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Operations and vector spaces on m-topological transformation semigroup

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Abstract. This research paper introduces the concept of m-topological transformation semigroup spaces and explores their fundamental set operations. Additionally, the study explores the properties of vector spaces defined on m-topological transformation semigroup spaces, examining how algebraic structures interact with the underlying spaces.

Keywords: Vector spaces, topological space, full transformation semigroup, partial transformation semigroup.

2010 AMS Subject Classification: 46A19, 47L07.

1. Introduction and preliminaries

Algebraic topology is one approach to studying algebraic structures in mathematics, specifically, the relationship between topological spaces and group theory. It utilizes algebraic tools to investigate topological spaces, aiming to transform topological problems into more amenable algebraic forms, such as groups. In semigroup theory, finite transformation semigroups play a significant role. These semigroups are functions that operate on a given set and preserve its structure Ganyushkin and Mazorchuk [3]. The standard definitions of terms regarding metric spaces can be found in Kreyzig [2], while topological space is covered in Sidney [4] and Munkre [5], and multiset topology by Girish [6]. Adeniji et al. [1] discussed the utilization of the Hamming distance function technique to assign a metric to any transformation within the semigroup. Their researchers considered the distance function within the full transformation semigroup as the aggregate sum of positional differences between its elements. Francis [7] considered the number of

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equivalent and non-equivalent homeomorphic topological spaces for $n \leqslant 4$. It is also possible to translate algebraic concepts into the realm of topology. Building upon this idea, our research introduces a novel concept called m-topological transformation semigroups, which leverages topology to address problems in semigroup theory. Our primary focus is on transformation semigroups and by employing topological principles, which is aimed at providing solutions within this domain. Transformation semigroups are mainly functions from a given set to itself and one of the important transformation in semigroup theory is the finite partial transformations semigroups. m-topological transformation semigroup spaces, denoted by M_δ, are set of transformation semigroups that admits the properties of topological spaces. Let consider two arbitrary transformations α and β in M_δ such that $\alpha = \begin{pmatrix} x_1 & x_2 & x_3 & ... & x_n \\ \alpha x_1 & \alpha x_2 & \alpha x_3 & ... & \alpha x_n \end{pmatrix}$ and $\beta = \begin{pmatrix} y_1 & y_2 & y_3 & ... & y_n \\ \beta y_1 & \beta y_2 & \beta y_3 & ... & \beta y_n \end{pmatrix}$.

In this paper, we explore the concept of vector spaces based on the absolute difference between two vectors: x and αx, where α is a transformation factor. We define a vector space $V = (\upsilon_1, \upsilon_2, \upsilon_3, ..., \upsilon_n)^T$, where $\upsilon_i = |x_i - (\alpha x)_i|$ and $\nu_i = |y_i - (\beta y)_i|$. To refer to the spaces of m-topological full transformation semigroups, we use the notation M_{T_n}. Additionally, we denote the spaces of m-topological partial transformation semigroups as M_{P_n}. The intersection, union and complement of α and β are given as follows $\alpha \cap \beta = \min\{\alpha x, \beta y\}$, $\alpha \cup \beta = \max\{\alpha x, \beta y\}$ and $\alpha^c = |n - \alpha x|$ where $n = \max(X)$ for $x \in Dom(\alpha)$ and $\alpha x \in Im(\alpha)$. In the context of the paper, if $\alpha \in M_\delta$ is an open set in the m-topological transformation semigroup space, then its complement $\alpha^c \in M_\delta$ is a closed set. We call $\{\alpha_i : i \in I\}$ and $\{\beta_i : i \in I\}$ an indexed family of sets of M_{T_n} and M_{P_n}, respectively sometimes denoted as $\{\alpha_i\}_{i\in I}$ and $\{\beta_i\}_{i\in I}$.

Definition 1.1 (Full and partial transformation semigroup). Let δ be the chart on $X_n = \{1, 2, 3, ...\}$. The map $\alpha : Dom(\alpha) \subseteq X_n \to Im(\alpha) \subseteq X_n$ is said to be a full transformation semigroup; denoted by T_n if $Dom(\alpha) = X_n$, and partial transformation if $Dom(\alpha) \subseteq X_n$; denoted by P_n.

Definition 1.2 (m-topological transformation semigroup). A set of transformations in δ is said to be m-topological transformation semigroup (shorten as M_δ) if it satisfies the following properties:
(i) α and $\emptyset$ are in M_δ;
(ii) α is closed under arbitrary union in M_δ;
(iii) α is closed under finite intersection in M_δ.

Definition 1.3 An m-topological transformation semigroup vector space is a set together with two operations of vector addition $\oplus$ and scalar multiplication ($\otimes$) satisfying the following conditions for $\upsilon, \nu, \omega \in M_\delta$ and $\lambda, \Lambda \in \mathbb{R}$:
(i) Vector addition: $\upsilon \oplus \nu \in M_\delta$;
(ii) Scalar multiplication: $\lambda \otimes \upsilon = \upsilon \otimes \lambda \in M_\delta$;
(iii) Scalar multiplication by 1: If the scalar $\lambda = 1$, then $\lambda \otimes \upsilon = \upsilon$;
(iv) Scalar distributive property: $(\lambda + \Lambda) \otimes \upsilon = \lambda \otimes \upsilon + \Lambda \otimes \upsilon$;
(v) Scalar multiplication is distributive over vector addition: $\lambda(\upsilon \oplus \nu) = \lambda\upsilon \oplus \lambda\nu$;
(vi) Scalar multiplication is associative: $(\lambda\Lambda) \otimes \upsilon = \lambda \otimes (\Lambda\upsilon)$;
(vii) Vector addition is commutative: $\upsilon \oplus \nu = \nu \oplus \upsilon$;
(viii) Vector addition is associative: $(\upsilon \oplus \nu) \oplus \omega = \nu \oplus (\upsilon \oplus \omega)$.

Addition and multiplication are performed in (mod r), where $r = n + 1$.

To ensure clarity and organization, we begin by presenting a list of elements in m-topological partial transformation semigroups M_{P_n}.

For $n = 2$, we have

$$M_{P_2} = \left\{ \begin{matrix} \begin{pmatrix} 1 & 2 \\ \emptyset & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 2 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \end{matrix} \right\}. \tag{1}$$

***Example* 1.4** Consider an arbitrary M_{P_2} on $X = \{1, 2\}$

$$M_{P_2} = \left\{ \begin{pmatrix} 1 & 2 \\ \emptyset & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\}$$

represented as $\{\beta_1, \beta_2, \beta_3, \alpha_1\}$, respectively.

Case 1.

$$\beta_2 \cup \beta_3 = \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix} \cup \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix} = \beta_2.$$

Case 2.

$$\beta_2 \cap \beta_3 = \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix} \cap \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix} = \beta_3.$$

Case 3.

$$\beta_3 \cup \alpha_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cup \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \alpha_1.$$

$\beta_2, \beta_3 \in M_{P_4}$ is a full transformation while β_3 is partial.

$$(M_{P_2})^c = \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix} \right\}.$$

2. Main results

Theorem 2.1 Let the triple (X, δ, M_δ) be m-topological transformation semigroup, where $\{\alpha_i\}_{i\in I}$ is a family of subsets of M_{T_n} and $\{\beta_i\}_{i\in I}$ is a family of subsets of M_{P_n}. Then the following holds:

(i) $\bigcap_{i\in I} \alpha_i = \alpha$;
(ii) $\bigcup_{i\in I} \alpha_i = \alpha$;
(iii) $\bigcap_{i\in I} \beta_i = \beta$

for $\alpha \in M_{T_n}$ and $\beta \in M_{P_n}$

Proof. (i) Let $\alpha x \in \alpha_m$ and $\alpha y \in \alpha_{m+1}$ for all $m \in I$. Since α_i is a family of subsets, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n \subset M_{T_n}$. By definition, $\alpha \cap \beta = \min\{\alpha x, \beta y\}$. The finite intersections $\alpha_1 \cap \alpha_2 \cap \alpha_3 \cap \alpha_4 \cap \ldots \cap \alpha_n = \alpha$. Since α_m and α_{m+1} are both full transformations, by

intersection $\alpha_m \cap \alpha_{m+1} = \alpha$, the minimum elements $\alpha y \in \alpha$ and $\alpha y \neq \emptyset$. Hence, α is itself a full transformation and therefore, $\alpha \in M_{T_n}$.
(ii) Suppose $\alpha x \in \alpha_m$ and $\alpha y \in \alpha_{m+1}$ where $m \in I$. Since α_i is a family of subsets, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...\alpha_n \subset M_{T_n}$. By definition, $\alpha \cup \beta = \max\{\alpha x, \beta y\}$. Similarly, we took the finite union $\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup ... \cup \alpha_n = \alpha$. Since α_m and α_{m+1} are both full transformations, by intersection $\alpha_m \cap \alpha_{m+1} = \alpha$, the maximum elements $\alpha y \in \alpha$ and $\alpha y \neq \emptyset$. Hence, α is itself a full transformation and therefore, $\alpha \in M_{T_n}$.
(iii) follows from (i). ■

Theorem 2.2 Let α be M_{T_n} and $\{\beta_i\}_{i\in I}$ be a family of subsets of M_{P_n}. Then, $\bigcup_{i\in I} \beta_i = \beta$ if and only if there exists a common point in $\beta_1, \beta_2, \ldots, \beta_n$ such that the union has an empty point, and $\bigcup_{i\in I} \beta_i = \alpha$ if the union has no not empty point.

Proof. Case 1: $\bigcup_{i\in I} \beta_i = \beta$ if and only if there exists a common point in $\beta_1, \beta_2, \ldots, \beta_n$ such that the union has an empty point. Assume that $\bigcup_{i\in I} \beta_i = \beta$. Since the union of the sets β_i is equal to β, every element of β must belong to at least one of the sets β_i. Therefore, there exists at least one common point in $\beta_1, \beta_2, \ldots, \beta_n$. Now, let's assume that the union $\bigcup_{i\in I} \beta_i$ does not have an empty point. This means that there exists an element, let's αx, such that αx belongs to every set β_i. However, since there is a common point among the sets β_i, this contradicts the assumption that the union has an empty point. Therefore, the forward direction is proven. Assume that there exists a common point, let's call it αx, in $\beta_1, \beta_2, \ldots, \beta_n$ such that the union has an empty point. Let's denote the empty point as $\emptyset$. Since αx is a common point in all the sets β_i, αx must belong to the union $\bigcup_{i\in I} \beta_i$. Moreover, since the union has an empty point, $\emptyset$ is also an element of the union. Therefore, every element in β and $\emptyset$ belongs to $\bigcup_{i\in I} \beta_i$, which implies that $\bigcup_{i\in I} \beta_i = \beta$.

Case 2: $\bigcup_{i\in I} \beta_i = \alpha$ if and only if the union has no non-empty point. Assume that $\bigcup_{i\in I} \beta_i = \alpha$. Since the union of the sets β_i is equal to α, every element of α must belong to at least one of the sets β_i. This implies that every element in α is in the union $\bigcup_{i\in I} \beta_i$. Therefore, the union has no non-empty point. Assume that the union has no non-empty point. This means that the union $\bigcup_{i\in I} \beta_i$ is either empty ($\emptyset$) or consists only of empty sets. If the union is empty, then $\bigcup_{i\in I} \beta_i = \emptyset$, which is not equal to α. Therefore, we consider the case where the union consists only of empty sets. In this case, every element in α is an empty set, which implies that every element in α is also in $\bigcup_{i\in I} \beta_i$. Hence, $\bigcup_{i\in I} \beta_i = \alpha$. ■

Theorem 2.3 Let the triple (X, δ, M_δ) be m-topological transformation semigroup, where $\{\alpha_i\}_{i\in I}$ is a family of subsets of M_{T_n} and $\{\beta_i\}_{i\in I}$ is a family of subsets of M_{P_n}. Then the following holds:

(i) $\left(\bigcap_{i\in I} \alpha_i\right) \cup \left(\bigcap_{i\in I} \beta_i\right) = \alpha$;

(ii) $\left(\bigcap_{i\in I} \alpha_i\right) \cap \left(\bigcap_{i\in I} \beta_i\right) = \beta$;

(iii) $\left(\bigcup_{i\in I} \alpha_i\right) \cup \left(\bigcup_{i\in I} \beta_i\right) = \alpha$;

(iv) $\left(\bigcup_{i\in I} \alpha_i\right) \cap \left(\bigcup_{i\in I} \beta_i\right) = \beta$

for $\alpha \in M_{T_n}$ and $\beta \in M_{P_n}$.

Proof. (i) From Theorem 2.1, we have that $\left(\bigcap_{i\in I}\alpha_i\right) = \alpha$. Since $\alpha\cup\beta = \max\{\alpha x, \beta y\}$ and α has the maximum point of the union, $\alpha\cup\beta = \alpha$
(ii) follows (i) with β with the minimum point of the intersection.
(iii) Suppose $\left(\bigcup_{i\in I}\beta_i\right) = \beta$, we have $\alpha\cup\beta = \alpha$
(iv) If $\left(\bigcup_{i\in I}\beta_i\right) = \beta$, then $\alpha\cap\beta = \beta$. Suppose $\left(\bigcup_{i\in I}\beta_i\right) \neq \beta$. We therefore prove by contradiction, since $\left(\bigcup_{i\in I}\beta_i\right) \neq \beta$, then follows that $\beta_1\cup\beta_2\cup\beta_3\cup....\cup\beta_n \neq \beta$. This implies that $\max\{\beta x, \beta y\} \neq \emptyset$, and $\left(\bigcup_{i\in I}\beta_i\right) = \alpha$. Since $\left(\bigcup_{i\in I}\alpha_i\right) = \alpha$, we have $\alpha\cap\alpha = \alpha$ ■

Theorem 2.4 Let $\{\upsilon_i\}_{i\in I}$ and $\{\nu_i\}_{i\in I}$ be the family of m-topological transformation semigroup vector. Then the following equations hold:

(i) $\bigcup\limits_{i=1}^{n}(\upsilon_i\cap\nu_i) = \bigcup\limits_{i=1}^{n}\upsilon_i\cap\bigcup\limits_{i=1}^{n}\nu_i$;
(ii) $\bigcap\limits_{i=1}^{n}(\upsilon_i\cup\nu_i) = \bigcap\limits_{i=1}^{n}\upsilon_i\cup\bigcap\limits_{i=1}^{n}\nu_i$.

Proof. (i) We first prove that $\bigcup\limits_{i=1}^{n}(\upsilon_i\cap\nu_i) \subseteq \bigcup\limits_{i=1}^{n}\upsilon_i\cap\bigcup\limits_{i=1}^{n}\nu_i$. Let $r\in\bigcup\limits_{i=1}^{n}(\upsilon_i\cap\nu_i)$, where r is the minimum value of $(\upsilon_i\cap\nu_i)$. Then there exists $j\in I$ such that $r\in\upsilon_j\cap\nu_j$. Thus, $r\in\upsilon_j$ and $r\in\nu_j$. It follows by definition that $r\bigcup\limits_{i=1}^{n}\in\upsilon_i$ and $r\bigcup\limits_{i=1}^{n}\in\nu_i$. Since r is in both, we have

$$\bigcup_{i=1}^{n}(\upsilon_i\cap\nu_i) \subseteq \bigcup_{i=1}^{n}\upsilon_i\cap\bigcup_{i=1}^{n}\nu_i. \tag{2}$$

Similarly, to prove that $\bigcup\limits_{i=1}^{n}\upsilon_i\cap\bigcup\limits_{i=1}^{n}\nu_i \subseteq \bigcup\limits_{i=1}^{n}(\upsilon_i\cap\nu_i)$, let $r\in\bigcup\limits_{i=1}^{n}\upsilon_i\cap\bigcup\limits_{i=1}^{n}\nu_i$. this implies that $r\in\bigcup\limits_{i=1}^{n}\upsilon_i$ and $r\in\bigcup\limits_{i=1}^{n}\nu_i$ there exist $j\in I$ such that $r\in\upsilon_j,\nu_j$. Therefore, $r\in\upsilon_j\cap\nu_j$. Let $r\in\bigcup\limits_{i=1}^{n}\upsilon_i\cap\bigcup\limits_{i=1}^{n}\nu_i$

$$\bigcup_{i=1}^{n}\upsilon_i\cap\bigcup_{i=1}^{n}\nu_i \subseteq \bigcup_{i=1}^{n}(\upsilon_i\cap\nu_i). \tag{3}$$

(ii) We first prove that $\bigcap\limits_{i=1}^{n}(\upsilon_i\cup\nu_i) \subseteq \bigcap\limits_{i=1}^{n}\upsilon_i\cup\bigcap_{i=1}^{n}\nu_i$. Therefore, let $t\in\bigcap\limits_{i=1}^{n}(\upsilon_i\cup\nu_i)$, where t is the maximum value of $(\upsilon_i\cup\nu_i)$. Then there exists $j\in I$ such that $t\in\upsilon_j\cup\nu_j$. Thus, $t\in\upsilon_j$ and $t\in\nu_j$. It follows by definition that $t\bigcap\limits_{i=1}^{n}\in\upsilon_i$ and $t\bigcap\limits_{i=1}^{n}\in\nu_i$. Since t is in both, we have

$$\bigcap_{i=1}^{n} (\upsilon_i \cup \nu_i) \subseteq \bigcap_{i=1}^{n} \upsilon_i \cup \bigcap_{i=1}^{n} \nu_i. \tag{4}$$

We prove that $\bigcap_{i=1}^{n} \upsilon_i \cup \bigcap_{i=1}^{n} \nu_i \subseteq \bigcap_{i=1}^{n} (\upsilon_i \cup \nu_i)$. Let $t \in \bigcap_{i=1}^{n} \upsilon_i \cup \bigcap_{i=1}^{n} \nu_i$. This implies that $r \in \bigcap_{i=1}^{n} \upsilon_i$ and $t \in \bigcap_{i=1}^{n} \nu_i$. Then, there exists $j \in I$ such that $t \in \upsilon_j, \nu_j$. Therefore, $t \in \upsilon_j \cup \nu_j$. Since $t \in \bigcap_{i=1}^{n} \upsilon_i \cup \bigcap_{i=1}^{n} \nu_i$, we have

$$\bigcap_{i=1}^{n} \upsilon_i \cup \bigcap_{i=1}^{n} \nu_i \subseteq \bigcap_{i=1}^{n} (\upsilon_i \cup \nu_i). \tag{5}$$

The desired result for (i) follows the following equations (2) and (3), and (ii) follows from (4) and (5) ■

Theorem 2.5 Let the triple (X, δ, M_δ) be m-topological transformation semigroup, where $\{\alpha\}_{i\in I}$ be indexed family of m-topological full transformation semigroup. Then,

(i) $\left(\bigcup_{i\in I} \alpha_i\right)^c = \bigcap_{i\in I} {\alpha_i}^c$;

(ii) $\left(\bigcap_{i\in I} \alpha_i\right)^c = \bigcup_{i\in I} {\alpha_i}^c$.

Proof.

(i) First, let's take a point αx in the left-hand side transformations, which means αx is not in the union of all α_i, i.e., $\alpha x \notin \bigcup_{i\in I} \alpha_i$. This implies that αx must be in the complement of the union, which is the intersection of the complements, i.e., $\alpha x \in \bigcap_{i\in I} \alpha_i^c$. Therefore, we have shown that every element in the left-hand side transformation is also in the right-hand side transformation.

Now, let's take another point αy in the right-hand side transformation, which means αy is not in any of the α_i, i.e., $\alpha y \notin \alpha_i$ for all $i \in I$. This implies that αy must be in the complement of each α_i, i.e., $\alpha y \in {\alpha_i}^c$ for all $i \in I$. Therefore, αy is not in the union of all α_i, i.e., $\alpha y \notin \bigcup_{i\in I} \alpha_i$. This means that y is in the complement of the union, which is the left-hand side set, i.e., $\alpha y \in \left(\bigcup_{i\in I} \alpha_i\right)^c$. Therefore, we have shown that every element in the right-hand side transformation is also in the left-hand side set. Since we have shown double containment, we can conclude that the left-hand side transformation and the right-hand side transformation are equal, i.e., $\left(\bigcup_{i\in I} \alpha_i\right)^c = \bigcap_{i\in I} {\alpha_i}^c$.

(ii) To prove this, we need to show that an element belongs to the left-hand side if and only if it belongs to the right-hand side. Let αx be an point of $\left(\bigcap_{i\in I} \alpha_i\right)^c$. This means that αx is not in the intersection of all the α_i, that is, αx does not belong to each α_i. Therefore, there exists at least one $i \in I$ such that αx is not in α_i. Hence, αx is in ${\alpha_i}^c$ for that particular i. Since this holds for at least one $i \in I$, αx is in the union of all the ${\alpha_i}^c$ for $i \in I$. Thus, we have shown that αx belongs to the right-hand side.

Conversely, let αx be an element of $\bigcup_{i \in I} {\alpha_i}^c$. This means that αx belongs to at least one ${\alpha_i}^c$ for some $i \in I$. Therefore, αx is not in α_i. Since this holds for at least one $i \in I$, αx is not in the intersection of all the α_i, that is, αx belongs to $\left(\bigcap_{i \in I} \alpha_i\right)^c$. Thus, we have shown that αx belongs to the left-hand side. Therefore, we have shown that an element belongs to the left-hand side if and only if it belongs to the right-hand side. Hence, we have proven that $\left(\bigcap_{i \in I} \alpha_i\right)^c = \bigcup_{i \in I} {\alpha_i}^c$. ■

Theorem 2.6 Let the triple (X, δ, M_δ) be m-topological transformation semigroup, where $\{\beta_i\}_{i \in I}$ is a family of subsets of M_{P_n}. Then,

(i) $\left(\bigcap_{i \in I} \beta_i\right)^c = \bigcup_{i \in I} {\beta_i}^c$;

(ii) $\left(\bigcup_{i \in I} \beta_i\right)^c = \bigcap_{i \in I} {\beta_i}^c$.

Proof. Proof follows from Theorem 2.5. ■

Theorem 2.7 Let $\{\upsilon_i\}_{i \in I}$ and $\{\nu_i\}_{i \in I}$ be the family of m-topological transformation semigroup vector. Then the following equations hold:

(i) $\bigcup_{i=1}^{n} \upsilon_i \oplus \bigcup_{i=1}^{n} \nu_i \leqslant \bigcup_{i=1}^{n} (\upsilon_i \oplus \nu_i)$;

(ii) $\bigcup_{i=1}^{n} \upsilon_i \otimes \bigcup_{i=1}^{n} \nu_i \leqslant \bigcup_{i=1}^{n} (\upsilon_i \otimes \nu_i)$;

(iii) $\bigcap_{i=1}^{n} \upsilon_i \oplus \bigcap_{i=1}^{n} \nu_i \geqslant \bigcap_{i=1}^{n} (\upsilon_i \oplus \nu_i)$;

(iv) $\bigcap_{i=1}^{n} \upsilon_i \otimes \bigcap_{i=1}^{n} \nu_i \geqslant \bigcap_{i=1}^{n} (\upsilon_i \otimes \nu_i)$.

Proof. (i) Let $t \in \upsilon_i$ and $s \in \nu_i$. There exists $j \in I$ such that $t \in \upsilon_j$ and $s \in \nu_j$. By definition, union gives the maximum element. Therefore $a = \bigcup_{i=1}^{n} \upsilon_i$ and $b = \bigcup_{i=1}^{n} \nu_i$, where a and b are the maximum elements. Thus, $\bigcup_{i=1}^{n} \upsilon_i \oplus \bigcup_{i=1}^{n} \nu_i = (a \oplus b) mod r = c(mod)r$ which implies that $c \leq n$.

We proceed to establish the R.H.S. Let $p \in \upsilon_i$ and $q \in \nu_i$. We have $(p+q)modr \in (\upsilon_i \oplus \nu_i)$ such that $(p+q)modr = t(mod)r$. Therefore $t(mod)r \in (\upsilon_i \oplus \nu_i)$. There exists $j \in I$ such that $t(mod)r \in (\upsilon_j \oplus \nu_j)$. Hence, $t(mod)r \in \bigcup_{i=1}^{n} (\upsilon_i \oplus \nu_i)$. Since t is the maximum singular element in $\bigcup_{i=1}^{n} (\upsilon_i \oplus \nu_i)$, then $c \leq t$, where $c \in \upsilon_i$. Hence, $\bigcup_{i=1}^{n} \upsilon_i \oplus \bigcup_{i=1}^{n} \nu_i \leqslant \bigcup_{i=1}^{n} (\upsilon_i \oplus \nu_i)$.

(ii), (iii), (iv) follows from (i). ■

Lemma 2.8 Let υ_i, ν_i and ω_i be the family of m-topological transformation semigroup vector. Then the following equations are associative:

$$\left(\bigcup_{i=1}^{n}\upsilon_i \oplus \bigcup_{i=1}^{n}\nu_i\right) \oplus \bigcup_{i=1}^{n}\omega_i = \bigcup_{i=1}^{n}\upsilon_i \oplus \left(\bigcup_{i=1}^{n}\nu_i \oplus \bigcup_{i=1}^{n}\omega_i\right),$$

$$\left(\bigcap_{i=1}^{n}\upsilon_i \otimes \bigcap_{i=1}^{n}\nu_i\right) \otimes \bigcap_{i=1}^{n}\omega_i = \bigcap_{i=1}^{n}\upsilon_i \otimes \left(\bigcap_{i=1}^{n}\nu_i \otimes \bigcap_{i=1}^{n}\omega_i\right),$$

$$\bigcap_{i=1}^{n}(\upsilon_i \oplus \nu_i) = \bigcap_{i=1}^{n}\upsilon_i \oplus \bigcap_{i=1}^{n}\nu_i,$$

$$\bigcup_{i=1}^{n}(\upsilon_i \otimes \nu_i) = \bigcup_{i=1}^{n}\upsilon_i \otimes \bigcup_{i=1}^{n}\nu_i.$$

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