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Strength of dynamic technique to rational type contraction in partially ordered metric spaces and the extension of outcomes of coupled fixed point

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Abstract. We investigate the mechanisms of dynamic technique to rational type contraction in the context of partially ordered complete metric spaces and obtain coupled fixed point theorems in this article. Our derived results extend and generalize some prominent outcomes in the literature. At last, we have presented an example and an application for a system of integral that preserve the main results.

Keywords: Coupled fixed point, mixed monotone property, rational type contraction, partially ordered complete metric space.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

In 1922, Banach [5] presented a fixed point theorem (FPT) for contraction mappings in metric space. The FPT is also known as the Banach contraction mapping theorem (BCMT) or the Banach fixed point theorem (BFPT). BFPT has been generalized for various mappings in various metric spaces by numerous researchers. In 1975, Wolk [27] and Monjardet [6] developed the partially ordered metric space theory and expanded the Banach contraction principle. Ran and Reurings [23] looked into the idea of fixed points and applied it to matrix equations in partially ordered metric spaces. Additionally, Nieto et al. [18–20] extended the outcomes of [23] to partially ordered sets with periodic conditions and applied this idea in equations of ordinary differential of the first order. Numerous authors have extensively generalized, expanded and discussed the results in partially ordered metric spaces and obtained unique fixed points (see [1–3, 11, 17]).

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In 2006, Bhaskar and Laxmikantham [17] defined mixed monotone functions, proved the existence and uniqueness of a solution for periodic boundary value problems and found the coupled fixed point theorem. Additionally, coupled coincidences or common coupled fixed points were added to nonlinear contractions in ordered metric spaces by Lakshmikantham and Ciric [15]. Many conclusions have been generalized in partially ordered metric spaces on coupled fixed points and common coupled fixed points (for instance, see [4, 7–10, 12–16, 21, 22, 25, 28]).

Very recently, Rao and Kalyani [24] generalized the outcomes of Singh and Chatterjee [26] in partially ordered metric space with properties of mixed monotone and got some results from coupled fixed points. Consequently, its investigation intends to obtain new results by constructing coupled fixed points for mappings satisfying rational type contraction. The obtained results expand and modify a range of results from the body of previous literature. We start with the following definitions and notations.

Definition 1.1 [24] A self-mapping $\zeta : R \to R$ is said to be increasing on partially ordered set $(R, \leqslant)$ if $\zeta(\alpha) \leqslant \zeta(\beta)$ for all $\alpha, \beta \in R$ when $\alpha \leqslant \beta$, and is also known as decreasing if $\zeta(\alpha) \geqslant \zeta(\beta)$ when $\alpha \leqslant \beta$.

Definition 1.2 [17] A self-mapping $\zeta : R \to R$ is said to be have strict mixed monotone property on partially ordered set $(R, \leqslant)$ if $\zeta(\alpha, \beta)$ is increasing in α and decreasing β as well, i.e., $\alpha_1, \alpha_2 \in R$ with $\alpha_1 \leqslant \alpha_2 \Rightarrow \zeta(\alpha_1, \beta) \leqslant \zeta(\alpha_2, \beta)$ for all $\beta \in R$ and $\beta_1, \beta_2 \in R$ with $\beta_1 \leqslant \beta_2 \Rightarrow \zeta(\alpha, \beta_1) \geqslant \zeta(\alpha, \beta_2)$ for all $\alpha \in R$.

Definition 1.3 [6] If $(R, \leqslant)$ is a partially ordered set in addition to (R, d) being a metric space, then triple $(R, d, \leqslant)$ is referred to as a partially ordered metric space.

Definition 1.4 [6] An element $(\tau, \varpi) \in R \times R$ is said to be coupled fixed point of the mapping $\zeta : R \times R \to R$ if $\zeta(\tau, \varpi) = \tau$ and $\zeta(\varpi, \tau)$.

2. Main results

Here, we display key outcomes and deliver their proofs.

Theorem 2.1 Let a mapping $\zeta : R \times R \to R$ having the mixed monotone property on $(R, d, \leqslant)$, where $(R, d, \leqslant)$ is a partially ordered complete metric space, satisfy the condition

$$\begin{aligned} d(\zeta(\tau, \varpi), \zeta(\eta, \rho)) \leqslant r_1 d(\tau, \eta) + r_2 d(\varpi, \rho) + r_3 \left[\frac{d(\tau, \zeta(\tau, \varpi)) d(\eta, \zeta(\eta, \rho))}{d(\tau, \eta)}\right] \\ + r_4 \left[\frac{d(\tau, \zeta(\eta, \rho)) d(\eta, \zeta(\tau, \varpi))}{d(\tau, \eta)}\right] + r_5 \left[\frac{d(\varpi, \zeta(\varpi, \tau)) d(\rho, \zeta(\rho, \eta))}{d(\varpi, \rho)}\right] \\ + r_6 \left[\frac{d(\varpi, \zeta(\rho, \eta)) d(\rho, \zeta(\varpi, \tau))}{d(\varpi, \rho)}\right] \end{aligned} \tag{1}$$

for all $\tau, \varpi, \eta, \rho \in R$ with $\tau \geqslant \eta$ and $\varpi \leqslant \rho$, where $r_i \in [0, 1]$ for $i = 1, 2, ..., 6$ such that $\sum_{i=1}^{6} r_i < 1$ and there exist $\tau_0, \varpi_0 \in R$ such that $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0)$. If ζ is continuous, then ζ has a coupled fixed point $(\tau, \varpi) \in R \times R$.

Proof. Given that ζ is a continuous and $\tau_0, \varpi_0 \in R$ through $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant$

(ϖ_0, τ_0). We construct $\{\tau_k\}$ and $\{\varpi_k\}$ in R such that

$$\tau_{k+1} = \zeta(\tau_k, \varpi_k) \text{ and } \varpi_{k+1} = \zeta(\varpi_k, \tau_{2k}) \text{ for all } k \geqslant 0. \tag{2}$$

Now, we have to show that

$$\tau_k \leqslant \tau_{k+1} \text{ and } \varpi_k \geqslant \varpi_{k+1} \text{ for all } k \geqslant 0 \tag{3}$$

by the process of mathematical induction for this. Now, from (2) with $k = 0$, we have $\tau_0 \leqslant \zeta(\tau_0, \varpi_0) = \tau_1$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0) = \varpi_1$. Because $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0)$. Thus, the inequalities in (3) holds for $k = 0$. Let (3) hold for k and according to the mixed monotone property of ζ, we have

$$\tau_{k+2} = \zeta(\tau_{k+1}, \varpi_{k+1}) \geqslant \zeta(\tau_k, \varpi_{k+1}) \geqslant \zeta(\tau_k, \varpi_k) = \tau_{k+1}. \tag{4}$$

and

$$\varpi_{k+2} = \zeta(\varpi_{k+1}, \tau_{k+1}) \leqslant \zeta(\varpi_k, \tau_{k+1}) \leqslant \zeta(\varpi_k, \tau_k) = \varpi_{k+1}. \tag{5}$$

Now, from (4) and (5), we get $\tau_{k+1} \leqslant \tau_{k+2}$ and $\varpi_{k+1} \geqslant \varpi_{k+2}$. Thus, the inequalities in (3) hold for $k+1$ and we get $\tau_0 \leqslant \tau_1 \leqslant .... \leqslant \tau_k \leqslant \tau_{k+1} \leqslant ...$ and $\varpi_0 \geqslant \varpi_1 \geqslant ... \geqslant \varpi_k \geqslant \varpi_{k+1} \geqslant ...$ Now, using (1), we get

$$\begin{aligned}
d(\tau_k, \tau_{k+1}) &= d(\zeta(\tau_{k-1}, \varpi_{k-1}), \zeta(\tau_k, \varpi_k)) \\
&\leqslant r_1 d(\tau_{k-1}, \tau_k) + r_2 d(\varpi_{k-1}, \varpi_k) + r_3[\frac{d(\tau_{k-1}, \zeta(\tau_{k-1}, \varpi_{k-1})).d(\tau_k, \zeta(\tau_k, \varpi_k))}{d(\tau_{k-1}, \tau_k)}] \\
&+ r_4[\frac{d(\tau_{k-1}, \zeta(\tau_k, \varpi_k))d(\tau_k, \zeta(\tau_{k-1}, \varpi_{k-1}))}{d(\tau_{k-1}, \tau_k)}] \\
&+ r_5[\frac{d(\varpi_{k-1}, \zeta(\varpi_{k-1}, \tau_{k-1}))d(\varpi_k, \zeta(\varpi_k, \tau_k))}{d(\varpi_{k-1}, \varpi_k)}] \\
&+ r_6[\frac{d(\varpi_{k-1}, \zeta(\varpi_k, \tau_k))d(\varpi_k, \zeta(\varpi_{k-1}, \tau_{k-1}))}{d(\varpi_{k-1}, \varpi_k)}] \\
&\leqslant r_1 d(\tau_{k-1}, \tau_k) + r_2 d(\varpi_{k-1}, \varpi_k) + r_3[\frac{d(\tau_{k-1}, \tau_k)d(\tau_k, \tau_{k+1})}{d(\tau_{k-1}, \tau_k)}] \\
&+ r_4[\frac{d(\tau_{k-1}, \tau_{k+1})d(\tau_k, \tau_k)}{d(\tau_{k-1}, \tau_k)}] + r_5[\frac{d(\varpi_{k-1}, \varpi_k)d(\varpi_k, \varpi_{k+1})}{d(\varpi_{k-1}, \varpi_k)}] \\
&+ r_6[\frac{d(\varpi_{k-1}, \varpi_{k+1})d(\varpi_k, \varpi_k)}{d(\varpi_{k-1}, \varpi_k}] \\
&\leqslant r_1 d(\tau_{k-1}, \tau_k) + r_2 d(\varpi_{k-1}, \varpi_k) + r_3 d(\tau_k, \tau_{k+1}) + r_5 d(\varpi_k, \varpi_{k+1})
\end{aligned} \tag{6}$$

Similarly, we can prove that

$$d(\varpi_k, \varpi_{k+1}) \leqslant r_1 d(\varpi_{k-1}, \varpi_k) + r_2 d(\tau_{k-1}, \tau_k) + r_3 d(\varpi_k, \varpi_{k+1}) + r_5 d(\tau_k, \tau_{k+1}). \tag{7}$$

Adding up (6) and (7), we have $C_k = aC_{k-1}$ where $C_k = A_k + B_K$, $C_{k-1} = A_{k-1} + B_{k-1}$, $A_k = d(\tau_k, \tau_{k+1})$, $B_k = d(\varpi_k, \varpi_{k+1})$, $A_{k-1} = d(\tau_{k-1}, \tau_k)$, $B_{k-1} = d(\varpi_{k-1}, \varpi_k)$ and

$a = \frac{(r_1+r_2)}{1-(r_3+r_5)} < 1$. Continuing this process, we have $C_k \leqslant aC_{k-1} \leqslant a^2C_{k-2} \leqslant ... \leqslant a^kC_0$ which implies that $\lim\limits_{n\to\infty} C_k = \lim\limits_{n\to\infty} A_k + B_k = 0$. Consequently, we have $\lim\limits_{n\to\infty} A_k = 0$ and $\lim\limits_{n\to\infty} B_k = 0$. Therefore, by the triangular inequality, we get for $m \leqslant k$ that

$$\begin{aligned} d(\tau_m, \tau_k) + d(\varpi_m, \varpi_k) &\leqslant C_{m-1} + C_{m-2} + ... + C_k \\ &\leqslant (a^{m-1} + a^{m-2} + ... + a^k)C_0 \\ &\leqslant \frac{a^k}{1-a}C_0 \qquad \text{as} \qquad m, k \to \infty, \end{aligned} \tag{8}$$

which implies that $\lim\limits_{n\to\infty} d(\tau_m, \tau_k) + d(\varpi_m, \varpi_k) = 0$. Hence, $\{\tau_k\}$ and $\{\varpi_k\}$ are Cauchy sequence in R. Since ζ be a continuous and $(R, d, \leqslant)$ is a partially ordered complete metric space, there exist $\varrho, \varsigma \in R$ such that $\tau_k \to \varrho$ and $\varpi_k \to \varsigma$ and we have

$$\begin{aligned} \varrho &= \lim_{n\to\infty} \tau_{k+1} = \lim_{n\to\infty} \zeta(\tau_k, \varpi_k) = \zeta(\lim_{n\to\infty} \tau_k, \lim_{n\to\infty} \varpi_k) = \zeta(\varrho, \varsigma) \\ \varsigma &= \lim_{n\to\infty} \varpi_{k+1} = \lim_{n\to\infty} \zeta(\varpi_k, \tau_k) = \zeta(\lim_{n\to\infty} \varpi_k, \lim_{k\to\infty} \tau_k) = \zeta(\varsigma, \varrho). \end{aligned}$$

Hence, (ϱ, ς) is a coupled fixed point of ζ. This completes the theorem. ■

Corollary 2.2 Let a mapping $\zeta : R \times R \to R$ having the mixed monotone property on $(R, d, \leqslant)$, where $(R, d, \leqslant)$ is a partially ordered complete metric space, satisfy the condition

$$d(\zeta(\tau, \varpi), \zeta(\eta, \rho)) \leqslant r_1 d(\tau, \eta) + r_2 d(\varpi, \rho)$$

for all $\tau, \varpi, \eta, \rho \in R$ with $\tau \geqslant \eta$ and $\varpi \leqslant \rho$, where $r_i \in [0, 1]$ for $i = 1, 2$ such that $\sum_{i=1}^{2} r_i < 1$ and there exist $\tau_0, \varpi_0 \in R$ such that $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0)$. If ζ is continuous, then ζ has a coupled fixed point $(\tau, \varpi) \in R \times R$.

Proof. Taking $r_3 = r_4 = r_5 = r_6 = 0$ in Theorem 2.1, we obtain Corollary 2.2. ■

Theorem 2.3 Adding comparable condition to the hypothesis of Theorem 2.1, we obtain the uniqueness of the coupled fixed point of ζ.

Proof. Suppose that $(\varrho^{'}, \varsigma^{'}) \in R \times R$ is another coupled fixed point of ζ; that is, $\zeta(\varrho^{'}, \varsigma^{'}) = \varsigma^{'}$ and $\zeta(\varrho^{'}, \varsigma^{'}) = \varrho^{'}$.We shall show that $\varrho = \varrho^{'}$ and $\varsigma = \varsigma^{'}$. Consider the following two cases.

Case1: If (ϱ, ς) and $(\varrho^{'}, \varsigma^{'})$ are comparable. Then we have

$$\begin{aligned} d(\varrho, \varrho^{'}) &= d(\zeta(\varrho, \varsigma), \zeta(\varrho^{'}, \varsigma^{'})) \\ &\leqslant r_1 d(\varrho, \varrho^{'}) + r_2 d(\varsigma^{'}, \varsigma^{'}) \\ &+ r_3[\frac{d(\varrho, \zeta(\varrho, \varsigma))d(\varrho^{'}, \zeta(\varrho^{'}, \varsigma^{'}))}{d(\varrho, \varrho^{'})}] + r_4[\frac{d(\varrho, \zeta(\varrho^{'}, \varsigma^{'}))d(\varrho^{'}, \zeta(\varrho, \varsigma))}{d(\varrho, \varrho^{'})}] \\ &+ r_5[\frac{d(\varsigma, \zeta(\varsigma, \varrho))d(\varsigma^{'}, \zeta(\varsigma^{'}, \varrho^{'}))}{d(\varsigma, \varsigma^{'}}] + r_6[\frac{d(\varsigma, \zeta(\varsigma^{'}, \varrho^{'}))d(\varsigma^{'}, \zeta(\varsigma, \varrho))}{d(\varsigma, \varsigma^{'}}] \\ &= r_1 d(\varrho, \varrho^{'}) + r_2 d(\varsigma, \varsigma^{'}) + r_4 d(\varrho^{'}, \varrho) + r_6 d(\varsigma^{'}, \varsigma) \end{aligned}$$

Thus, $d(\varrho, \varrho^{'}) \leqslant (r_1 + r_4)d(\varrho, \varrho^{'}) + (r_2 + r_6)d(\varsigma, \varsigma')$. Similarly, $d(\varsigma, \varsigma^{'}) \leqslant (r_1 + r_4)d(\varsigma, \varsigma^{'}) + (r_2 + r_6)d(\varrho, \varrho')$. Now, we have

$$d(\varrho, \varrho^{'}) + d(\varsigma, \varsigma^{'}) \leqslant (r_1 + r_2 + r_4 + r_6)[d(\varrho, \varrho') + d(\varsigma, \varsigma^{'})]$$

Since $r_1 + r_2 + r_4 + r_6 < 1$, $d(\varrho, \varrho^{'}) + d(\varsigma, \varsigma^{'}) \leqslant 0$. Hence, $d(\varrho, \varrho^{'}) = 0$ and $d(\varsigma, \varsigma^{'}) = 0$, which implies that $\varrho = \varrho^{'}$ and $\varsigma = \varsigma^{'}$. Thus, (ϱ, ς) is a unique coupled fixed point of ζ.
Case 2: If (ϱ, ς) and $(\varrho^{'}, \varsigma^{'})$ are not comparable. By assumption, there exists $(s, t) \in R \times R$ comparable with both of them. We define sequence $\{s_k\}$ and $\{t_k\}$ as follows: $s_0 = s$, $t_0 = t$, $s_{k+1} = \zeta(s_k, t_k)$ and $t_{k+1} = \zeta(t_k, s_k)$ for all k. Since (s, t) is comparable with (ϱ, ς), we may assume that $(\varrho, \varsigma) \geqslant (s, t) = (s_0, t_0)$. By using the mathematical induction, it is easy to prove that $(\varrho, \varsigma) \geqslant (s_k, t_k)$ for all k. Now, from (1), we have

$$\begin{aligned} d(\varrho, s_{k+1}) = d(\zeta(\varrho, \varsigma), \zeta(s_k, t_k)) \\ &\leqslant r_1 d(\varrho, s_k) + r_2 d(\varsigma, t_k) + r_3[\frac{d(\varrho, \zeta(\varrho, \varsigma)).d(s_k, \zeta(s_k, t_k))}{d(\varrho, s_k)}] \\ &+ r_4[\frac{d(\varrho, \zeta(s_k, t_k)).d(s_k, \zeta(\varrho, \varsigma))}{d(\varrho, s_k)}] + r_5[\frac{d(\varsigma, \zeta(\varsigma, \varrho)).d(t_k, \zeta(t_k, s_k))}{d(\varsigma, t_k)}] \\ &+ r_6[\frac{d(\varsigma, \zeta(t_k, s_k)).d(t_k, \zeta(\varsigma, \varrho))}{d(\varsigma, t_k)}], \end{aligned}$$

which implies that

$$d(\varrho, s_{k+1}) \leqslant r_1 d(\varrho, s_k) + r_2 d(\varsigma, t_k) + r_4 d(\varrho, s_{k+1}) + r_6 d(\varsigma, t_{k+1}). \tag{9}$$

Similarly, we obtain

$$d(t_{k+1}, \varsigma) \leqslant r_1 d(t_k, \varsigma) + r_2 d(s_k, \varrho) + r_4 d(\varsigma, t_{k+1}) + r_6 d(\varrho, s_{k+1}). \tag{10}$$

Adding (9) and (10), we get

$$d(\varrho, s_{k+1}) + d(t_{k+1}, \varsigma) \leqslant \frac{(r_1 + r_2)}{1 - (r_4 + r_6)}[d(\varrho, s_k) + d(\varsigma, t_k,)]$$

Suppose $h = \frac{(r_1+r_2)}{1-(r_4+r_6)} < 1$. Then from above equation, we have

$$\begin{aligned} d(\varrho, s_{k+1}) + d(t_{k+1}, \varsigma) &\leqslant h[d(\varrho, s_k) + d(t_k, \varsigma)] \\ &\leqslant h^2[d(\varrho, s_{k-1}) + d(\varsigma, t_{k-1})] \\ &\vdots \\ &\leqslant h^k[d(\varrho, s_0) + d(\varsigma, t_0)] \to 0 \qquad \text{as} \qquad k \to \infty. \end{aligned}$$

Thus, we obtain $\lim_{n\to\infty}[d(\varrho, s_{k+1}) + d(\varsigma, t_{k=1})] = 0$, which implies that

$$\lim_{n\to\infty} d(\varrho, s_{k+1}) = \lim_{n\to\infty} d(t_{k+1}, \varsigma) = 0. \tag{11}$$

Similarly, we can show that

$$\lim_{n\to\infty} d(\varrho^{'}, s_{k+1}) = \lim_{n\to\infty} d(t_{k+1}, \varsigma^{'}) = 0. \tag{12}$$

From (11) and (12), we get $\varrho = \varrho^{'}$ and $\varsigma = \varsigma^{'}$. The proof is complete. ■

Theorem 2.4 Along with the assertion of Theorem 2.1, if τ_0 and ϖ_0 are comparable, then $\tau = \varpi$.

Proof. By Theorem 2.1, (τ, ϖ) is a coupled fixed point of ζ and $\{\tau_k\}$ and $\{\varpi_k\}$ in R are two sequence such that $\tau_k \to \tau$ and $\varpi_k \to \varpi_k$. Presume that $\tau_0 \leqslant \varpi_0$, then we will show that $\tau_k \leqslant \varpi_k$, where $\tau_k = \zeta(\tau_{k-1}, \varpi_{k-1})$ and $\varpi_k = \zeta(\varpi_{k-1}, \tau_{k-1})$ for all k. Let it holds for some $k \geqslant 0$. Then, by the mixed monotone property of ζ, we have

$$\tau_{k+1} = \zeta(\tau_k, \varpi_k) \leqslant \zeta(\varpi_k, \tau_k) = \varpi_{k+1}.$$

From (1), we have

$$\begin{aligned}
d(\tau_{k+1}, \varpi_{k+1}) &= d(\zeta(\tau_k, \varpi_k), \zeta(\varpi_k, \tau_k)) \\
&\leqslant r_1 d(\tau_k, \varpi_k) + r_2 d(\varpi_k, \tau_k) \\
&+ r_3[\frac{d(\tau_k, \zeta(\tau_k, \varpi_k))d(\varpi_k, \zeta(\varpi_k, \tau_k))}{d(\tau_k, \varpi_k)}] + r_4[\frac{d(\tau_k, \zeta(\varpi_k, \tau_k))d(\varpi_k, \zeta(\tau_k, \varpi_k))}{d(\tau_k, \varpi_k)}] \\
&+ r_5[\frac{d(\varpi_k, \zeta(\varpi_k, \tau_k))d(\tau_k, \zeta(\tau_k, \varpi_k))}{d(\varpi_k, \tau_k)}] + r_6[\frac{d(\varpi_k, \zeta(\tau_k, \varpi_k))d(\tau_k, \zeta(\varpi_k, \tau_k))}{d(\varpi_k, \tau_k)}].
\end{aligned}$$

Taking $k \to \infty$, we obtain $d(\varpi, \tau) \leqslant (r_1 + r_2 + r_4 + r_6) d(\varpi, \tau)$. Since $r_1 + r_2 + r_4 + r_6 < 1$, we have $d(\varpi, \tau) = 0$. Hence, $\zeta(\tau, \varpi) = \tau = \varpi = \zeta(\varpi, \tau)$. Similarly, one can also show this by considering $\varpi_0 \leqslant \tau_0$. ■

***Example* 2.5** Suppose $R = [0, 1]$, $d : R \times R \to R$ is metric with $d(\tau, \varpi) = |\tau - \varpi|$ for all $\tau, \varpi \in R$ and $\leqslant$ is usual order. Then, $(R, d, \leqslant)$ is a partially ordered complete metric space. Define the mapping $\zeta : R \times R \to R$ by $\zeta(\tau, \varpi) = \frac{\tau}{10} + \frac{\varpi}{15}$ for all $\tau, \varpi \in [0, 1]$. Clearly, ζ is continuous and has mixed monotone property. Also,

$$0 = \tau_0 \leqslant \zeta(0, 0) = \zeta(\tau_0, \varpi_0) \text{ and } \varpi_0 = 0 \geqslant \zeta(0, 0) = \zeta(\varpi_0, \tau_0).$$

Then it is obvious that $(0, 0)$ is the coupled fixed point of ζ. Now we have following possibility for value of (τ, ϖ) an (η, ρ) such that $\tau \geqslant \eta$ and $\varpi \leqslant \rho$. Then, we have

$$\begin{aligned}
d(\zeta(\tau, \varpi), \zeta(\eta, \rho)) &= |(\frac{\tau}{10} + \frac{\varpi}{15}) - (\frac{\eta}{10} + \frac{\rho}{15})| \\
&\leqslant |(\frac{\tau}{10} - \frac{\eta}{10}) + (\frac{\varpi}{15} + \frac{\rho}{15}| \\
&\leqslant \frac{1}{10}|\tau - \eta| + \frac{1}{15}|\varpi - \rho| \\
&\leqslant \frac{1}{10} d(\tau, \eta) + \frac{1}{15} d(\varpi, \rho),
\end{aligned}$$

where $r_1 = \frac{1}{10}$ and $r_2 = \frac{1}{15}$ for all $(\tau, \varpi) \leqslant (\eta, \rho)$. Hence, the condition of Corollary 2.2 is satisfied and also other conditions of inequality (1) of Theorem 2.1 are satisfied.

Therefore, ζ has a coupled fixed point in R.

Corollary 2.6 Let a mapping $\zeta^k : R \times R \to R$ having the mixed monotone property on $(R, d, \leqslant)$, where $(R, d, \leqslant)$ is a partially ordered complete metric space, satisfy the following condition for some positive integer k:

$$
\begin{aligned}
d(\zeta^k(\tau,\varpi),\zeta^k(\eta,\rho)) \leqslant r_1 d(\tau,\eta) + r_2 d(\varpi,\rho) + r_3\ &[\frac{d(\tau,\zeta^k(\tau,\varpi))d(\eta,\zeta^k(\eta,\rho))}{d(\tau,\eta)}\ ] \\
+ r_4\ [\frac{d(\tau,\zeta^k(\eta,\rho))d(\eta,\zeta^k(\tau,\varpi))}{d(\tau,\eta)}\ ] &+ r_5\ [\frac{d(\varpi,\zeta^k(\varpi,\tau))d(\rho,\zeta^k(\rho,\eta))}{d(\varpi,\rho)}\ ] \\
+ r_6\ [\frac{d(\varpi,\zeta^k(\rho,\eta))d(\rho,\zeta^k(\varpi,\tau))}{d(\varpi,\rho)}\ ].
\end{aligned}
$$

for all $\tau, \varpi, \eta, \rho \in R$ with $\tau \geqslant \eta$ and $\varpi \leqslant \rho$, where $r_i \in [0, 1]$ for $i = 1, 2, ..., 6$ such that $\sum_{i=1}^{6} r_i < 1$ and there exist $\tau_0, \varpi_0 \in R$ such that $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0)$. If ζ^k is continuous for some positive integer k, then ζ has a coupled fixed point $(\tau, \varpi) \in R \times R$.

Proof. By Theorem 2.1, there exists $(\eta, \rho) \in R \times R$ such that $\eta = \zeta^k(\eta, \rho)$ and $\rho = \zeta^k(\rho, \eta)$. It is easy to show that

$$\zeta(\zeta^k(\eta,\rho),\zeta^k(\rho,\eta)) = \zeta^{k+1}(\eta,\rho) = \zeta^k(\zeta(\eta,\rho),\zeta(\rho,\eta)).$$

Thus,

$$
\begin{aligned}
d(\zeta(\eta,\rho),\eta) &= d(\zeta(\zeta^k(\eta,\rho),\zeta^k(\rho,\eta)),\zeta^k(\eta,\rho)) \\
&= d(\zeta^k(\zeta(\eta,\rho),\zeta(\rho,\eta)),\zeta^k(\eta,\rho)) \\
&\leqslant r_1 d(\zeta(\eta,\rho),\eta) + r_2 d(\zeta(\rho,\eta),\rho) + r_3[\frac{d(\zeta(\eta,\rho),\zeta^k(\zeta(\eta,\rho),\zeta(\rho,\eta)))d(\eta,\zeta^k(\eta,\rho))}{d(\zeta(\eta,\rho),\eta)}] \\
&+ r_4[\frac{d(\zeta(\eta,\rho),\zeta^k(\eta,\rho))d(\eta,\zeta^k(\zeta(\eta,\rho),\zeta(\rho,\eta)))}{d(\zeta(\eta,\rho),\eta)}] \\
&+ r_5[\frac{d(\zeta(\rho,\eta),\zeta^k(\zeta(\rho,\eta),\zeta(\eta,\rho)))d(\rho,\zeta^k(\rho,\eta))}{d(\zeta(\rho,\eta),\rho)}] \\
&+ r_6[\frac{d(\zeta(\rho,\eta),\zeta^k(\rho,\eta))d(\rho,\zeta^k(\zeta(\rho,\eta),\zeta(\eta,\rho)))}{d(\zeta(\rho,\eta),\rho)}].
\end{aligned}
$$

Since $d(\zeta^k(\eta,\rho),\eta) = 0 = d(\zeta^k(\rho,\eta),\rho)$, we get

$$
\begin{aligned}
d(\zeta(\eta,\rho),\eta) &\leqslant r_1 d(\zeta(\eta,\rho),\eta) + r_2 d(\zeta(\rho,\eta),\rho) + r_4 d(\eta,\zeta(\eta,\rho)) + r_6 d(\rho,\zeta(\rho,\eta)) \\
&\leqslant (r_1 + r_4) d(\eta,\zeta(\eta,\rho)) + (r_2 + r_6) d(\rho,\zeta(\rho,\eta)) \qquad (13)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d(\zeta(\rho,\eta),\rho) &\leqslant r_1 d(\zeta(\rho,\eta),\rho)) + r_2 d(\zeta(\eta,\rho),\eta)) + r_4 d(\rho,\zeta(\rho,\eta)) + r_6 d(\eta,\zeta(\eta,\rho)) \\
&\leqslant (r_1 + r_4) d(\rho,\zeta(\rho,\eta)) + (r_2 + r_6) d(\eta,\zeta(\eta,\rho)). \qquad (14)
\end{aligned}
$$

Now, from (13) and (14), we obtain

$$d(\zeta(\eta,\rho),\eta)+d(\zeta(\rho,\eta),\rho)\leqslant(r_1+r_2+r_4+r_6)[d(\zeta(\eta,\rho),\eta)+(\zeta(\rho,\eta),\rho)].$$

Since $r_1+r_2+r_4+r_6<1$, we have $d(\zeta(\eta,\rho),\eta)+(\zeta(\rho,\eta),\rho)<0$. Thus, $d(\zeta(\eta,\rho),\eta)=0$. Similarly, we can get $d(\zeta(\rho,\eta),\rho)=0$. ■

3. Applications

Applying our recent results to integral-type functions involving rational contraction is the main objective of this section. To this effect, we indicate Γ the set of all functions $\lambda:[0,\infty)\to[0,\infty)$ which satisfy the following hypothesis:

(i) each λ must be a Lebesgue-integrable mapping on every compact subset of $[0,\infty)$;

(ii) we have $\int_0^\epsilon \lambda(t)>0$ for any $\epsilon>0$.

Theorem 3.1 Let a mapping $\zeta:R\times R\to R$ having the mixed monotone property on $(R,d,\leqslant)$, where $(R,d,\leqslant)$ is a partially ordered complete metric space, satisfy the condition

$$\begin{aligned}\int_0^{d(\zeta(\tau,\varpi),\zeta(\eta,\rho))}\varphi(t)dt\leqslant r_1\int_0^{d(\tau,\eta)}\varphi(t)dt+r_2\int_0^{d(\varpi,\rho)}\varphi(t)dt\\ +r_3\int_0^{\frac{d(\tau,\zeta(\tau,\varpi))d(\eta,\zeta(\eta,\rho))}{d(\tau,\eta)}}\varphi(t)dt+r_4\int_0^{\frac{d(\tau,\zeta(\eta,\rho))d(\eta,\zeta(\tau,\varpi))}{d(\tau,\eta)}}\varphi(t)dt\\ +r_5\int_0^{\frac{d(\varpi,\zeta(\varpi,\tau))d(\rho,\zeta(\rho,\eta))}{d(\varpi,\rho)}}\varphi(t)dt+r_6\int_0^{\frac{d(\varpi,\zeta(\rho,\eta))d(\rho,\zeta(\varpi,\tau))}{d(\varpi,\rho)}}\varphi(t)dt\end{aligned}$$

for all $\tau,\varpi,\eta,\rho\in R$ with $\tau\geqslant\eta$ and $\varpi\leqslant\rho$, where $r_i\in[0,1]$ for $i=1,2,...,6$ such that $\sum_{i=1}^6 r_i<1$, and $\varphi\in\Gamma$ and there exist $\tau_0,\varpi_0\in R$ such that $\tau_0\leqslant\zeta(\tau_0,\varpi_0)$ and $\varpi_0\geqslant\zeta(\varpi_0,\tau_0)$. If ζ is continuous, then ζ has a coupled fixed point $(\tau,\varpi)\in R\times R$.

If $r_4=r_5=r_6=0$ in Theorem 3.1, then we get the following result:

Theorem 3.2 Let a mapping $\zeta:R\times R\to R$ having the mixed monotone property on $(R,d,\leqslant)$, where $(R,d,\leqslant)$ is a partially ordered complete metric space, satisfy the condition

$$\int_0^{d(\zeta(\tau,\varpi),\zeta(\eta,\rho))}\varphi(t)dt\leqslant r_1\int_0^{d(\tau,\eta)}\varphi(t)dt+r_2\int_0^{d(\varpi,\rho)}\varphi(t)dt+r_3\int_0^{\frac{d(\tau,\zeta(\tau,\varpi))d(\eta,\zeta(\eta,\rho))}{d(\tau,\eta)}}\varphi(t)dt$$

for all $\tau,\varpi,\eta,\rho\in R$ with $\tau\geqslant\eta$ and $\varpi\leqslant\rho$, where $r_i\in[0,1]$ for $i=1,2,3$ such that $\sum_{i=1}^3 r_i<1$, $\varphi\in\Gamma$ and there exist $\tau_0,\varpi_0\in R$ such that $\tau_0\leqslant\zeta(\tau_0,\varpi_0)$ and $\varpi_0\geqslant\zeta(\varpi_0,\tau_0)$. If ζ is continuous, then ζ has a coupled fixed point $(\tau,\varpi)\in R\times R$.

If $r_3=0$ in Theorem 3.2, then we have the following result:

Theorem 3.3 Let a mapping $\zeta:R\times R\to R$ having the mixed monotone property on $(R,d,\leqslant)$, where $(R,d,\leqslant)$ is a partially ordered complete metric space, satisfy the condition

$$\int_0^{d(\zeta(\tau,\varpi),\zeta(\eta,\rho))}\varphi(t)dt \leqslant r_1\int_0^{d(\tau,\eta)}\varphi(t)dt + r_2\int_0^{d(\varpi,\rho)}\varphi(t)dt$$

for all $\tau, \varpi, \eta, \rho \in R$ with $\tau \geqslant \eta$ and $\varpi \leqslant \rho$, where $r_1, r_2 \in [0,1]$ such that $r_1 + r_2 < 1$, $\varphi \in \Gamma$ and there exist $\tau_0, \varpi_0 \in R$ such that $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0)$. If ζ is continuous, then ζ has a coupled fixed point $(\tau, \varpi) \in R \times R$.

If $r_1 = r_2 = \lambda$ in Theorem 3.3, then we have the following result:

Theorem 3.4 Let a mapping $\zeta : R \times R \to R$ having the mixed monotone property on $(R, d, \leqslant)$, where $(R, d, \leqslant)$ is a partially ordered complete metric space, satisfy the condition

$$\int_0^{d(\zeta(\tau,\varpi),\zeta(\eta,\rho))}\varphi(t)dt \leqslant \lambda[\int_0^{d(\tau,\eta)}\varphi(t)dt + \int_0^{d(\varpi,\rho)}\varphi(t)dt]$$

for all $\tau, \varpi, \eta, \rho \in R$ with $\tau \geqslant \eta$ and $\varpi \leqslant \rho$, where $\lambda \in [0,1]$ such that $\lambda < 1$, $\varphi \in \Gamma$ and there exist $\tau_0, \varpi_0 \in R$ such that $\tau_0 \leqslant \zeta(\tau_0, \varpi_0)$ and $\varpi_0 \geqslant \zeta(\varpi_0, \tau_0)$. If ζ is continuous, then ζ has a coupled fixed point $(\tau, \varpi) \in R \times R$.

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