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Solvability of the infinite systems of nonlinear third-order differential equations in the weighted sequence space $\mathbf{m}_\omega(\boldsymbol{\Delta}_{\mathfrak{v}}^{\varsigma}, \psi, \mathbf{q})$

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Abstract. In this work, we first introduce the concept of weighted sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$. Then, we construct a Hausdorff measure of noncompactness on this sequence space. Furthermore, by employing this measure of noncompactness we discuss the solvability of an infinite system of nonlinear third-order differential equations with initial conditions in the weighted sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$. Eventually, we demonstrate an example to show the usefulness of the obtained result.

Keywords: Infinite system of third-order boundary value problem, measure of noncompactness, Meir–Keeler condensing operator, weighted sequence space.

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1. Introduction and preliminaries

Third-order differential equations occur in some fields of physics like electromagnetic waves, the deflection of curved beams with varying cross or constant sections, gravity driven flows and three-layer beams [13]. Therefore, third-order differential equations with different initial conditions have been attracted a lot of attention during the recent several decades (see [6, 8–10, 12, 18] and the references therein). On the other hand, we encounter many problems in the mechanics, the branching processes and neural nets, and so on [7, 23]. These problems can be modelled and described using infinite systems of ordinary differential equations (IODEs). The measure of noncompactness (MNC), which was first

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introduced by Kuratowski [16], is a powerful tool for studying IODEs. In recent times, the MNC has been effectively applied in sequence spaces for some classes of differential equations [5, 15, 20, 22, 24]. Ghanenia et al. [11] studied the existing results for an infinite system of second-order BVP in the space $m(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$. Motivated by the above papers, in this work, we first introduce the concept of weighted sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$. Then, we construct a Hausdorff measure of noncompactness in this sequence space. Employing this Hausdorff MNC, we study the existence of solutions of the infinite system of third-order differential equations with initial conditions (IDE for short)

$$\begin{cases} -Z_i'''(\tau) = a_i(\tau) f_i(\tau, Z(\tau), W(\tau)), \ 0 < \tau < 1 \\ -W_i'''(\tau) = b_i(\tau) g_i(\tau, Z(\tau), W(\tau)), \ 0 < \tau < 1 \\ Z_i(0) = Z_i'(0) = 0, \ Z_i'(1) = \alpha Z_i'(\zeta), \\ W_i(0) = W_i'(0) = 0, \ W_i'(1) = \alpha W_i'(\zeta), \ i = 1, 2, \dots \end{cases} \tag{1}$$

in the weighted sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$, where $f_i, g_i \in C([0,1] \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty, \mathbb{R}_+)$, $i = 1, 2, \dots$, $0 < \zeta < 1$, $1 < \alpha < \frac{1}{\zeta}$, $a_i, b_i \in C([0,1], \mathbb{R}_+)$ such that they are different from zero on any subinterval of $[0,1]$. Eventually, we present an example illustrating the main result. Here, we preliminarily collect some definitions and auxiliary facts applied throughout this paper.

Suppose that $(\Lambda, \|\cdot\|)$ is a real Banach space containing zero element. We mean by $D(z, r)$ the closed ball centered at z with radius r. For $\emptyset \neq \mathcal{U} \subset \Lambda$, the symbols $\overline{\mathcal{U}}$ and $\mathrm{Conv}\mathcal{U}$ denote the closure and closed convex hull of $\mathcal{U}$, respectively. We denote by $\mathfrak{M}_\Lambda$ the family of all non-empty, bounded subsets of Λ and by $\mathfrak{N}_\Lambda$ its subfamily consisting of non-empty relatively compact subsets of Λ.

Definition 1.1 [1] The function $\tilde{\mu} : \mathfrak{M}_\Lambda \to \mathbb{R}_+ = [0, +\infty)$ is called a measure of noncompactness (MNC) in Λ if for any $\mathcal{U}, \mathcal{V}_1, \mathcal{V}_2 \in \mathfrak{M}_\Lambda$, the following conditions hold:

(i) $\emptyset \neq \ker \tilde{\mu} = \{\mathcal{U} \in \mathfrak{M}_\Lambda : \tilde{\mu}(\mathcal{U}) = 0\} \subseteq \mathfrak{N}_\Lambda$.
(ii) If $\mathcal{V}_1 \subset \mathcal{V}_2$, then $\tilde{\mu}(\mathcal{V}_1) \leqslant \tilde{\mu}(\mathcal{V}_2)$.
(iii) $\tilde{\mu}(\overline{\mathcal{U}}) = \tilde{\mu}(\mathrm{Conv}\mathcal{U}) = \tilde{\mu}(\mathcal{U})$.
(iv) For each $\ell \in [0,1]$, $\tilde{\mu}(\ell \mathcal{V}_1 + (1-\ell)\mathcal{V}_2) \leqslant \ell \tilde{\mu}(\mathcal{V}_1) + (1-\ell)\tilde{\mu}(\mathcal{V}_2)$.
(v) If for each natural number n $\mathcal{U}_n$ is a closed set in $\mathfrak{M}_\Lambda$, $\mathcal{U}_{n+1} \subset \mathcal{U}_n$, and $\lim_{n\to\infty} \tilde{\mu}(\mathcal{U}_n) = 0$, then $\mathcal{U}_\infty = \bigcap_{n=1}^{\infty} \mathcal{U}_n$ is non-empty.

In the sequel, $\mathfrak{M}_Y$ is the family of bounded subsets of the metric space (Y, d).

Definition 1.2 [4] Suppose that (Y, d) is a metric space. Also, suppose that $\mathcal{P} \in \mathfrak{M}_Y$. The Kuratowski MNC of $\mathcal{P}$, which is denoted by $\alpha(\mathcal{P})$, is defined by

$$\alpha(\mathcal{P}) = \inf \Big\{ \varepsilon > 0 : \mathcal{P} \subset \bigcup_{i=1}^{n} K_i, K_i \subset Y, \mathrm{diam}(K_i) < \varepsilon \ (i = 1, \dots, n); \ n \in \mathbb{N} \Big\},$$

where $\mathrm{diam}(K_i) = \sup\{d(\varsigma, \nu) : \varsigma, \nu \in K_i\}$.

The Hausdorff MNC (ball MNC) of the bounded set $\mathcal{P}$, which is denoted by $\beta(\mathcal{P})$, is defined by

$$\beta(\mathcal{P}) = \inf\Big\{\varepsilon > 0 : \mathcal{P} \subset \bigcup_{i=1}^{n} D(z_i, r_i), z_i \in Y, r_i < \varepsilon \ (i = 1, \ldots, n);\ n \in \mathbb{N}\Big\}.$$

Here, we quote the following result contained in [4].

Lemma 1.3 Let (Y, d) be a metric space and $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{M}_Y$. Then

(*i*) $\beta(\mathcal{P}) = 0 \Leftrightarrow \mathcal{P}$ is totally bounded,
(*ii*) $\mathcal{P}_1 \subset \mathcal{P}_2 \Rightarrow \beta(\mathcal{P}_1) \leqslant \beta(\mathcal{P}_2)$,
(*iii*) $\beta(\overline{\mathcal{P}}) = \beta(\mathcal{P})$,
(*iv*) $\beta(\mathcal{P}_1 \cup \mathcal{P}_2) = \max\{\beta(\mathcal{P}_1), \beta(\mathcal{P}_2)\}$.

The notion of Meir–Keeler contractive mapping was first introduced by Meir and Keeler [19]. They studied some fixed point theorems using such mappings. After that Aghajani et al. [2] generalized this notion via MNC.

Definition 1.4 [2] Suppose that Λ is a Banach space and $\emptyset \neq \mathfrak{F} \subset \Lambda$. Also, suppose that $\tilde{\mu}$ is an arbitrary MNC on Λ. An operator $S : \mathfrak{F} \to \mathfrak{F}$ is said to be a Meir–Keeler condensing operator if for each $\varepsilon > 0$, $\delta > 0$ exists such that $\varepsilon \leqslant \tilde{\mu}(\mathcal{U}) < \varepsilon + \delta$ implies $\tilde{\mu}(S(\mathcal{U})) < \varepsilon$ for each bounded subset $\mathcal{U}$ of $\mathfrak{F}$.

Theorem 1.5 [2] Assume that $\mathfrak{D}$ is a non-empty closed, bounded and convex subset of a Banach space Λ, $\tilde{\mu}$ is a MNC in Λ and $S : \mathfrak{D} \to \mathfrak{D}$ is a continuous Meir–Keeler condensing operator. Then S has fixed point and the set of fixed points of S is compact.

Suppose that $K = [0, s]$ is a closed bounded interval, and Λ is a Banach space. Consider the Banach space $C(K, \Lambda)$ with the norm $\|z\|_{C(K,\Lambda)} := \sup\{\|z(\rho)\| : \rho \in K,\ z \in C(K, \Lambda)\}$.

Proposition 1.6 [4] Suppose that $\Omega \subseteq C(K, \Lambda)$ is equicontinuous and bounded. Then $\tilde{\mu}(\Omega(.))$ is continuous on K and

$$\tilde{\mu}(\Omega) = \sup_{\rho \in K} \tilde{\mu}(\Omega(\rho)), \quad \tilde{\mu}\Big(\int_0^{\rho} \Omega(\varrho) d\varrho\Big) \leqslant \int_0^{\rho} \tilde{\mu}(\Omega(\varrho)) d\varrho.$$

We terminate this section with a remark concerning the construction of a MNC in a product space.

Remark 1 *[3] Suppose that $\widetilde{\mu}$ is a MNC on a Banach space Λ. Then, $\overline{\mu}(\mathcal{U}) = \widetilde{\mu}(\mathcal{U}_1) + \widetilde{\mu}(\mathcal{U}_2)$ is a MNC in the product space $\Lambda \times \Lambda$ where $\mathcal{U}_1, \mathcal{U}_2$ denote the natural projections of $\mathcal{U}$.*

2. Weighted Sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$

Suppose that $\mathfrak{S}$ denote the set of real sequences and c_0 is the set of null sequences $z = (z_k)$ with complex terms, normed by $\|z\|_\infty = \sup_{k \in \mathbb{N}} |z_k|$. Let $1 \leqslant q < \infty$. By a weight we mean a positive, measurable, and locally q-summable function on the locally compact group $\mathbb{Z}$. Assume that $\mathfrak{F}$ is the family of finite subsets of different natural numbers. For each element ϑ of $\mathfrak{F}$, we consider the sequence $c(\vartheta) = (c_n(\vartheta))$, where the terms of the sequence are given by $c_n(\vartheta) = 1$ if $n \in \vartheta$ and $c_n(\vartheta) = 0$, otherwise. Moreover, take $\mathfrak{F}_\varrho = \{\vartheta \in \mathfrak{F} : \sum_{n=1}^{\infty} c_n(\vartheta) \leqslant \varrho\}$ and

$$\Psi = \{\psi = (\psi_k) \in \mathfrak{S} : \ \psi_1 > 0, \ \Delta\psi_k \geqslant 0 \text{ and } \Delta\big(\frac{\psi_k}{k}\big) \leqslant 0 \ (k = 1, 2, \ldots)\},$$

when $\Delta\psi_k = \psi_k - \psi_{k-1}$ [21]. Now, suppose that $\varsigma \in \mathbb{N}$, $\mathfrak{v} = (\mathfrak{v}_k)$ is a sequence of nonzero complex numbers and $\psi \in \Psi$. The weighted sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$ is defined by

$$m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q) = \{z = (z_k) \in \mathfrak{S} : \ \sup_{\varrho_1 \geqslant 1} \sup_{\vartheta \in \mathfrak{F}_{\varrho_1}} \big(\frac{1}{\psi_{\varrho_1}} \sum_{k \in \vartheta} |\Delta_{\mathfrak{v}}^{\varsigma} z_k|^q \omega_k^q\big) < \infty, \ 1 \leqslant q < \infty\},$$

where ω is a weight, $\omega_k = \omega(k)$ for each $k \in \vartheta$ and

$$\begin{aligned} \Delta_{\mathfrak{v}}^{0} z_k &= \mathfrak{v}_k z_k, \\ \Delta_{\mathfrak{v}}^{1} z_k &= \mathfrak{v}_k z_k - \mathfrak{v}_{k+1} z_{k+1}, \\ \Delta_{\mathfrak{v}}^{\varsigma} z_k &= \Delta_{\mathfrak{v}}^{\varsigma-1} z_k - \Delta_{\mathfrak{v}}^{\varsigma-1} z_{k+1}, \end{aligned}$$

such that

$$\Delta_{\mathfrak{v}}^{\varsigma} z_k = \sum_{i=0}^{\varsigma} (-1)^i \begin{bmatrix} \varsigma \\ i \end{bmatrix} \mathfrak{v}_{k+i} z_{k+i}.$$

Similar to procedure presented in [25], we get the following result.

Theorem 2.1 Suppose that $\psi \in \Psi$ and $1 \leqslant q < \infty$. Then the weighted sequence space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$ is a Banach space with the norm given by

$$\|z\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)} = \sum_{i=1}^{\varsigma} |z_i| \omega_i + \sup_{\varrho_1 \geqslant 1} \sup_{\vartheta \in \mathfrak{F}_{\varrho_1}} \big(\frac{1}{\psi_{\varrho_1}} \sum_{k \in \vartheta} |\Delta_{\mathfrak{v}}^{\varsigma} z_k|^q \omega_k^q\big)^{\frac{1}{q}}.$$

From now on, it is supposed that $1 \leqslant q < \infty$. We describe the Hausdorff MNC χ in the Banach space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$. For, we quote the following result.

Theorem 2.2 [21] Suppose that Υ is a normed space and $\emptyset \neq \mathcal{P} \subset \Upsilon$ is bounded, where Υ is c_0 or l_q (the space of all absolutely q-summable series). Also, suppose that $T_n : \Upsilon \to \Upsilon$ is the operator given by $T_n(z) = (z_0, z_1, \ldots, z_n, 0, \ldots)$, then

$$\chi(\mathcal{P}) = \lim_{n \to \infty} \{\sup_{z \in \mathcal{P}} \|(I - T_n)z\|\}.$$

Hence, for $\mathcal{P} \in \mathfrak{M}_{l_q}$, we get

$$\chi(\mathcal{P}) = \lim_{n \to \infty} \{\sup_{z \in \mathcal{P}} (\sum_{k \geqslant n} |z_k|^q)^{\frac{1}{q}}\}.$$

Theorem 2.3 Suppose that $\emptyset \neq \mathcal{P} \subset m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)$ is bounded. Then the Huasdorff

MNC χ on $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$ can be defined as the following form:

$$\chi(\mathcal{P}) := \lim_{n\to\infty}\Big\{\sup_{z\in\mathcal{P}}\big(\sup_{\varrho_1\geqslant n}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}z_k|^q\omega_k^q\big)^{\frac{1}{q}}\big)\Big\}. \tag{2}$$

Proof. It can be achieved with slight modification from [11, Theorem 2.3]. ■

We terminate this section by describing the unique positive solution of the IDE (1). Set $I=[0,1]$. Suppose that $C^3(I,\mathbb{R})$ is the space of functions with continuous third derivative defined on I. According to [17],

$$(Z,W)=\Big((Z_i),(W_i)\Big)\in\Big(C^3(I,\mathbb{R}_+)\Big)^{\infty}\times\Big(C^3(I,\mathbb{R}_+)\Big)^{\infty}$$

is a solution of (1) if and only if (Z,W) is a solution of the following infinite system of integral equations

$$\begin{cases} Z(\tau)=\Big(\int_0^1 A(\tau,\varrho)a_i(\varrho)f_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big),\\ W(\tau)=\Big(\int_0^1 A(\tau,\varrho)b_i(\varrho)g_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big),\end{cases} \tag{3}$$

when the Green's function A associated with (1) is given by

$$A(\tau,\varrho)=\frac{1}{2(1-\alpha\zeta)}\begin{cases}(2\tau\varrho-\varrho^2)(1-\alpha\zeta)+\tau^2\varrho(\alpha-1), & \varrho\leqslant\min\{\zeta,\tau\},\\ \tau^2(1-\alpha\zeta)+\tau^2\varrho(\alpha-1), & \tau\leqslant\varrho\leqslant\zeta,\\ (2\tau\varrho-\varrho^2)(1-\alpha\zeta)+\tau^2(\alpha\zeta-\varrho), & \zeta\leqslant\varrho\leqslant\tau,\\ \tau^2(1-\varrho), & \max\{\zeta,\tau\}\leqslant\varrho.\end{cases} \tag{4}$$

Now, we reveal a property of the function A which will be needed later.

Lemma 2.4 [14] For all $(\tau,\varrho)\in I\times I$, $0\leqslant A(\tau,\varrho)\leqslant\beta(\tau)$, when $\beta(\tau)=\frac{1+\alpha}{1-\alpha\zeta}\varrho(1-\varrho)$.

3. Solvability of infinite systems of third-order differential equations in $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$

In this section, we establish some sufficient conditions to discuss the existence of solutions of IDE (1) in the space $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$.

Here, we consider some assumptions.
$(B1)$ Suppose that $f_i,g_i\in C(I\times\mathbb{R}_+^{\infty}\times\mathbb{R}_+^{\infty},\mathbb{R}_+)$, $i\in\mathbb{N}$. The mapping $\Lambda: I\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\to m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$ is defined by

$$(\varrho,Z(\varrho),W(\varrho))\to\Lambda(Z,W)(\varrho)=\bigg(\Big(f_i(\varrho,Z(\varrho),W(\varrho))\Big),\Big(g_i(\varrho,Z(\varrho),W(\varrho))\Big)\bigg)$$

in which the family $\Big(\Lambda(Z,W)(\varrho)\Big)_{\varrho\in I}$ is equicontinuous at any point of $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$.

$(B2)$ The following inequalities hold:

$$|f_k(\varrho, Z(\varrho), W(\varrho))| \leqslant |h_k(\varrho)|(|Z_k(\varrho)| + |W_k(\varrho)|), k \in \mathbb{N},$$
$$|g_k(\varrho, Z(\varrho), W(\varrho))| \leqslant |h_k(\varrho)|(|Z_k(\varrho)| + |W_k(\varrho)|), k \in \mathbb{N},$$
$$\Big(\sum_{k\in\vartheta} \omega_k^q |\Delta_{\mathfrak{v}}^{\varsigma} f_k(\varrho, Z(\varrho), W(\varrho))|^q\Big)^{\frac{1}{q}} \leqslant \Big(\sum_{k\in\vartheta} \omega_k^q |\phi_k(\varrho)|^q |\Delta_{\mathfrak{v}}^{\varsigma} Z_k(\varrho)|^q\Big)^{\frac{1}{q}} + \Big(\sum_{k\in\vartheta} \omega_k^q |\phi_k(\varrho)|^q |\Delta_{\mathfrak{v}}^{\varsigma} W_k(\varrho)|^q\Big)^{\frac{1}{q}},$$
$$\Big(\sum_{k\in\vartheta} \omega_k^q |\Delta_{\mathfrak{v}}^{\varsigma} g_k(\varrho, Z(\varrho), W(\varrho))|^q\Big)^{\frac{1}{q}} \leqslant \Big(\sum_{k\in\vartheta} \omega_k^q |\phi_k(\varrho)|^q |\Delta_{\mathfrak{v}}^{\varsigma} Z_k(\varrho)|^q\Big)^{\frac{1}{q}} + \Big(\sum_{k\in\vartheta} \omega_k^q |\phi_k(\varrho)|^q |\Delta_{\mathfrak{v}}^{\varsigma} W_k(\varrho)|^q\Big)^{\frac{1}{q}},$$

where $\vartheta \in \mathfrak{F}$, $h_k, \phi_k : I \to \mathbb{R}$ are continuous and the sequences $(h_k(\varrho))$ and $(\phi_k(\varrho))$ are equibounded on I.
$(B3)$ Assume that the sequence $(a_i(\tau))$ and $(b_i(\tau))$ are Riemann integrable on I and they are equibounded. Put

$$M' = \max\{\sup_{i\in\mathbb{N}} \sup_{\varrho\in I} a_i(\varrho), \sup_{i\in\mathbb{N}} \sup_{\varrho\in I} b_i(\varrho)\},$$
$$H = \sup_{i\in\mathbb{N}} \sup_{\varrho\in I} |h_i(\varrho)|,$$
$$\Phi = \sup_{i\in\mathbb{N}} \sup_{\varrho\in I} |\phi_i(\varrho)|.$$

Theorem 3.1 Assume that the IDE (1) fulfills the hypotheses $(B1)$-$(B3)$, and $\frac{(1+\alpha)M'(\Phi+H)}{2(1-\alpha\zeta)} < 1$, then it has at least one solution

$$(Z, W) \in C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)).$$

Proof. Suppose that $(Z, W) = ((Z_i), (W_i))$ satisfies the initial conditions of the IDE (1) and also, suppose that each Z_i and W_i is continuous on I. Take the mapping $F : C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \to C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q) \times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))$ defined by

$$(F(Z, W))(\tau) = \Big(\Big(\int_0^1 A(\tau, \varrho) a_i(\varrho) f_i(\varrho, Z(\varrho), W(\varrho)) d\varrho\Big), \Big(\int_0^1 A(\tau, \varrho) b_i(\varrho) g_i(\varrho, Z(\varrho), W(\varrho)) d\varrho\Big)\Big).$$

The product space $C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))$ is furnished with the sum norm

$$\|(Z, W)\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))} = \|Z\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))} + \|W\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))}$$

for each $(Z, W) \in C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))$. Applying the assumptions

$(B1)$-$(B3)$ and Lemma 2.4, we get

$$\begin{aligned}
&\|F(Z,W)(\tau)\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}\\
&=\Big\|\Big(\int_0^1 A(\tau,\varrho)a_i(\varrho)f_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big)\Big\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}+\Big\|\Big(\int_0^1 A(\tau,\varrho)b_i(\varrho)g_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big)\Big\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}\\
&=\sum_{i=1}^{\varsigma}\Big|\int_0^1 A(\tau,\varrho)a_i(\varrho)f_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big|\omega_i+\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}\Big|\Delta_{\mathfrak{v}}^{\varsigma}\Big(\int_0^1 A(\tau,\varrho)a_k(\varrho)f_k(\varrho,Z(\varrho),W(\varrho))d\varrho\Big|^q\omega_k^q\Big)^{\frac{1}{q}}\\
&\quad+\sum_{i=1}^{\varsigma}\Big|\int_0^1 A(\tau,\varrho)b_i(\varrho)g_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big|\omega_i+\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}\Big|\Delta_{\mathfrak{v}}^{\varsigma}\Big(\int_0^1 A(\tau,\varrho)b_k(\varrho)g_k(\varrho,Z(\varrho),W(\varrho))d\varrho\Big|^q\omega_k^q\Big)^{\frac{1}{q}}\\
&\leqslant\beta(\varrho)M'\Big(\Big(\sum_{i=1}^{\varsigma}\omega_i\int_0^1|f_i(\varrho,Z(\varrho),W(\varrho))|d\varrho+\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\int_0^1\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}f_k(\varrho,Z(\varrho),W(\varrho))|^q\omega_k^q d\varrho\Big)^{\frac{1}{q}}\Big)\\
&\quad+\beta(\varrho)M'\Big(\Big(\sum_{i=1}^{\varsigma}\omega_i\int_0^1|g_i(\varrho,Z(\varrho),W(\varrho))|d\varrho+\sup_{\varrho_1\geqslant 1}\sup_{\varrho\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\int_0^1\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}g_k(\varrho,Z(\varrho),W(\varrho))|^q\omega_k^q d\varrho\Big)^{\frac{1}{q}}\Big)\\
&\leqslant\frac{2(1+\alpha)M'}{4(1-\alpha\zeta)}\Big(\sum_{i=1}^{\varsigma}\omega_i\int_0^1|h_i(\varrho)|(|Z_i(\varrho)|+|W_i(\varrho)|)d\varrho+\sup_{\tau\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\psi_k(\tau)|^q|\Delta_{\mathfrak{v}}^{\varsigma}Z_k(\tau)|^q\omega_k^q\Big)^{\frac{1}{q}}\\
&\quad+\sup_{\tau\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\phi_k(\tau)|^q|\Delta_{\mathfrak{v}}^{\varsigma}W_k(\tau)|^q\omega_k^q\Big)^{\frac{1}{q}}\Big)\\
&\leqslant\frac{(1+\alpha)M'}{2(1-\alpha\zeta)}\Big(H\Big(\sup_{\tau\in I}\sum_{i=1}^{\varsigma}\omega_i|Z_i(\tau)|+\sup_{\tau\in I}\sum_{i=1}^{\varsigma}\omega_i|W_i(\tau)|\Big)\\
&\quad+\Phi\Big(\sup_{\tau\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}Z_k(\tau)|^q\times\omega_k^q\Big)^{\frac{1}{q}}+\sup_{\tau\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}W_k(\tau)|^q\times\omega_k^q\Big)^{\frac{1}{q}}\Big)\Big)\\
&\leqslant\frac{(1+\alpha)M'}{2(1-\alpha\zeta)}(H+\Phi)(\|Z\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}+\|W\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))})\\
&=\frac{(1+\alpha)M'}{2(1-\alpha\zeta)}(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}.
\end{aligned}$$

Accordingly, we obtain

$$\|F(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}\leqslant\frac{(1+\alpha)M'}{2(1-\alpha\zeta)}(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}.$$

It implies that

$$r\leqslant\frac{(1+\alpha)M'}{2(1-\alpha\zeta)}(H+\Phi)r. \tag{5}$$

Let r_0 denote the optimal solution of the inequality (5). Take

$$\begin{aligned}
D=D((u^0,u^0),r_0)=\Big\{&(Z,W):Z=(Z_i)\text{ and }W=(W_i)\text{ are in }C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)),\\
&\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}\leqslant r_0,\text{ and }Z_i(0)=Z_i'(0)=0,\\
&Z_i'(1)=\alpha Z_i'(\zeta),W_i(0)=W_i'(0)=0,\ W_i'(1)=\alpha W_i'(\zeta),\ \forall i\in\mathbb{N}\Big\}
\end{aligned}$$

where $u^0(\tau)=(u_i^0(\tau))$ and $u_i^0(\tau)=0$ for any $\tau\in I$. Evidently, D is bounded, closed and convex, and also F is bounded on D. We prove that F is continuous. For, let $(Z_1,W_1)\in D\times D$ and let $\varepsilon>0$ be arbitrarily fixed. Employing assumption $(B1)$, a real number $\delta > 0$ exists such that if $(Z_2, W_2) \in D \times D$ and

$$\|(Z_1, W_1) - (Z_2, W_2)\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))} \leqslant \delta,$$

then

$$\|\Lambda(Z_1, W_1) - \Lambda(Z_2, W_2)\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q) \times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))} \leqslant \frac{4\varepsilon(1-\alpha\zeta)}{(1+\alpha)M'}.$$

Therefore, for any τ in I, we have

$$\begin{aligned}
&\|F(Z_1, W_1)(\tau) - F(Z_2, W_2)(\tau)\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q) \times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)} \\
&= \sum_{i=1}^{\varsigma} \omega_i \Big| \int_0^1 A(\tau, \varrho) a_i(\varrho) \big(f_i(\varrho, Z_1(\varrho), W_1(\varrho)) - f_i(\varrho, Z_2(\varrho), W_2(\varrho))\big) d\varrho \Big| \\
&+ \sup_{\varrho_1 \geqslant 1} \sup_{\vartheta \in \mathfrak{F}_{\varrho_1}} \Big(\frac{1}{\psi_{\varrho_1}} \sum_{k \in \vartheta} \Big| \int_0^1 A(\tau, \varrho) \Delta_{\mathfrak{v}}^{\varsigma} \big(a_k(\varrho)(f_k(\varrho, Z_1(\varrho), W_1(\varrho)) - f_k(\varrho, Z_2(\varrho), W_2(\varrho)))\big) ds \Big|^q \omega_k^q \Big)^{\frac{1}{q}} \\
&+ \sum_{i=1}^{\varsigma} \omega_i \Big| \int_0^1 A(\tau, \varrho) b_i(\varrho) \big(g_i(\varrho, Z_1(\varrho), W_1(\varrho)) - g_i(\varrho, Z_2(\varrho), W_2(\varrho))\big) d\varrho \Big| \\
&+ \sup_{\varrho_1 \geqslant 1} \sup_{\vartheta \in \mathfrak{F}_{\varrho_1}} \Big(\frac{1}{\psi_{\varrho_1}} \sum_{k \in \vartheta} \Big| \int_0^1 A(\tau, \varrho) \Delta_{\mathfrak{v}}^{\varsigma} \big(b_k(\varrho)(g_k(\varrho, Z_1(\varrho), W_1(\varrho)) - g_k(\varrho, Z_2(\varrho), W_2(\varrho)))\big) d\varrho \Big|^q \omega_k^q \Big)^{\frac{1}{q}} \\
&\leqslant \frac{(1+\alpha)M'}{4(1-\alpha\zeta)} \Big(\sum_{i=1}^{\varsigma} \omega_i \sup_{\tau \in I} \Big(f_i(\tau, Z_1(\tau), W_1(\tau)) - f_i(\tau, Z_2(\tau), W_2(\tau)) \Big) \\
&+ \sup_{\tau \in I} \sup_{\varrho_1 \geqslant 1} \sup_{\vartheta \in \mathfrak{F}_{\varrho_1}} \Big(\frac{1}{\psi_{\varrho_1}} \sum_{k \in \vartheta} |\Delta_{\mathfrak{v}}^{\varsigma} \big(f_k(\tau, Z_1(\tau), W_1(\tau)) - f_k(\tau, Z_2(\tau), W_2(\tau))\big)|^q \omega_k^q \Big)^{\frac{1}{q}} \Big) \\
&+ \frac{(1+\alpha)M'}{4(1-\alpha\zeta)} \Big(\sum_{i=1}^{\varsigma} \omega_i \sup_{\tau \in I} \Big(g_i(\tau, Z_1(\tau), W_1(\tau)) - g_i(\tau, Z_2(\tau), W_2(\tau)) \Big) \\
&+ \sup_{\tau \in I} \sup_{\varrho_1 \geqslant 1} \sup_{\vartheta \in \mathfrak{F}_{\varrho_1}} \Big(\frac{1}{\psi_{\varrho_1}} \sum_{k \in \vartheta} |\Delta_{\mathfrak{v}}^{\varsigma} \big(g_k(\tau, Z_1(\tau), W_1(\tau)) - g_k(\tau, Z_2(\tau), W_2(\tau))\big)|^q \omega_k^q \Big)^{\frac{1}{q}} \Big) \\
&= \frac{(1+\alpha)M'}{4(1-\alpha\zeta)} \|\Lambda(Z_1, W_1) - \Lambda(Z_2, W_2)\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q) \times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))} \\
&\leqslant \varepsilon.
\end{aligned}$$

Accordingly, we get

$$\|F(Z_1, W_1) - F(Z_2, W_2)\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q) \times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))} \leqslant \varepsilon.$$

Therefore, F is continuous. Now, we are going to show that $(F(Z, W))$ is continuous on $(0, 1)$. For this, let $\tau_1 \in (0, 1)$ and let $\varepsilon > 0$. Applying the continuity of $A(\tau, \varrho)$ w.r.t. τ, we are able to find $\delta = \delta(\tau_1, \varepsilon) > 0$ such that if $|\tau - \tau_1| < \delta$, then

$$|A(\tau, \varrho) - A(\tau_1, \varrho)| < \frac{\varepsilon}{2M'(H + \Phi)\|(Z, W)\|_{C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q)) \times C(I, m_\omega(\Delta_{\mathfrak{v}}^{\varsigma}, \psi, q))}}.$$

Using Minkowski's inequality, we can write

$$
\begin{aligned}
&\|(F(Z,W))(\tau)-(F(Z,W))(\tau_1)\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}\\
&=\sum_{i=1}^{\varsigma}\omega_i\Big|\int_0^1\big(A(\tau,\varrho)-A(\tau_1,\varrho)\big)a_i(\varrho)f_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big|\\
&\quad+\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}\Big|\int_0^1(A(\tau,\varrho)-A(\tau_1,\varrho))\Delta_{\mathfrak{v}}^{\varsigma}\big(a_k(\varrho)f_k(\varrho,Z(\varrho),W(\varrho))\big)d\varrho\Big|^q\omega_k^q\big)^{\frac{1}{q}}\\
&\quad+\sum_{i=1}^{\varsigma}\omega_i\Big|\int_0^1(A(\tau,\varrho)-A(\tau_1,\varrho))b_i(\varrho)g_i(\varrho,Z(\varrho),W(\varrho))d\varrho\Big|\\
&\quad+\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}\Big|\int_0^1(A(\tau,\varrho)-A(\tau_1,\varrho))\Delta_{\mathfrak{v}}^{\varsigma}\big(b_k(\varrho)g_k(\varrho,Z(\varrho),W(\varrho))\big)d\varrho\Big|^q\omega_k^q\big)^{\frac{1}{q}}\\
&<\frac{2M'H\varepsilon}{2M'(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\big(\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|Z_i(\tau_0)|+\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|W_i(\tau_0)|\big)\\
&\quad+\frac{M'\varepsilon}{2M'(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\times\\
&\quad\Big(\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\int_0^1\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}f_k(\varrho,Z(\varrho),W(\varrho))|^q d\varrho\,\omega_k^q\big)^{\frac{1}{q}}\\
&\quad+\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\int_0^1\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}g_k(\varrho,Z(\varrho),W(\varrho))|^q d\varrho\,\omega_k^q\big)^{\frac{1}{q}}\Big)\\
&\leqslant\frac{\varepsilon}{\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\big(\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|Z_i(\tau_0)|+\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|W_i(\tau_0)|\big)\\
&\quad+\frac{M'\varepsilon}{2M'(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,p))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\times\\
&\quad\Big(\sup_{\tau_0\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}f_k(\tau_0,Z(\tau_0),W(\tau_0))|^q\omega_k^q\big)^{\frac{1}{q}}\\
&\quad+\sup_{\tau_0\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta_{\mathfrak{v}}^{\varsigma}g_k(\tau_0,Z(\tau_0),W(\tau_0))|^q\omega_k^q\big)^{\frac{1}{q}}\Big)\\
&\leqslant\frac{\varepsilon}{\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\big(\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|Z_i(\tau_0)|+\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|W_i(\tau_0)|\big)\\
&\quad+\frac{2\varepsilon}{2(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\times\\
&\quad\sup_{\tau_0\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\big)^{\frac{1}{q}}\Big(\big(\sum_{k\in\vartheta}\omega_k^q|\psi_k(\tau_0)|^q|\Delta_{\mathfrak{v}}^{\varsigma}Z_k(\tau_0)|^q\big)^{\frac{1}{q}}\\
&\quad+\big(\sum_{k\in\vartheta}\omega_k^q|\phi_k(\tau_0)|^q|\Delta_{\mathfrak{v}}^{\varsigma}W_k(\tau_0)|^q\big)^{\frac{1}{q}}\Big)\\
&\leqslant\frac{\varepsilon}{\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\big(\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|Z_i(\tau_0)|+\sum_{i=1}^{\varsigma}\omega_i\sup_{\tau_0\in I}|W_i(\tau_0)|\big)\\
&\quad+\frac{\varepsilon\Phi}{(H+\Phi)\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\Big(\sup_{\tau_0\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}\omega_k^q|\Delta_{\mathfrak{v}}^{\varsigma}Z_k(\tau_0)|^q\big)^{\frac{1}{q}}\\
&\quad+\sup_{\tau_0\in I}\sup_{\varrho_1\geqslant 1}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}\omega_k^q|\Delta_{\mathfrak{v}}^{\varsigma}W_k(\tau_0)|^q\big)^{\frac{1}{q}}\Big)\\
&\leqslant\frac{\varepsilon}{\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\Big(\|Z\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}+\|W\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}\Big)\\
&=\frac{\varepsilon}{\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}}\big(\|(Z,W)\|_{C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))}\big)\\
&=\varepsilon.
\end{aligned}
$$

Eventually, we are going to verify that F is a Meir–Keeler condensing operator w.r.t. the Hausdorff MNC χ on the space $C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))$. Due to the formula (2) and Proposition 1.6, it can be concluded that the Hausdorff MNC for $D\subset C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))$ is defined as

$$\chi_{C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))}(D)=\sup_{\tau\in I}\chi_{m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)\times m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)}(D(\tau)),$$

where $D(\tau)=\{(Z,W)(\tau):\ (Z,W)\in D\}$. Therefore, we deduce

$$\begin{aligned}
&\chi_{m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)\times m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)}(F(D))(\tau)\\
&=\lim_{n\to\infty}\Big\{\sup_{(Z,W)\in B}\Big(\sup_{\varrho_1\geqslant n}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta^\varsigma_\mathfrak{v}\Big(\int_0^1 A(\tau,\varrho)a_k(\varrho)f_k(\varrho,Z(\varrho),W(\varrho))d\varrho|^q\omega_k^q\Big)^{\frac{1}{q}}\Big)\Big\}\\
&\quad+\lim_{n\to\infty}\Big\{\sup_{(Z,W)\in B}\Big(\sup_{\varrho_1\geqslant n}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta^\varsigma_\mathfrak{v}\Big(\int_0^1\big(A(\tau,\varrho)b_k(\varrho)g_k(\varrho,Z(\varrho),W(\varrho))d\varrho\big)|^q\omega_k^q\Big)^{\frac{1}{q}}\Big)\Big\}\\
&\leqslant\frac{(1+\alpha)M'}{4(1-\alpha\zeta)}\Bigg(\lim_{n\to\infty}\Big\{\sup_{\tau_0\in I}\sup_{(Z,W)\in B}\Big(\sup_{\varrho_1\geqslant n}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta^\varsigma_\mathfrak{v}f_k(\tau_0,Z_k(\tau_0),W_k(\tau_0))|^q\omega_k^q\Big)^{\frac{1}{q}}\Big)\Big\}\Bigg)\\
&\quad+\frac{(1+\alpha)M'}{4(1-\alpha\zeta)}\Bigg(\lim_{n\to\infty}\Big\{\sup_{\tau_0\in I}\sup_{(Z,W)\in B}\Big(\sup_{\varrho_1\geqslant n}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\sum_{k\in\vartheta}|\Delta^\varsigma_\mathfrak{v}g_k(\tau_0,Z_k(\tau_0),W_k(\tau_0))|^q\omega_k^q\Big)^{\frac{1}{q}}\Big)\Big\}\Bigg)\\
&\leqslant\frac{2(1+\alpha)M'}{4(1-\alpha\zeta)}\Bigg(\lim_{n\to\infty}\Big\{\sup_{\tau_0\in I}\sup_{(Z,W)\in B}\Big(\sup_{\varrho_1\geqslant n}\sup_{\vartheta\in\mathfrak{F}_{\varrho_1}}\Big(\frac{1}{\psi_{\varrho_1}}\Big)^{\frac{1}{q}}\Big(\Big(\sum_{k\in\vartheta}|\phi_k(\tau_0)|^p|\Delta^\varsigma_\mathfrak{v}Z_k(\tau_0)|^q\omega_k^q\Big)^{\frac{1}{q}}\\
&\quad+\Big(\sum_{k\in\vartheta}|\phi_k(\tau_0)|^q|\Delta^\varsigma_\mathfrak{v}W_k(\tau_0)|^q\omega_k^q\Big)^{\frac{1}{q}}\Big)\Big)\Big\}\Bigg)\\
&\leqslant\frac{(1+\alpha)M'\Phi}{2(1-\alpha\zeta)}\big(\chi_{C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))}(D)\big).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\chi_{m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)\times m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)}(F(D))&=\sup_{\tau\in I}\chi_{m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)\times m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q)}(F(D))(\tau)\\
&\leqslant\frac{(1+\alpha)M'\Phi}{2(1-\alpha\zeta)}\chi_{C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))}(D)\\
&<\varepsilon.
\end{aligned}$$

Then

$$\chi_{C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))}(D)<\frac{2(1-\alpha\zeta)\varepsilon}{(1+\alpha)M'\Phi}.$$

Let us take $\delta=\varepsilon(\frac{2(1-\alpha\zeta)}{(1+\alpha)M'\Phi}-1)$. It easily can be verified that F is a Meir–Keeler condensing operator on $D\subset C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))$. Owing to the Theorem 1.5, we conclude that F has a fixed point in D, and hence the IDE (1) admits at least one solution in $C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))\times C(I,m_\omega(\Delta^\varsigma_\mathfrak{v},\psi,q))$. ∎

***Example* 3.2** Consider the IDE

$$\begin{cases} -Z_i'''(\tau) = \frac{1}{\tau^2+240000}\sum_{j=i}^{\infty}\arctan\big(\frac{4}{1+(4j+3)(4j-1)}\big)e^{-2\tau}\ln\big(\frac{9}{10}+|Z_j(\tau)+W_j(\tau)|\big),\ 0<\tau<1 \\ -W_i'''(\tau) = \frac{1}{\tau^2+240001}\sum_{j=i}^{\infty}\frac{3\pi}{(4j-3)(4j+1)}|\sin(Z_j(\tau)+W_j(\tau)+\frac{1}{8})-\frac{1}{8}|\cos^2(2\tau+1),\ 0<\tau<1 \\ Z_i(0)=Z_i'(0)=0,\ Z_i'(1)=99.9Z_i'(\frac{1}{100}), \\ W_i(0)=W_i'(0)=0,\ W_i'(1)=99.9W_i'(\frac{1}{100}),\ i=1,2,\ldots. \end{cases} \tag{6}$$

By taking $a_i(\tau)=\frac{1}{\tau^2+240000}$, $b_i(\tau)=\frac{1}{\tau^2+240001}$, $\alpha=99.9$, $\zeta=\frac{1}{100}$,

$$f_i(\tau,Z(\tau),W(\tau)) = \sum_{j=i}^{\infty}\arctan\big(\frac{4}{1+(4j+3)(4j-1)}\big)e^{-2\tau}\ln\big(\frac{9}{10}+|Z_j(\tau)+W_j(\tau)|\big),$$

and

$$g_i(\tau,Z(\tau),W(\tau)) = \sum_{j=i}^{\infty}\frac{3\pi}{(4j-3)(4j+1)}|\sin(Z_j(\tau)+W_j(\tau)+\frac{1}{8})-\frac{1}{8}|\cos^2(2\tau+1),$$

the system (6) is a special case of IDE (1). Clearly, for each $i\in\mathbb{N}$, $f_i,g_i\in C([0,1]\times(\mathbb{R}_+)^\infty\times(\mathbb{R}_+)^\infty,\mathbb{R}_+)$. Notice that, for each $\tau\in I$, if $(Z(\tau),W(\tau))\in m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$, then $\Lambda(Z,W)(\tau)=\big((f_i(\tau,Z(\tau),W(\tau)),(g_i(\tau,Z(\tau),W(\tau))\big)$ is in $m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$. We claim that the hypothesis $(B1)$ of Theorem 3.1 holds. Indeed, let $\varepsilon>0$ be given and let $(Z(\tau),W(\tau))=((Z_i(\tau)),(W_i(\tau)))\in m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$. Then, by taking $(Z_1(\tau),W_1(\tau))\in m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)$ with

$$\|(Z(\tau),W(\tau))-(Z_1(\tau),W_1(\tau))\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}\leqslant\delta(\varepsilon):=\frac{2\varepsilon}{3\pi},$$

we get

$$\begin{aligned}&\|\Lambda(Z(\tau),W(\tau))-\Lambda(Z_1(\tau),W_1(\tau))\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}\\ &\leqslant\frac{3\pi}{2}\|(Z(\tau),W(\tau))-(Z_1(\tau),W_1(\tau))\|_{m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)\times m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q)}=\varepsilon,\end{aligned}$$

which implies the equicontinuity of Λ. Now, we prove condition $(B2)$. We can write

$$\begin{aligned}|f_i(\tau,Z(\tau),W(\tau))| &= |\sum_{j=i}^{\infty}\arctan\big(\frac{4}{1+(4j+3)(4j-1)}\big)e^{-2\tau}\ln\big(\frac{9}{10}+|Z_j(\tau)+W_j(\tau)|\big)|\\ &\leqslant\frac{3\pi}{4}e^{-2\tau}\big(|Z_i(\tau)|+|W_i(\tau)|\big),\end{aligned}$$

and

$$\Big(\sum_{k\in\varrho}\omega_k^q|\Delta_{\mathfrak{v}}^{\varsigma}f_i(\tau,Z(\tau),W(\tau))|^q\Big)^{\frac{1}{q}} = \Big(\sum_{k\in\vartheta}\omega_k^q|\Delta_{\mathfrak{v}}^{\varsigma}\sum_{j=i}^{\infty}\arctan\big(\frac{4}{1+(4j+3)(4j-1)}\big)e^{-2\tau}\ln\big(\frac{9}{10}+|Z_j(\tau)+W_j(\tau)|\big)|^q\Big)^{\frac{1}{q}}$$

$$\leqslant \Big(\sum_{k\in\vartheta}\omega_k^q(\frac{3\pi}{4}e^{-2\tau})^p\Delta_{\mathfrak{v}}^{\varsigma}\big(|Z_i(\tau)|+|W_i(\tau)|\big)^q\Big)^{\frac{1}{q}}$$

$$\leqslant \frac{3\pi}{4}e^{-2\tau}\Big(\big(\sum_{k\in\vartheta}\omega_k^p\Delta_{\mathfrak{v}}^{\varsigma}|Z_i(\tau)|^q\big)^{\frac{1}{q}}+\big(\sum_{k\in\vartheta}\omega_k^q\Delta_{\mathfrak{v}}^{\varsigma}|W_i(\tau)|^q\big)^{\frac{1}{q}}\Big).$$

Hence, the function $h_i(\tau)=\frac{3\pi}{4}e^{-2\tau}$ is continuous and the sequence $(h_i(\tau))$ is equibounded on I and also $H=\frac{3\pi}{4}$. Also, we have

$$|g_i(\tau,Z(\tau),W(\tau))| = |\sum_{j=i}^{\infty}\frac{3\pi}{(4j-3)(4j+1)}|\sin(Z_j(\tau)+W_j(\tau)+\frac{1}{8})-\frac{1}{8}|\cos^2(2\tau+1)|$$

$$\leqslant \frac{3\pi}{4}\cos^2(2\tau+1)\big(|Z_i(\tau)|+|W_i(\tau)|\big).$$

Moreover, we obtain

$$\Big(\sum_{k\in\vartheta}\omega_k^q|\Delta_{\mathfrak{v}}^{\varsigma}g_i(\tau,Z(\tau),W(\tau))|^q\Big)^{\frac{1}{q}} = \Big(\sum_{k\in\vartheta}\omega_k^q|\Delta_{\mathfrak{v}}^{\varsigma}\sum_{j=i}^{\infty}\frac{3\pi}{(4j-3)(4j+1)}|\sin(z_j(\tau)+w_j(\tau)+\frac{1}{8})-\frac{1}{8}|\cos^2(2\tau+1))|^q\Big)^{\frac{1}{q}}$$

$$\leqslant \Big(\sum_{k\in\vartheta}\omega_k^q(\frac{3\pi}{4}\cos^2(2\tau+1))^q\Delta_{\mathfrak{v}}^{\varsigma}\big(|Z_i(\tau)|+|W_i(\tau)|\big)^q\Big)^{\frac{1}{q}}$$

$$\leqslant (\frac{3\pi}{4}\cos^2(2\tau+1))\Big(\big(\sum_{k\in\vartheta}\omega_k^q\Delta_{\mathfrak{v}}^{\varsigma}|Z_i(\tau)|^q\big)^{\frac{1}{q}}+\big(\sum_{k\in\vartheta}\omega_k^q\Delta_{\mathfrak{v}}^{\varsigma}|W_i(\tau)|^q\big)^{\frac{1}{q}}\Big)$$

when $\phi_i(\tau)=\frac{3\pi}{4}\cos^2(2\tau+1)$ is continuous and the sequence $(\phi_i(\tau))$ is equibounded on I and also $\Phi=\frac{3\pi}{4}$. Trivially the condition $(B3)$ holds. On the other hand, we get $\frac{(1+\alpha)M'(H+\Phi)}{2(1-\alpha\zeta)}=\frac{4752.39}{4800}<1$. Thus, infinite system (6) fulfils the hypotheses of Theorem 3.1. So the (6) has at least one solution in $C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))\times C(I,m_\omega(\Delta_{\mathfrak{v}}^{\varsigma},\psi,q))$.

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References

[1] A. Aghajani, J. Banaś, Y. Jalilian, Existence of solution for a class of nonlinear Volterra singular integral equation, Comput. Math. Appl. 62 (2011), 1215-1227.
[2] A. Aghajani, M. Mursaleen, A. Shole Haghighi, Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, Acta. Math. Sci. 35B (2015), 552-566.
[3] A. Aghajani, N. Sabzali, Existence of coupled fixed points via measure of noncompactness and applications, J. Nonlinear Convex Anal. 15 (2014), 941-952.
[4] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, 60 Marcel Dekker, New York, 1980.
[5] J. Banaś, M. Mursaleen, Sequence Spaces and Measure of Noncompactness with Applications to Differential and Integral Equation, Springer, India, 2014.
[6] A. Bhrawy, W. Abd-Elhameed, New algorithm for the numerical solutions of nonlinear third-order differential equations using Jacobi-Gauss collocation method, Math. Probl. Eng. Art. (2011), 2011:837218.
[7] K. Deimling, Ordinary Differential Equations in Banach Spaces, Lecture Notes in Mathematics, Springer, Berlin, 1977.

[8] Z. Du, B. Zhao, Z. Bai, Solvability of a third-order multipoint boundary value problem at resonance, Abstr. Appl. Anal. (2014), 2014:931217.

[9] X. Feng, H. Feng, D. Bai, Eigenvalue for a singular third-order three-point boundary value problem, Appl. Math. Comput. 219 (2013), 9783-9790.

[10] Y. Feng, S. Liu, Solvability of a third-order two-point boundray value problem, Appl. Math. Lett. 18 (2005), 1034-1040.

[11] A. Ghanenia, M. Khanehgir, M. Mehrabinezhad, R. Allahyari, H. Amiri Kayvanloo, Solvability of infinite systems of second order differential equations in the sequence space, Rend. Circ. Mat. Palermo. II. Ser. 69 (2020), 1-11.

[12] J. Graef, B. Yang, Positive solutions of a nonlinear third order eigenvalue problem, Dyn. Syst. Appl. 15 (2006), 97-110.

[13] M. Gregus, Third Order Linear Differential Equations, in: Math. Appl., Reidel, Dordrecht, 1987.

[14] L. Guo, L. J. Sun, Y. Zhao, Existence of positive solutions for nonlinear third-order three-point boundary value problems, Nonlinear Anal. 68 (2008), 3151-3158.

[15] B. Hazarika, A. Das, R. Arab, M. Mursaleen, Solvability of the infinite system of integral equations in two variables in the sequence spaces c_0 and l_1, J. Comput. Appl. Math. 326 (2012), 183-192.

[16] K. Kuratowski, Sur les espaces complets, Fund Math. 15 (1930), 301-309.

[17] Y. Li, Y. Guo, G. Li, Existence of positive solutions for systems of nonlinear third-order differential equations, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009), 3792-3797.

[18] X. Lin, Z. Zhao, Iterative technique for a third-order differential equation with three-point nonlinear boundary value conditions, Elect. J. Qual. Theory. Diff. Equ. (2016), 2016:1-10.

[19] A. Meir, E. A. Keeler, Theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326-329.

[20] M. Mursaleen, Application of measure of noncompactness to infinite system of differential equations, Can. Math. Bull. 56 (2013), 388-394.

[21] M. Mursaleen, Some geometric properties of a sequence space related to l_p, Bull. Aust. Math. Soc. 67 (2003), 343-347.

[22] M. Mursaleen, B. Bilalov, S. M. H. Rizvi, Applications of measures of noncompactness to infinite system of fractional differential equations, Filomat. 31 (2017), 3421-3432.

[23] M. N. Oguzt Poreli, On the neural equations of Cowan and Stein, Utilitas Math. 2 (1972), 305-315.

[24] N. Sapkota, R. Das, Application of measure of non-compactness for the existence of solutions of an infinite system of differential equations in the sequence spaces of convergent and bounded series, Adv. Intell. Syst. Comput. 1262 (2021), 167-177.

[25] A. Vakeel, A. Khan, New type of difference sequence spaces, Applied Sci. 12 (2010), 102-108.