}) \Rightarrow e\text{-}cl(\{x\}) \neq e\text{-}cl(\{y\}) \\ X \text{ is } e\text{-}R_1 \end{array}\right\} \Rightarrow$$

$$\begin{aligned} &\Rightarrow (\exists U, V \in eO(X))(U \cap V = \emptyset)(e\text{-}cl(\{x\}) \subseteq U)(e\text{-}cl(\{y\}) \subseteq V) \\ &\Rightarrow (U \in eO(X,x))(V \in eO(X,y))(e\text{-}cl(\{x\}) \cap e\text{-}cl(\{y\}) \subseteq e\text{-}cl(\{x\}) \cap V \\ &\qquad \subseteq e\text{-}cl(\{x\}) \cap e\text{-}cl(V) \\ &\qquad \subseteq e\text{-}cl(U) \cap e\text{-}cl(V) = \emptyset) \\ &\Rightarrow (V \in eO(X,y))(\{x\} \cap e\text{-}cl(V) \subseteq e\text{-}cl(\{x\}) \cap e\text{-}cl(V) = \emptyset) \\ &\Rightarrow y \notin e\text{-}cl_\theta(\{x\}) \end{aligned}$$

Then we have $e\text{-}cl_\theta(\{x\}) \subseteq e\text{-}cl(\{x\}) \ldots (2)$
$(1),(2) \Rightarrow e\text{-}cl(\{x\}) = e\text{-}cl_\theta(\{x\})$.

$(\Leftarrow)$: Let $x, y \in X$ and $e\text{-}cl(\{x\}) \neq e\text{-}cl(\{y\})$.

$$\left.\begin{array}{r} e\text{-}cl(\{x\}) \neq e\text{-}cl(\{y\}) \Rightarrow (\exists z \in X)(z \in e\text{-}cl(\{x\}))(z \notin e\text{-}cl(\{y\})) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\begin{aligned} &\Rightarrow (z \in e\text{-}cl(\{x\}) = e\text{-}cl_\theta(\{x\}))(z \notin e\text{-}cl(\{y\}) = e\text{-}cl_\theta(\{y\})) \\ &\Rightarrow (\forall W \in eR(X,z))(W \cap \{x\} \neq \emptyset)(\exists U \in eR(X,z))(U \cap \{y\} = \emptyset) \end{aligned}$$

$$\left.\begin{array}{r} \Rightarrow (U \in eR(X,z) \subseteq eO(X,z))(\{x\} \subseteq U)(\{y\} \subseteq \setminus U) \\ V := \setminus U \end{array}\right\} \Rightarrow$$

$\Rightarrow (U, V \in eO(X,z))(U \cap V = \emptyset)(e\text{-}cl(\{x\}) \subseteq U)(e\text{-}cl(\{y\}) \subseteq V)$. ■

Theorem 5.22 Let X be a topological space. Then, X is e-R_1 if and only if for each *e-open* set A and each $x \in A$, $e\text{-}cl_\theta(\{x\}) \subseteq A$.

Proof. $(\Rightarrow)$: Let $A \in eO(X,x)$ and $y \notin A$.

$$\left.\begin{array}{r} y \notin A \in eO(X,x) \\ X \text{ is } e\text{-}R_1 \end{array}\right\} \Rightarrow x \notin e\text{-}cl_\theta(\{y\}) = e\text{-}cl(\{y\}) \subseteq X \setminus A$$

$$\left.\begin{array}{r} \Rightarrow (\exists V \in eO(X,x))(e\text{-}cl(V) \cap \{y\} = \emptyset) \\ U := \setminus e\text{-}cl(V) \end{array}\right\} \Rightarrow (U \in eO(X,y))(e\text{-}cl(U) \cap \{x\} = \emptyset)$$

$\Rightarrow y \notin e\text{-}cl_\theta(\{x\})$.

$(\Leftarrow)$: Let $x, y \in X$ and $y \in e\text{-}cl_\theta(\{x\}) \setminus e\text{-}cl(\{x\})$.
$y \in e\text{-}cl_\theta(\{x\}) \setminus e\text{-}cl(\{x\}) \Rightarrow (y \in e\text{-}cl_\theta(\{x\}))(y \notin e\text{-}cl(\{x\}))$

$$\left.\begin{array}{r} \Rightarrow (y \in e\text{-}cl_\theta(\{x\}))(\exists A \in eO(X,y))(A \cap \{x\} = \emptyset) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow$$

$$\begin{aligned} &\Rightarrow (y \in e\text{-}cl_\theta(\{x\}))(e\text{-}cl_\theta(\{y\}) \cap \{x\} = \emptyset) \\ &\Rightarrow (y \in e\text{-}cl_\theta(\{x\})(x \notin e\text{-}cl_\theta(\{y\})) \\ &\overset{\text{Theorem 4.17}}{\Rightarrow} (y \in e\text{-}cl_\theta(\{x\}))(y \notin e\text{-}cl_\theta(\{x\})). \end{aligned}$$

This is a contradiction. ■

Theorem 5.23 Let X and Y be two topological spaces. If $f : X \to Y$ is a *θ-R-e-continuous* surjection, then Y is an e-R_1 space.

Proof. Let $V \in eO(Y,y)$.

$$\left.\begin{array}{r} V \in eO(Y,y) \\ f \text{ is surjective} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists x \in X)(y = f(x))(V \in eO(Y, f(x))) \\ f \text{ is } \theta\text{-}R\text{-}e\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in O(X,x))(e\text{-}cl_\theta(\{y\}) \subseteq e\text{-}cl_\theta(f[U]) \subseteq V).$ ■

Now, we discuss some fundamental properties of θ-R-e-continuous functions related to composition and restriction.

Theorem 5.24 Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If f is continuous and g is θ-R-e-continuous, then $g \circ f : X \to Z$ is θ-R-e-continuous.

Proof. Let $x \in X$ and $W \in eO(Z, g(f(x)))$.

$$\left.\begin{array}{r} W \in eO(Z,g(f(x))) \\ g \text{ is } \theta\text{-}R\text{-}e\text{-continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists V \in O(Y,f(x))(e\text{-}cl_\theta(g[V]) \subseteq W) \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists U \in O(X,x))(e\text{-}cl_\theta(g(f[U])) \subseteq e\text{-}cl_\theta(g[V]) \subseteq W).$$ ■

Theorem 5.25 Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If $g \circ f$ is θ-R-e-*continuous* and f is an open surjection, then g is θ-R-e-*continuous*.

Proof. Let $y \in Y$ and $W \in eO(Z, g(y))$.

$$\left.\begin{array}{r} (y \in Y)(W \in eO(Z,g(y))) \\ f \text{ is surjective} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists x \in X)(y = f(x))(W \in eO(Z,g(f(x)))) \\ g \circ f \text{ is } \theta\text{-}R\text{-}e\text{-continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r} (\exists U \in O(X,x))(e\text{-}cl_\theta(g(f[U])) \subseteq W) \\ f \text{ is open} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r} (f[U] \in O(Y,y))(e\text{-}cl_\theta(g[f[U]]) \subseteq W) \\ V := f[U] \end{array}\right\} \Rightarrow (V \in O(Y,y))(e\text{-}cl_\theta(g[V]) \subseteq W).$$ ■

Theorem 5.26 Let $f : X \to Y$ be a function and $A \subseteq X$. If f is θ-R-e-continuous, then $f|_A : A \to Y$ is θ-R-e-continuous.

Proof. Let $x \in A$ and $V \in eO(Y, f(x))$.

$$\left.\begin{array}{r} (x \in A)(V \in eO(Y,f(x))) \\ A \subseteq X \end{array}\right\} \Rightarrow \left.\begin{array}{r} (x \in X)(V \in eO(Y,f(x))) \\ f \text{ is } \theta\text{-}R\text{-}e\text{-continuous} \end{array}\right\} \Rightarrow$$
$$\Rightarrow \left.\begin{array}{r} (\exists W \in O(X,x))(e\text{-}cl_\theta(f[W]) \subseteq V) \\ U := W \cap A \end{array}\right\} \Rightarrow$$
$$\Rightarrow (W \in O(A,x))(e\text{-}cl_\theta(f|_A[U]) = e\text{-}cl_\theta(f[W \cap A]) \subseteq e\text{-}cl_\theta(f[W]) \subseteq V).$$ ■

6. Conclusion

One of the most studied objects of general topology is undoubtedly open set types. Two of them are the notions of b-open and e-open sets which are independent of each other. Similarities and differences between these notions in the literature are examined by several authors. In this paper, we study further properties of the notion of e-θ-open set which is stronger than the notion of e-open set. Since the notions of b-open and e-open sets are the same notions in regular topological spaces, nearly all results obtained in the scope of this present paper coincide with the results obtained in the article [11]. We believe that this study will help researchers to upgrade and support further studies related to some types of open sets. Furthermore, this work may be even more useful to enrich the class of continuous functions.

Acknowledgements

We would like to thank the anonymous reviewer(s) for their careful reading of our manuscript and their insightful comments and suggestions which improved the paper. Also, this study has been supported by the Scientific Research Project Fund of Muğla Sıtkı Koçman University.

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