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Existence of fixed point theorems for complex partial b-metric spaces using S-contractive mapping

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Abstract. In this paper, we prove some results on complex partial b-metric space $(\Re, p_b^c)$, which are more generalization of S-contractive mappings. Also, we expand weakly increasing mappings of S-contractive for two self-mappings and prove some common fixed point theorems with supported examples in complete partial b-metric spaces $(\Re, p_b^c)$.

Keywords: S-contraction, weakly increasing mapping, complex partial b-metric space, fixed point.

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1. Introduction

In fixed point theory, usual metric spaces and their generalizations are significant topics for many researchers. It have vast literature proven in many papers. Fixed point theory is one of the most important and significant theory in mathematics, since it has many applications in area of sciences. One of many generalizations of metric spaces, which is introduced by Matthews [11, 12], is partial metric space such that the distance of a point to itself need not be equal to zero.

Bakhtin [3] and Czerwick [6] introduced a b-metric space which is generalization of a metric space and proved some results with contraction mappings. Furthermore, Gunaseelan [7, 9] introduced complex partial b-metric spaces and gave some more results.

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Also, we recite some few research papers [1, 2, 8, 10, 19] for the right direction from metric space to complex partial b-metric spaces.

Shukla and Tiwari [16] generalized the contraction mapping known as S-contraction mapping and proved many results [13, 14, 17]. In this paper, we prove some results using S-contraction in complex partial b-metric spaces.

2. Preliminaries and definitions

Rao et al. [15] has defined partial order relations $\prec$ and $\preceq$ in complex numbers c. Also, they considered the set of non-negative complex numbers as follows: $\mathrm{C}^+ = \{(\alpha, \beta)|\alpha, \beta \in R^+\}$, where R^+ is the set of non-negative real numbers.

Definition 2.1 [16] Let $\zeta : \Re \to \Re$, where $(\Re, d)$ be a complete metric space. Then ζ is called a S-contraction if there exist $0 \leqslant \kappa < \frac{1}{3}$ such that

$$d(\zeta\alpha, \zeta\beta) \leqslant \kappa[d(\alpha, \zeta\beta) + d(\zeta\alpha, \beta) + d(\alpha, \beta)]$$

for all $\alpha, \beta \in \Re$.

Lemma 2.2 [5] Let $(\Re, p)$ be a partial metric space. If a sequence $\{\alpha_n\}$ is convergent to a point $\alpha \in \Re$, then $\lim\limits_{n\to\infty} p(\alpha_n, \alpha) \leqslant p(\alpha, \gamma)$ for all $\gamma \in \Re$. Also, if $p(\alpha, \alpha) = 0$, then $\lim\limits_{n\to\infty} p(\alpha_n, \gamma) = p(\alpha, \gamma)$.

Definition 2.3 [15] A function $d^c_b : \Re \times \Re \to \mathrm{C}$ is said to be a complex valued b-metric on a nonempty set $\Re$ if for all $\alpha, \beta, \gamma \in \Re$ and $\mu \geqslant 1$, it satisfies the following conditions:

(1) (d^c_{b1}) $0 \preceq d^c_b(\alpha, \beta)$ and $d^c_b(\alpha, \beta) = 0 \preceq$ if and only if $\alpha = \beta$,
(2) (d^c_{b2}) $d^c_b(\alpha, \beta) = d^c_b(\beta, \alpha)$, (symmetry)
(3) (d^c_{b3}) $d^c_b(\alpha, \beta) \preceq \mu[d^c_b(\alpha, \gamma) + d^c_b(\gamma, \beta)]$. (triangularity)

The pair $(\Re, d^c_b)$ is said to be a complex valued b-metric space.

Definition 2.4 [9] A function $p^c_b : \Re \times \Re \to \mathrm{C}^+$ is said to be a complex partial b-metric on a non-void set $\Re$ if for all $\alpha, \beta, \gamma \in \Re$, it satisfies the following conditions:

- (p^c_{b1}) $p^c_b(\alpha, \beta) = p^c_b(\beta, \alpha)$, (symmetry)
- (p^c_{b2}) $p^c_b(\alpha, \alpha) = p^c_b(\alpha, \beta) = p^c_b(\beta, \beta) \Leftrightarrow \alpha = \beta$, (equality)
- (p^c_{b3}) $0 \preceq p^c_b(\alpha, \alpha) \leqslant p^c_b(\alpha, \beta)$, (small self distance)
- (p^c_{b4}) $\exists$ a real number $\mu \geqslant 1$ and μ is an independent of α, β, γ such that $p^c_b(\alpha, \beta) \leqslant \mu\big[p^c_b(\alpha, \gamma) + p^c_b(\gamma, \beta)\big] - p^c_b(\gamma, \gamma)$. (triangularity)

The pair $(\Re, p^c_b)$ is said to be complex partial b-metric space.

***Example* 2.5** [4, 9] Let $\Re^+$ be the set of nonnegative real numbers and C^+ be the set of nonnegative complex numbers. We define a mapping $p^c_b : \Re^+ \times \Re^+ \to \mathrm{C}^+$ such that

$$p^c_b(\alpha, \beta) = \Big[(\max\{\alpha, \beta\})^r + |\alpha - \beta|^r\Big](1 + i)$$

for all $\alpha, \beta \in \Re^+$. Then, for coefficient $\mu = 2^r > 1$, $(\Re^+, p^c_b)$ is a complete complex partial b-metric space but it is neither a b-metric space nor a partial metric space.

Definition 2.6 [4] Let $(\Re, p^b_c)$ be a complex partial b-metric space with coefficient $\mu \geqslant 1$ and $(\Re, \preceq)$ be a partially ordered set. A pair (ζ, ξ) of self-mappings of $\Re$ is said to be weakly increasing if $\zeta\alpha \preceq \zeta\xi\alpha$ and $\xi\alpha \preceq \xi\zeta\alpha$ for all $\alpha \in \Re$.

Now, we define Cauchy sequence and convergent sequence in complex partial b-metric spaces.

Definition 2.7 [18] Let $(\Re, p_b^c)$ be a complex partial b-metric space with coefficient $\mu \geqslant 1$. Then

- (i) The sequence $\{\alpha_n\}$ in $\Re$ converges to $\alpha \in \Re$ if $\lim_{n\to\infty} p_b^c(\alpha_n, \alpha) = p_b^c(\alpha, \alpha)$.
- (ii) The sequence $\{\alpha_n\}$ is said to be a Cauchy sequence in $(\Re, p_b^c)$ if $\lim_{m,n\to\infty} p_b^c(\alpha_m, \alpha_n)$ exists and is finite.
- (iii) The space $(\Re, p_b^c)$ is said to be a complete complex partial b-metric space if for every Cauchy sequence $\{\alpha_n\}$ in $\Re$, there exists $\alpha \in \Re$ such that

$$\lim_{m,n\to\infty} p_b^c(\alpha_m, \alpha_n) = \lim_{m\to\infty} p_b^c(\alpha_m, \alpha) = p_b^c(\alpha, \alpha).$$

- (iv) A mapping $\zeta : \Re \to \Re$ is said to be continuous at $\alpha_0 \in \Re$ if for every $\epsilon > 0$, there exists $t > 0$ such that $\zeta(Bp_b^c(\alpha_0, t)) \subset Bp_b^c(\zeta(\alpha_0, \epsilon))$.

3. Main results

Theorem 3.1 Let $(\Re, p_b^c)$ be a complete partial b-metric space with coefficient $\mu \geqslant 1$ and ζ be a self-mapping on $\Re$ satisfying the following condition:

$$p_b^c(\zeta\alpha, \zeta\beta) \leqslant \kappa\big[p_b^c(\alpha, \zeta\beta) + p_b^c(\zeta\alpha, \beta) + p_b^c(\alpha, \beta)\big] \tag{1}$$

for all $\alpha, \beta \in \Re$ and $\kappa \in \big[0, \frac{1}{3}\big)$. Then ζ has a unique fixed point in $\Re$.

Proof. Let ϵ_0 be an arbitrary point of $\Re$ and a sequence $\{\epsilon_n\}$ in $\Re$ such that $\zeta\epsilon_n = \epsilon_{n+1}$. Now, putting $\alpha = \epsilon_{n-1}$ and $\beta = \epsilon_n$ in (1), we have

$$\begin{aligned} p_b^c(\epsilon_n, \epsilon_{n+1}) &= p_b^c(\zeta\epsilon_{n-1}, \zeta\epsilon_n) \\ &\leqslant \kappa\big[p_b^c(\epsilon_{n-1}, \zeta\epsilon_n) + p_b^c(\zeta\epsilon_{n-1}, \epsilon_n) + p_b^c(\epsilon_{n-1}, \epsilon_n)\big] \\ &= \kappa\big[p_b^c(\epsilon_{n-1}, \epsilon_{n+1}) + p_b^c(\epsilon_n, \epsilon_n) + p_b^c(\epsilon_{n-1}, \epsilon_n)\big] \quad (\text{using } p_{b4}^c) \\ &\leqslant \kappa\big[\mu\big\{p_b^c(\epsilon_{n-1}, \epsilon_n) + p_b^c(\epsilon_n, \epsilon_{n+1})\big\} - p_b^c(\epsilon_n, \epsilon_n) + p_b^c(\epsilon_n, \epsilon_n) + p_b^c(\epsilon_{n-1}, \epsilon_n)\big], \end{aligned}$$

which implies that

$$(1 - \kappa\mu)p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant (\kappa\mu + \kappa)p_b^c(\epsilon_{n-1}, \zeta\epsilon_n).$$

Thus, $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \Big(\frac{\kappa\mu+\kappa}{1-\kappa\mu}\Big)p_b^c(\epsilon_{n-1}, \zeta\epsilon_n)$ and hence, $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \hbar p_b^c(\epsilon_{n-1}, \zeta\epsilon_n)$, where $\hbar = \Big(\frac{\kappa\mu+\kappa}{1-\kappa\mu}\Big) < 1$ if $\kappa < \frac{1}{1+2\mu}$. Then it follows that

$$p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \hbar p_b^c(\epsilon_{n-1}, \zeta\epsilon_n) \leqslant .... \leqslant \hbar^n p_b^c(\epsilon_0, \epsilon_1).$$

Now, we have to show that the sequence $\{\epsilon_n\}$ is a Cauchy sequence. For any $m, n \in \mathrm{N}$

with $m < n$, we get

$$\begin{aligned}
p_b^c(\epsilon_m,\epsilon_n) &\leqslant \mu\big[p_b^c(\epsilon_m,\epsilon_{m+1})+p_b^c(\epsilon_{m+1},\epsilon_n)\big]-p_b^c(\epsilon_{m+1},\epsilon_{m+1})\\
&\leqslant \mu p_b^c(\epsilon_m,\epsilon_{m+1})+\mu^2\big[p_b^c(\epsilon_{m+1},\epsilon_{m+2})+p_b^c(\epsilon_{m+2},\epsilon_n)\big]\\
&\quad -\mu p_b^c(\epsilon_{m+2},\epsilon_{m+2})-p_b^c(\epsilon_{m+1},\epsilon_{m+1})\\
&\leqslant \mu p_b^c(\epsilon_m,\epsilon_{m+1})+\mu^2 p_b^c(\epsilon_{m+1},\epsilon_{m+2})+\mu^3 p_b^c(\epsilon_{m+2},\epsilon_{m+3})+.....+\mu^{n-m}p_b^c(\epsilon_{n-1},\epsilon_n)\\
&\leqslant \mu\hbar^m p_b^c(\epsilon_1,\epsilon_0)+\mu^2\hbar^{m+1}p_b^c(\epsilon_1,\epsilon_0)+\mu^3\hbar^{m+2}p_b^c(\epsilon_1,\epsilon_0)+.....+\mu^{n-m}\hbar^{n-1}p_b^c(\epsilon_1,\epsilon_0)\\
&=\mu\hbar^m[1+\mu\hbar+\mu^2\hbar^2+.....+\mu^{n-m-1}\hbar^{n-m-1}]p_b^c(\epsilon_1,\epsilon_0),
\end{aligned}$$

which implies that

$$p_b^c(\epsilon_m,\epsilon_n)\leqslant \mu\hbar^m\Big[\frac{1-(\mu\hbar)^{n-m}}{1-\mu\hbar}\Big]p_b^c(\epsilon_1,\epsilon_0)$$

Thus, $|p_b^c(\epsilon_m,\epsilon_n)|\leqslant \mu\hbar^m\Big[\frac{1-(\mu\hbar)^{n-m}}{1-\mu\hbar}\Big]|p_b^c(\epsilon_1,\epsilon_0)|\to 0$ as $m,n\to\infty$, which implies that

$$\lim_{m,n\to\infty}p_b^c(\epsilon_m,\epsilon_n)=0.$$

Hence, $\{\epsilon_n\}$ is a Cauchy sequence in $\Re$. By completeness of $\Re$, there exists a $\alpha\in\Re$ such that $\epsilon_n\to\alpha$ and

$$\lim_{m,n\to\infty}p_b^c(\epsilon_m,\epsilon_n)=\lim_{m\to\infty}p_b^c(\epsilon_m,\alpha)=p_b^c(\alpha,\alpha)=0. \tag{2}$$

Next, we have to show that ζ has a fixed point in $\Re$. Let us assume that $\alpha\in\Re$ and for any $n\in$ N, we obtain

$$\begin{aligned}
p_b^c(\alpha,\zeta\alpha) &\leqslant \mu\big[p_b^c(\alpha,\epsilon_{n+1})+p_b^c(\epsilon_{n+1},\zeta\alpha)\big]-p_b^c(\epsilon_{n+1},\epsilon_{n+1})\\
&\leqslant \mu\big[p_b^c(\alpha,\epsilon_{n+1})+p_b^c(\zeta\epsilon_n,\zeta\alpha)\big]\\
&\leqslant \mu\big[p_b^c(\alpha,\epsilon_{n+1})+\kappa\big\{p_b^c(\epsilon_n,\zeta\alpha)+p_b^c(\zeta\epsilon_n,\alpha)+p_b^c(\epsilon_n,\alpha)\big\}\big]
\end{aligned}$$

By completeness of $\Re$ and using equation (2), we have $p_b^c(\alpha,\epsilon_n)$ and $p_b^c(\epsilon_n,\zeta\alpha)\to 0$ as $n\to\infty$. Therefore, $p_b^c(\alpha,\zeta\alpha)\leqslant 0$, but by definition of partial metric $p_b^c(m,n)\geqslant 0$. Thus, $p_b^c(\alpha,\zeta\alpha)=0$. Hence, α is a fixed point of ζ. Now, we have to show that α is a unique fixed point of ζ. We assume a contradictory that $\alpha,\beta\in\Re$ be two distinct fixed points of ζ. Then $p_b^c(\alpha,\alpha)=0$ and $p_b^c(\beta,\beta)=0$. Now, consider

$$\begin{aligned}
p_b^c(\alpha,\beta)=p_b^c(\zeta\alpha,\zeta\beta) &\leqslant \kappa\big[p_b^c(\alpha,\zeta\beta)+p_b^c(\zeta\alpha,\beta)+p_b^c(\alpha,\beta)\big]\\
&=\kappa\big[p_b^c(\alpha,\beta)+p_b^c(\alpha,\beta)+p_b^c(\alpha,\beta)\big]\\
&=3\kappa p_b^c(\alpha,\beta),
\end{aligned}$$

which implies that $(1-3\kappa)p_b^c(\alpha,\beta)\leqslant 0$. Since $1-3\kappa>0$, therefore $p_b^c(\alpha,\beta)\leqslant 0$, which is contradiction. Hence our assumption is false. Thus, $p_b^c(\alpha,\beta)=0\Rightarrow\alpha=\beta$. Hence, the fixed point is unique. ■

Theorem 3.2 Let $(\Re, p_b^c)$ be a complete partial b-metric space with coefficient $\mu \geqslant 1$ and ζ be a self-mapping on $\Re$ satisfying the following condition:

$$p_b^c(\zeta\alpha, \zeta\beta) \leqslant \kappa \max\left\{p_b^c(\alpha, \zeta\beta), p_b^c(\zeta\alpha, \beta), p_b^c(\alpha, \beta)\right\} \tag{3}$$

for all $\alpha, \beta \in \Re$ and $\kappa \in \left[0, \frac{1}{3}\right)$. Then ζ has a unique fixed point in $\Re$.

Proof. Let ϵ_0 be an arbitrary point of $\Re$ and a sequence $\{\epsilon_n\}$ in $\Re$, such that $\zeta\epsilon_n = \epsilon_{n+1}$. Now, putting $\alpha = \epsilon_{n-1}$ and $\beta = \epsilon_n$ in (3), we have

$$\begin{aligned}
p_b^c(\epsilon_n, \epsilon_{n+1}) &= p_b^c(\zeta\epsilon_{n-1}, \zeta\epsilon_n)\\
&\leqslant \kappa \max\left\{p_b^c(\epsilon_{n-1}, \zeta\epsilon_n), p_b^c(\zeta\epsilon_{n-1}, \epsilon_n), p_b^c(\epsilon_{n-1}, \epsilon_n)\right\}\\
&= \kappa \max\left\{p_b^c(\epsilon_{n-1}, \epsilon_{n+1}), p_b^c(\epsilon_n, \epsilon_n), p_b^c(\epsilon_{n-1}, \epsilon_n)\right\}.
\end{aligned}$$

Case-I If $\max\left\{p_b^c(\epsilon_{n-1}, \epsilon_{n+1}), p_b^c(\epsilon_n, \epsilon_n), p_b^c(\epsilon_{n-1}, \epsilon_n)\right\} = p_b^c(\epsilon_n, \epsilon_n)$, then $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant p_b^c(\epsilon_n, \epsilon_n)$, which is contradiction.

Case-II If $\max\left\{p_b^c(\epsilon_{n-1}, \epsilon_{n+1}), p_b^c(\epsilon_n, \epsilon_n), p_b^c(\epsilon_{n-1}, \epsilon_n)\right\} = p_b^c(\epsilon_{n-1}, \epsilon_{n+1})$, then

$$\begin{aligned}
p_b^c(\epsilon_n, \epsilon_{n+1}) &\leqslant \kappa\left[\mu\left\{p_b^c(\epsilon_{n-1}, \epsilon_n) + p_b^c(\epsilon_n, \epsilon_{n+1})\right\} - p_b^c(\epsilon_n, \epsilon_n)\right]\\
&\leqslant \kappa\mu\left[p_b^c(\epsilon_{n-1}, \epsilon_n) + p_b^c(\epsilon_n, \epsilon_{n+1})\right] - \kappa p_b^c(\epsilon_n, \epsilon_n)\\
&\leqslant \kappa\mu\left[p_b^c(\epsilon_{n-1}, \epsilon_n) + p_b^c(\epsilon_n, \epsilon_{n+1})\right],
\end{aligned}$$

which implies that $(1 - \kappa\mu)p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \kappa\mu p_b^c(\epsilon_{n-1}, \epsilon_n)$ Thus,

$$p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \left(\frac{\kappa\mu}{1 - \kappa\mu}\right) p_b^c(\epsilon_{n-1}, \epsilon_n)$$

and hence, $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \hbar p_b^c(\epsilon_{n-1}, \zeta\epsilon_n)$, where $\hbar = \left(\frac{\kappa\mu}{1-\kappa\mu}\right) < 1$, if $\kappa < \frac{1}{2\mu}$. Then it follows that

$$p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \hbar p_b^c(\epsilon_{n-1}, \zeta\epsilon_n) \leqslant .... \leqslant \hbar^n p_b^c(\epsilon_0, \epsilon_1).$$

Thus, $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \hbar^n p_b^c(\epsilon_0, \epsilon_1)$.

Case-III If $\max\left\{p_b^c(\epsilon_{n-1}, \epsilon_{n+1}), p_b^c(\epsilon_n, \epsilon_n), p_b^c(\epsilon_{n-1}, \epsilon_n)\right\} = p_b^c(\epsilon_{n-1}, \epsilon_n)$, then $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \kappa p_b^c(\epsilon_{n-1}, \epsilon_n)$. Continuing this process, we obtain $p_b^c(\epsilon_n, \epsilon_{n+1}) \leqslant \kappa^n p_b^c(\epsilon_0, \epsilon_1)$. Now, we have to show that the sequence $\{\epsilon_n\}$ is a Cauchy sequence. For any $m, n \in \mathrm{N}$ with $m < n$, we have

$$\begin{aligned}
p_b^c(\epsilon_m, \epsilon_n) &\leqslant \mu\big[p_b^c(\epsilon_m, \epsilon_{m+1}) + p_b^c(\epsilon_{m+1}, \epsilon_n)\big] - p_b^c(\epsilon_{m+1}, \epsilon_{m+1})\\
&\leqslant \mu p_b^c(\epsilon_m, \epsilon_{m+1}) + \mu^2\big[p_b^c(\epsilon_{m+1}, \epsilon_{m+2}) + p_b^c(\epsilon_{m+2}, \epsilon_n)\big]\\
&\quad - \mu p_b^c(\epsilon_{m+2}, \epsilon_{m+2}) - p_b^c(\epsilon_{m+1}, \epsilon_{m+1})\\
&\leqslant \mu p_b^c(\epsilon_m, \epsilon_{m+1}) + \mu^2 p_b^c(\epsilon_{m+1}, \epsilon_{m+2}) + \mu^3 p_b^c(\epsilon_{m+2}, \epsilon_{m+3}) + \ldots.. + \mu^{n-m} p_b^c(\epsilon_{n-1}, \epsilon_n)\\
&\leqslant \mu\hbar^m p_b^c(\epsilon_1, \epsilon_0) + \mu^2\hbar^{m+1} p_b^c(\epsilon_1, \epsilon_0) + \mu^3\hbar^{m+2} p_b^c(\epsilon_1, \epsilon_0) + \ldots.. + \mu^{n-m}\hbar^{n-1} p_b^c(\epsilon_1, \epsilon_0)\\
&= \mu\hbar^m[1 + \mu\hbar + \mu^2\hbar^2 + \ldots.. + \mu^{n-m-1}\hbar^{n-m-1}] p_b^c(\epsilon_1, \epsilon_0)\\
&= \mu\hbar^m\Big[\frac{1 - (\mu\hbar)^{n-m}}{1 - \mu\hbar}\Big] p_b^c(\epsilon_1, \epsilon_0)
\end{aligned}$$

Thus, $|p_b^c(\epsilon_m, \epsilon_n)| \leqslant \mu\hbar^m\Big[\frac{1-(\mu\hbar)^{n-m}}{1-\mu\hbar}\Big]|p_b^c(\epsilon_1, \epsilon_0)| \to 0$ as $m, n \to \infty$, which implies that

$$\lim_{m,n\to\infty} p_b^c(\epsilon_m, \epsilon_n) = 0.$$

Hence $\{\epsilon_n\}$ is a Cauchy sequence in $\Re$. By completeness of $\Re$, there exists $\alpha \in \Re$ such that $\epsilon_n \to \alpha$ and

$$\lim_{m,n\to\infty} p_b^c(\epsilon_m, \epsilon_n) = \lim_{m\to\infty} p_b^c(\epsilon_m, \alpha) = p_b^c(\alpha, \alpha) = 0. \tag{4}$$

Next, we have to show that ζ has a fixed point in $\Re$. Let us assume that $\alpha \in \Re$ and for any $n \in \mathrm{N}$, we obtain

$$\begin{aligned}
p_b^c(\alpha, \zeta\alpha) &\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + p_b^c(\epsilon_{n+1}, \zeta\alpha)\big] - p_b^c(\epsilon_{n+1}, \epsilon_{n+1})\\
&\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + p_b^c(\zeta\epsilon_n, \zeta\alpha)\big]\\
&\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + \kappa \max\big\{p_b^c(\epsilon_n, \zeta\alpha), p_b^c(\zeta\epsilon_n, \alpha), p_b^c(\epsilon_n, \alpha)\big\}\big]\\
&\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + \kappa \max\big\{p_b^c(\epsilon_n, \zeta\alpha), p_b^c(\epsilon_{n+1}, \alpha), p_b^c(\epsilon_n, \alpha)\big\}\big].
\end{aligned}$$

By completeness of $\Re$ and using equation (4), we have $p_b^c(\alpha, \epsilon_n)$ and $p_b^c(\epsilon_n, \zeta\alpha) \to 0$ as $n \to \infty$. Therefore, $p_b^c(\alpha, \zeta\alpha) \leqslant 0$, but by definition of partial metric $p_b^c(m, n) \geqslant 0$. Thus, $p_b^c(\alpha, \zeta\alpha) = 0$. Hence, α is a fixed point of ζ. Now, we have to show that α is a unique fixed point of ζ. We assume a contradictory that $\alpha, \beta \in \Re$ are two distinct fixed points of ζ. Then $p_b^c(\alpha, \alpha) = 0$ and $p_b^c(\beta, \beta) = 0$. Now, we have

$$\begin{aligned}
p_b^c(\alpha, \beta) &= p_b^c(\zeta\alpha, \zeta\beta)\\
&\leqslant \kappa \max\big\{p_b^c(\alpha, \zeta\beta), p_b^c(\zeta\alpha, \beta), p_b^c(\alpha, \beta)\big\}\\
&= \kappa \max\big\{p_b^c(\alpha, \beta), p_b^c(\alpha, \beta), p_b^c(\alpha, \beta)\big\}\\
&= \kappa p_b^c(\alpha, \beta),
\end{aligned}$$

which implies that $(1 - \kappa)p_b^c(\alpha, \beta) \leqslant 0$. Since $1 - \kappa > 0$, therefore $p_b^c(\alpha, \beta) \leqslant 0$, which is contradiction. Hence our assumption is false. Thus, $p_b^c(\alpha, \beta) = 0 \Rightarrow \alpha = \beta$. Hence, the fixed point is unique. ■

***Example* 3.3** [4, 9] As earlier, we have discussed in Example 2.5. We define a mapping $\zeta : \Re^+ \to \Re^+$ such that $\zeta(s) = \frac{s}{5}$. Now, we choose $\alpha = 2, \beta = 3$ and $r = 2$. We have

$$p_b^c(\zeta(2), \zeta(3)) = p_b^c\Big(\frac{2}{5}, \frac{3}{5}\Big) = \Big(\frac{9}{25} + \frac{1}{25}\Big)(1+i) = \frac{2}{5}(1+i) \tag{5}$$

Case-I: Now, we calculate

$$\begin{aligned} p_b^c(2, \zeta(3)) + p_b^c(\zeta(2), 3) + p_b^c(2,3) &= p_b^c\Big(2, \frac{3}{5}\Big) + p_b^c\Big(\frac{2}{5}, 3\Big) + p_b^c(2,3) \\ &= \Big[\Big(4 + \frac{49}{25}\Big) + \Big(9 + \frac{169}{25}\Big) + (9+1)\Big](1+i) \\ &= \Big(23 + \frac{49}{25} + \frac{169}{25}\Big)(1+i) = \frac{793}{25}(1+i). \end{aligned}$$

From equations (1) and (5) with above value, we have

$$p_b^c(\zeta(2), \zeta(3)) \leqslant \kappa\Big[p_b^c(2, \zeta(3)) + p_b^c(\zeta(2), 3) + p_b^c(2,3)\Big].$$

Thus, ζ satisfies all conditions of Theorem 3.1 in complex partial b-metric space and hence, 0 is unique fixed point of ζ.
Case-II: Again we calculate

$$\begin{aligned} \max\Big[p_b^c(2, \zeta(3)), p_b^c(\zeta(2), 3), p_b^c(2,3)\Big] &= \max\Big[p_b^c\Big(2, \frac{3}{5}\Big), p_b^c\Big(\frac{2}{5}, 3\Big), p_b^c(2,3)\Big] \\ &= \max\Big[\Big(4 + \frac{49}{25}\Big), \Big(9 + \frac{169}{25}\Big), (9+1)\Big](1+i) \\ &= \Big(9 + \frac{169}{25}\Big)(1+i) = \frac{394}{25}(1+i). \end{aligned}$$

From equations (3) and (5) with above value, we have

$$p_b^c(\zeta(2), \zeta(3)) \leqslant \kappa \max\Big[p_b^c(2, \zeta(3)), p_b^c(\zeta(2), 3), p_b^c(2,3)\Big].$$

Similarly, ζ satisfies all conditions of Theorem 3.2 in complex partial b-metric space. Therefore, 0 is unique fixed point of ζ.

Theorem 3.4 Let $(\Re, p_b^c)$ be a complete partial b-metric space with coefficient $\mu \geqslant 1$ and $\zeta, \xi : \Re \to \Re$ be two continuous and weakly increasing self-mappings satisfy the following condition

$$p_b^c(\zeta\alpha, \xi\beta) \leqslant \kappa\big[p_b^c(\alpha, \xi\beta) + p_b^c(\zeta\alpha, \beta) + p_b^c(\alpha, \beta)\big] \tag{6}$$

for all $\alpha, \beta \in \Re$ and $\kappa \in \big[0, \frac{1}{3}\big)$. Then ζ and ξ have a unique common fixed point in $\Re$.

Proof. Let ϵ_0 be an arbitrary point of $\Re$ and a sequence $\{\epsilon_n\}$ in $\Re$ such that $\epsilon_{2n+1} = \zeta\epsilon_{2n}$ and $\epsilon_{2n+2} = \xi\epsilon_{2n+1}$ for $n = 0, 1, 2, \cdots$. Then $\epsilon_1 = \zeta\epsilon_0$ and $\xi\epsilon_1 = \epsilon_2$. Since ζ and ξ are weakly increasing mapping. Therefore, we obtain

$$\begin{aligned}\epsilon_1 = \zeta\epsilon_0 \leqslant \xi\zeta\epsilon_0 = \xi\epsilon_1 = \epsilon_2,\\ \epsilon_2 = \xi\epsilon_1 \leqslant \zeta\xi\epsilon_1 = \zeta\epsilon_2 = \epsilon_3.\end{aligned}$$

Continuing this process, we have

$$\epsilon_1 \leqslant \epsilon_2 \leqslant \epsilon_3 \leqslant ..... \leqslant \epsilon_n \leqslant \epsilon_{n+1} \leqslant ...$$

Now, consider $p_b^c(\epsilon_{2n}, \epsilon_{2n+1}) = 0$ for all $n \in$ N. Then $\epsilon_{2n} = \epsilon_{2n+1}$. Therefore, by the definition of ζ and ξ, we have

$$\epsilon_{2n} = \epsilon_{2n+1} = \zeta\epsilon_{2n} \quad , \quad \epsilon_{2n+1} = \epsilon_{2n+2} = \xi\epsilon_{2n+1}.$$

Therefore, ζ and ξ have a common fixed point in $\Re$. Hence, the proof is complete. Next, we assume that $p_b^c(\epsilon_{2n}, \epsilon_{2n+1}) > 0$ for all $n \in$ N. Since ϵ_{2n} and ϵ_{2n+1} are comparable, we have

$$\begin{aligned}p_b^c(\epsilon_{2n+1}, \epsilon_{2n+2}) &= p_b^c(\zeta\epsilon_{2n}, \xi\epsilon_{2n+1})\\ &\leqslant \kappa\big[p_b^c(\zeta\epsilon_{2n}, \epsilon_{2n+1}) + p_b^c(\epsilon_{2n}, \xi\epsilon_{2n+1}) + p_b^c(\epsilon_{2n}, \epsilon_{2n+1})\big]\\ &= \kappa\big[p_b^c(\epsilon_{2n+1}, \epsilon_{2n+1}) + p_b^c(\epsilon_{2n}, \epsilon_{2n+2}) + p_b^c(\epsilon_{2n}, \epsilon_{2n+1})\big]\\ &\leqslant \kappa\big[p_b^c(\epsilon_{2n+1}, \epsilon_{2n+1}) + \mu\big\{p_b^c(\epsilon_{2n}, \epsilon_{2n+1}) + p_b^c(\epsilon_{2n+1}, \epsilon_{2n+2})\big\}\\ &\quad - p_b^c(\epsilon_{2n+1}, \epsilon_{2n+1}) + p_b^c(\epsilon_{2n}, \epsilon_{2n+1})\big],\end{aligned}$$

which implies that $(1-\kappa\mu)p_b^c(\epsilon_{2n+1}, \epsilon_{2n+2}) \leqslant (\kappa+\kappa\mu)p_b^c(\epsilon_{2n}, \epsilon_{2n+1})$. Thus,

$$p_b^c(\epsilon_{2n+1}, \epsilon_{2n+2}) \leqslant \Big(\frac{\kappa+\kappa\mu}{1-\kappa\mu}\Big)p_b^c(\epsilon_{2n}\epsilon_{2n+1})$$

and hence, $p_b^c(\epsilon_{2n+1}, \epsilon_{2n+2}) \leqslant \hbar p_b^c(\epsilon_{2n}, \epsilon_{2n+1})$, where $\hbar = \Big(\frac{\kappa+\kappa\mu}{1-\kappa\mu}\Big) < 1$ if $\kappa < \frac{1}{1+2\mu}$. Continuing in the following way, we obtain

$$p_b^c(\epsilon_{2n+1}, \epsilon_{2n+2}) \leqslant \hbar p_b^c(\epsilon_{2n}, \epsilon_{2n+1}) \leqslant \hbar^2 p_b^c(\epsilon_{2n-1}, \epsilon_{2n}) \leqslant ...... \leqslant \hbar^{2n+1}p_b^c(\epsilon_0, \epsilon_1).$$

Now, we have to show that the sequence $\{\epsilon_n\}$ is a Cauchy sequence. For any $m, n \in$ N with $m < n$, we have

$$\begin{aligned}p_b^c(\epsilon_m, \epsilon_n) &\leqslant \mu\big[p_b^c(\epsilon_m, \epsilon_{m+1}) + p_b^c(\epsilon_{m+1}, \epsilon_n)\big] - p_b^c(\epsilon_{m+1}, \epsilon_{m+1})\\ &\leqslant \mu p_b^c(\epsilon_m, \epsilon_{m+1}) + \mu^2\big[p_b^c(\epsilon_{m+1}, \epsilon_{m+2}) + p_b^c(\epsilon_{m+2}, \epsilon_n)\big]\\ &\quad - \mu p_b^c(\epsilon_{m+2}, \epsilon_{m+2}) - p_b^c(\epsilon_{m+1}, \epsilon_{m+1})\\ &\leqslant \mu p_b^c(\epsilon_m, \epsilon_{m+1}) + \mu^2 p_b^c(\epsilon_{m+1}, \epsilon_{m+2}) + \mu^3 p_b^c(\epsilon_{m+2}, \epsilon_{m+3}) + ... + \mu^{n-m}p_b^c(\epsilon_{n-1}, \epsilon_n)\\ &\leqslant \mu\hbar^m p_b^c(\epsilon_1, \epsilon_0) + \mu^2\hbar^{m+1}p_b^c(\epsilon_1, \epsilon_0) + \mu^3\hbar^{m+2}p_b^c(\epsilon_1, \epsilon_0) + ... + \mu^{n-m}\hbar^{n-1}p_b^c(\epsilon_1, \epsilon_0)\\ &= \mu\hbar^m[1 + \mu\hbar + \mu^2\hbar^2 + ... + \mu^{n-m-1}\hbar^{n-m-1}]p_b^c(\epsilon_1, \epsilon_0),\end{aligned}$$

which implies that

$$p_b^c(\epsilon_m, \epsilon_n) \leqslant \mu\hbar^m \Big[\frac{1-(\mu\hbar)^{n-m}}{1-\mu\hbar}\Big] p_b^c(\epsilon_1, \epsilon_0).$$

Thus, $|p_b^c(\epsilon_m, \epsilon_n)| \leqslant \mu\hbar^m \Big[\frac{1-(\mu\hbar)^{n-m}}{1-\mu\hbar}\Big] |p_b^c(\epsilon_1, \epsilon_0)| \to 0$ as $m, n \to \infty$, which implies that

$$\lim_{m,n\to\infty} p_b^c(\epsilon_m, \epsilon_n) = 0.$$

Hence, $\{\epsilon_n\}$ is a Cauchy sequence in $\Re$. By completeness of $\Re$, there exists a $\alpha \in \Re$ such that $\epsilon_n \to \alpha$ and

$$\lim_{m,n\to\infty} p_b^c(\epsilon_m, \epsilon_n) = \lim_{m\to\infty} p_b^c(\epsilon_m, \alpha) = p_b^c(\alpha, \alpha) = 0. \tag{7}$$

Since ζ and ξ are continuous in $(\Re, p_b^c)$, then by property of continuity, $\zeta\epsilon_{2n} \to \zeta\alpha$ and $\xi\epsilon_{2n+1} \to \xi\alpha$ as $n \to \infty$. Therefore,

$$p_b^c(\zeta\alpha, \zeta\alpha) = \lim_{n\to\infty} p_b^c(\zeta\alpha, \zeta\epsilon_{2n}) = \lim_{n\to\infty} p_b^c(\zeta\epsilon_{2n}, \zeta\epsilon_{2n}),$$

but

$$p_b^c(\zeta\alpha, \zeta\alpha) = \lim_{n\to\infty} p_b^c(\zeta\epsilon_{2n}, \zeta\epsilon_{2n}) = \lim_{n\to\infty} p_b^c(\epsilon_{2n+1}, \epsilon_{2n+1}) = 0. \tag{8}$$

Similarly,

$$p_b^c(\xi\alpha, \xi\alpha) = \lim_{n\to\infty} p_b^c(\xi\alpha, \xi\epsilon_{2n+1}) = \lim_{n\to\infty} p_b^c(\xi\epsilon_{2n+1}, \xi\epsilon_{2n+1}),$$

but

$$p_b^c(\xi\alpha, \xi\alpha) = \lim_{n\to\infty} p_b^c(\xi\epsilon_{2n+1}, \xi\epsilon_{2n+1}) = \lim_{n\to\infty} p_b^c(\epsilon_{2n+2}, \epsilon_{2n+2}) = 0.$$

Next, we have to show that ζ and ξ have a common fixed point in $\Re$. Let us assume that $\alpha \in \Re$ and for any $n \in \mathrm{N}$, we have

$$\begin{aligned}
p_b^c(\alpha, \zeta\alpha) &\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + p_b^c(\epsilon_{n+1}, \zeta\alpha)\big] - p_b^c(\epsilon_{n+1}, \epsilon_{n+1}) \\
&\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + p_b^c(\zeta\epsilon_n, \zeta\alpha)\big] \\
&\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + \kappa\big\{p_b^c(\epsilon_n, \zeta\alpha) + p_b^c(\zeta\epsilon_n, \alpha) + p_b^c(\epsilon_n, \alpha)\big\}\big] \\
&\leqslant \mu\big[p_b^c(\alpha, \epsilon_{n+1}) + \kappa\big\{p_b^c(\epsilon_n, \zeta\alpha) + p_b^c(\epsilon_{n+1}, \alpha) + p_b^c(\epsilon_n, \alpha)\big\}\big].
\end{aligned}$$

By completeness of $\Re$ and using equation (7) and (8), we have $p_b^c(\alpha, \epsilon_n)$ and $p_b^c(\epsilon_n, \zeta\alpha) \to 0$ as $n \to \infty$. Therefore, $p_b^c(\alpha, \zeta\alpha) \leqslant 0$, but by definition of partial metric $p_b^c(m, n) \geqslant 0$. Thus, $p_b^c(\alpha, \zeta\alpha) = 0$. Therefore, $\zeta\alpha = \alpha$. Hence, α is a fixed point of ζ. Similarly, we can prove that α is fixed point of ξ. Hence, $\zeta\alpha = \xi\alpha = \alpha$. Thus, α is a common fixed point of ζ and ξ. Now, we have to show that α is a unique common fixed point of ζ and ξ. We assume a contradictory that $\alpha, \beta \in \Re$ are two distinct common fixed points of ζ and ξ. Then $p_b^c(\alpha,\alpha) = 0$ and $p_b^c(\beta,\beta) = 0$. Now, consider

$$\begin{aligned} p_b^c(\alpha,\beta) &= p_b^c(\zeta\alpha,\xi\beta) \\ &\leqslant \kappa\big\{p_b^c(\alpha,\xi\beta) + p_b^c(\zeta\alpha,\beta) + p_b^c(\alpha,\beta)\big\} \\ &= \kappa\big\{p_b^c(\alpha,\beta) + p_b^c(\alpha,\beta) + p_b^c(\alpha,\beta)\big\} \\ &= 3\kappa p_b^c(\alpha,\beta) \end{aligned}$$

which implies that $(1-3\kappa)p_b^c(\alpha,\beta) \leqslant 0$. Since $1-3\kappa > 0$, therefore $p_b^c(\alpha,\beta) \leqslant 0$, which is contradiction. Hence, our assumption is false. Thus, $p_b^c(\alpha,\beta) = 0 \Rightarrow \alpha = \beta$. Hence, α is a unique common fixed point of ζ and ξ. ■

***Example* 3.5** Let $\Re = \{a,b,c,d\}$ be a non-void set with partial order relation $\alpha \preceq \beta$ if and only if $\alpha \leqslant \beta$. Consider a mapping $p_b^c : \Re \times \Re \to$ C as follows:

(α,β)	$p_b^c(\alpha,\beta)$
(a,a), (b,b)	0
(a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (c,c)	$e^{i\theta}$
(a,d), (d,a), (b,d), (d,b), (c,d), (d,c),(d,d)	$de^{i\theta}$

Here $(\Re, p_b^c)$ satisfies all properties of complete complex partial b-metric space with the coefficient $\mu \geqslant 1$ and $\theta \in [0,\pi]$. Define two self-mappings ζ, ξ on $\Re$ such that

$$\zeta\alpha = a \qquad \text{and} \qquad \xi\beta = \begin{cases} a & \texttt{if} \quad \beta \in \{a,b,c\} \\ b & \texttt{if} \quad \beta = d. \end{cases}$$

Then ζ and ξ are continuous and weakly increasing mappings. Now, we create different cases as follows:
Case-I: For $\alpha = a$ and $\beta = a$, we have $\zeta\alpha = a$ and $\xi\beta = a$. Then (6) is trivially true.
Case-II: For $\alpha = b$ and $\beta = d$, we have $\zeta\alpha = a$ and $\xi\beta = b$. So, for value of α and β, we have $p_b^c(\zeta\alpha,\xi\beta) = p_b^c(\zeta b,\xi d) = p_b^c(a,b) = e^{i\theta}$. Putting value of α and β in (6) and using above values, we have

$$p_b^c(\zeta\alpha,\xi\beta) \leqslant \kappa\big[p_b^c(\alpha,\xi\beta) + p_b^c(\zeta\alpha,\beta) + p_b^c(\alpha,\beta)\big],$$

which implies that

$$p_b^c(\zeta b,\xi d) \leqslant \kappa\big[p_b^c(b,\xi d) + p_b^c(\zeta b,d) + p_b^c(b,d)\big].$$

Thus, $e^{i\theta} \leqslant \kappa\big[p_b^c(b,b) + p_b^c(a,d) + p_b^c(b,d)\big]$ and hence, $e^{i\theta} \leqslant \kappa\big[0 + de^{i\theta} + de^{i\theta}\big]$. So $e^{i\theta} \leqslant \kappa(2d)e^{i\theta}$, which implies that $1 \leqslant \kappa(2d)$ and hence, $\kappa \geqslant \frac{1}{2d}$.
Case-III: For $\alpha = d$ and $\beta = d$, then $\zeta\alpha = a$ and $\xi\beta = b$. Therefore, for value of α and β, we have $p_b^c(\zeta\alpha,\xi\beta) = p_b^c(\zeta d,\xi d) = p_b^c(a,b) = e^{i\theta}$. Similar process of case-II and using (6), we have

$$p_b^c(\zeta d,\xi d) \leqslant \kappa\big[p_b^c(d,\xi d) + p_b^c(\zeta d,d) + p_b^c(d,d)\big],$$

which implies that

$$e^{i\theta} \leqslant \kappa\big[p_b^c(d,b) + p_b^c(a,d) + p_b^c(d,d)\big] = \kappa\big[de^{i\theta} + de^{i\theta} + de^{i\theta}\big].$$

Thus, $e^{i\theta} \leqslant \kappa(3d)e^{i\theta}$, which implies that $1 \leqslant \kappa(3d)$ and hence, $\kappa \geqslant \frac{1}{3d}$.

Case-IV: (i) for $\alpha = a$ and $\beta = d$, then $\zeta\alpha = a$ and $\xi\beta = b$ and (ii) for $\alpha = c$ and $\beta = d$, then $\zeta\alpha = a$ and $\xi\beta = b$. So, for all values of α and β, we have $p_b^c(\zeta\alpha, \xi\beta) = p_b^c(a,b) = e^{i\theta}$. As similar manner of above cases, putting for all values of α and β in equation (6) and using above values, we get $e^{i\theta} \leqslant \kappa\big[e^{i\theta} + de^{i\theta} + de^{i\theta}\big]$ and so, $e^{i\theta} \leqslant \kappa(1+2d)e^{i\theta}$. Thus,

$$1 \leqslant \kappa(1+2d) \Rightarrow \kappa \geqslant \frac{1}{1+2d}.$$

Case-V: (i) for $\alpha = b$ and $\beta = c$, then $\zeta\alpha = a$ and $\xi\beta = a$ and (ii) for $\alpha = c$ and $\beta = c$, then $\zeta\alpha = a$ and $\xi\beta = a$. Therefore, for all values of α and β, we have $p_b^c(\zeta\alpha, \xi\beta) = p_b^c(a,a) = 0$. Similarly as above, putting for all values of α and β in (6), we have $0 \leqslant \kappa\big[e^{i\theta} + e^{i\theta} + e^{i\theta}\big]$ and hence, $0 \leqslant \kappa(3)e^{i\theta}$. Thus, $0 \leqslant \kappa(3)$ and $\kappa \geqslant 0$.

Case-VI: For (i) $\alpha = a$, $\beta = b$ then $\zeta\alpha = a$ and $\xi\beta = a$.
(ii) $\alpha = a$, $\beta = c$ then $\zeta\alpha = a$ and $\xi\beta = a$.
(iii) $\alpha = b$, $\beta = b$ then $\zeta\alpha = a$ and $\xi\beta = a$.

So, for all values of α and β, we have $p_b^c(\zeta\alpha, \xi\beta) = p_b^c(a,a) = 0$. Similarly, putting for all values of α and β in equation (6) and using required values given above, we get $0 \leqslant \kappa\big[e^{i\theta} + e^{i\theta} + 0\big]$, which implies that $0 \leqslant \kappa(2)e^{i\theta}$. Thus, $0 \leqslant \kappa(2)$ and hence, $\kappa \geqslant 0$. Here, ζ and ξ satisfy all conditions of Theorem 3.4. Hence, $\boldsymbol{a}$ is a unique common fixed point of ζ and ξ. ■

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