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Integral type contractions and best proximity points

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Abstract. In the present work, Banach and Kannan integral type contractions in metric spaces endowed with a graph are considered and the existence and uniqueness of best proximity points for mappings satisfying in these contractions are proved.

Keywords: Best proximity point, integral type contraction, Lebesgue-integrable function, graph proximal.

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1. Introduction and preliminaries

In 2002, Branciari [3] used Lebesgue integrals in metric fixed point theory. Indeed, he considered a mapping Ω from a complete metric space $(\mathcal{X}, \lceil)$ into itself so that

$$\int_0^{\lceil(\Omega\mathfrak{a},\Omega\mathfrak{b})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \iota \int_0^{\lceil(\mathfrak{a},\mathfrak{b})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$, where $\iota \in (0,1)$ and $\psi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable function and Lebesgue-integral is finite on each compact subset of $[0,+\infty)$ and also, $\int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} > 0$ for all $\varepsilon > 0$. Then he established the existence and uniqueness of fixed points in for such mappings. On the other hand, Ran and Reurings [12] and Nieto and Rodríguez-López [9] reviewed the Banach contraction principle distinctly from another perspective and imposed a partial order to a metric space and discussed on the existence

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and uniqueness of fixed points for contractive conditions and for the comparable elements of $\mathcal{X}$ (see [10], too).

On the other hand, let $(\mathcal{X}, \lceil)$ be a metric space, $\mathfrak{E}$ and $\mathfrak{F}$ two non-empty subsets of $\mathcal{X}$ and $\Omega : \mathcal{E} \to \mathcal{F}$ be a non-self mapping. The best proximity point(s) (in short, bpp(s)) of Ω are the set all points $\mathfrak{a} \in \mathfrak{E}$ provided that $\lceil(\mathfrak{a}, \Omega\mathfrak{a}) = \lceil(\mathfrak{E}, \mathfrak{F})$, where $\lceil(\mathfrak{E}, \mathfrak{F}) = \inf\{\lceil(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a} \in \mathfrak{E}, \mathfrak{b} \in \mathfrak{F}\}$. The bpp theory prepares sufficient conditions that certify the existence of such points. The existence and uniqueness of bpp(s) for so many different contractive mappings have been considered by in metric and partially ordered metric spaces (for instance, read [5, 6, 11, 13, 14] and references therein).

In this paper, some results about the existence and uniqueness of bbp(s) for Banach [2] and Kannan [7] integral type contractions in metric spaces endowed with graph are proved. First, let us review few basic notions in bpp theory used in the sequel. Consider a pair $(\mathfrak{E}, \mathfrak{F})$ of nonempty subsets of $(\mathcal{X}, \lceil)$ and put

$$\mathfrak{E}_0 = \{\mathfrak{a} \in \mathfrak{E} : \lceil(\mathfrak{a}, \mathfrak{b}) = \lceil(\mathfrak{E}, \mathfrak{F}) \text{ for some } \mathfrak{b} \in \mathfrak{F}\},$$
$$\mathfrak{F}_0 = \{\mathfrak{b} \in \mathfrak{F} : \lceil(\mathfrak{a}, \mathfrak{b}) = \lceil(\mathfrak{E}, \mathfrak{F}) \text{ for some } \mathfrak{a} \in \mathfrak{E}\}.$$

Definition 1.1 [11] A pair $(\mathfrak{E}, \mathfrak{F})$ of nonempty subsets of a metric space $(\mathcal{X}, \lceil)$ is said to have the P-property if

$$\left.\begin{array}{l}\lceil(\mathfrak{a}_1, \mathfrak{b}_1) = \lceil(\mathfrak{E}, \mathfrak{F})\\ \lceil(\mathfrak{a}_2, \mathfrak{b}_2) = \lceil(\mathfrak{E}, \mathfrak{F})\end{array}\right\} \Longrightarrow \lceil(\mathfrak{a}_1, \mathfrak{a}_2) = \lceil(\mathfrak{b}_1, \mathfrak{b}_2)$$

for all $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{E}_0$ and $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathfrak{F}_0$.

Given an arbitrary graph $\mathfrak{G}$, a link is an edge of $\mathfrak{G}$ with distinct ends and a loop is an edge of $\mathfrak{G}$ with identical ends. Two or more links of $\mathfrak{G}$ with the same pairs of ends are called parallel edges of $\mathfrak{G}$. Let $(\mathcal{X}, \lceil)$ be a metric space and $\mathfrak{G}$ be a directed graph whose vertex set $V(\mathfrak{G})$ coincides with $\mathcal{X}$ and edge set $E(\mathfrak{G})$ contains all loops. Also, assume $\mathfrak{G}$ has no parallel edges. $(\mathcal{X}, \lceil)$ is called a metric space with the graph $\mathfrak{G}$. Let $\mathfrak{G}^{-1}$ be the conversion of $\mathfrak{G}$; that is, a directed graph obtained from $\mathfrak{G}$ by reversing the directions of the edges of $\mathfrak{G}$. Also, consider $\tilde{\mathfrak{G}}$ as the undirected graph obtained from $\mathfrak{G}$ by ignoring the directions of the edges $\mathfrak{G}$. Obviously, $V(\mathfrak{G}^{-1}) = V(\tilde{\mathfrak{G}}) = V(\mathfrak{G}) = X$, $E(\mathfrak{G}^{-1}) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathcal{X} \times \mathcal{X} : (\mathfrak{b}, \mathfrak{a}) \in E(\mathfrak{G})\}$ and $E(\tilde{\mathfrak{G}}) = E(\mathfrak{G}) \cup E(\mathfrak{G}^{-1})\}$.

Note that a weaker type of continuity defined in metric spaces with a graph is introduced by Jachymski [8]. Fallahi and Aghanians [4] defined orbitally $\mathfrak{G}$-continuous as follow, too.

Definition 1.2 [4] Let $(\mathcal{X}, \lceil)$ be a metric space with a graph $\mathfrak{G}$. A mapping $\Omega : \mathcal{X} \to \mathcal{X}$ is called orbitally $\mathfrak{G}$-continuous on $\mathcal{X}$ if $\Omega^{\kappa_n}\mathfrak{a} \to \mathfrak{b}$ implies $\Omega(\Omega^{\kappa_n}\mathfrak{a}) \to \Omega\mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ and all sequences $\{\kappa_n\}$ of positive integers such that $(\Omega^{\kappa_n}\mathfrak{a}, \Omega^{\kappa_n+1}\mathfrak{a}) \in E(\mathfrak{G})$ for all $n \in \mathbb{N}$.

In the sequel, assume $(\mathcal{X}, \lceil)$ is a metric space endowed with graph $\mathfrak{G}$, $(\mathfrak{E}, \mathfrak{F})$ is a pair of non-empty closed subsets of $\mathcal{X}$ unless otherwise stated. Also, suppose λ is the Lebesgue measure on the Borel σ-algebra of $[0, +\infty)$. For a Borel set $\mathcal{E} = [\mathfrak{a}, \mathfrak{b}]$, we apply $\int_{\mathfrak{a}}^{\mathfrak{b}} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}$ to show the Lebesgue integral of a function ψ on $\mathcal{E}$. Considering Ψ as a class consisting of all functions $\psi : [0, +\infty) \to [0, +\infty)$, we have the following properties:

(Ψ1) ψ is Lebesgue-integrable on $[0, +\infty)$;
(Ψ2) The value of the Lebesgue integral $\int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}$ is positive and finite for all $\varepsilon > 0$.

Lemma 1.3 [1] Let $\psi : [0,+\infty) \to [0,+\infty)$ be a function in the class Ψ and $\{\mathfrak{a}_n\}$ be a sequence of nonnegative real numbers. Then the following statements hold:

1. If $\int_0^{\mathfrak{a}_n} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \to 0$ as $n \to +\infty$, then $\mathfrak{a}_n \to 0$ as $n \to +\infty$;
2. If $\{\mathfrak{a}_n\}$ is monotone and converges to some $\mathfrak{a} \geqslant 0$, then $\int_0^{\mathfrak{a}_n} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \to \int_0^{\mathfrak{a}} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}$ as $n \to +\infty$.

2. Main results

Continuing the idea of Sadiq Basha [13], the concept of a proximal mapping in partially ordered metric spaces can be defined as follows:

Definition 2.1 A mapping $\Omega : \mathfrak{E} \to \mathfrak{F}$ is named an graph proximal if

$$\left.\begin{array}{r} (\mathfrak{b}_1,\mathfrak{b}_2) \in E(\mathfrak{G}) \\ \lceil(\mathfrak{a}_1,\Omega\mathfrak{b}_1) = \lceil(\mathfrak{E},\mathfrak{F}) \\ \lceil(\mathfrak{a}_2,\Omega\mathfrak{b}_2) = \lceil(\mathfrak{E},\mathfrak{F}) \end{array}\right\} \Longrightarrow (\mathfrak{a}_1,\mathfrak{a}_2) \in E(\mathfrak{G})$$

for all $\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{b}_1,\mathfrak{b}_2 \in \mathcal{X}$.

Definition 2.2 A mapping $\Omega : \mathfrak{E} \to \mathfrak{F}$ is named an integral Banach graph type contraction if

(*i*) Ω is graph proximal;
(*ii*) there exists $\psi \in \Psi$ and $\iota \in (0,1)$ provided that

$$\int_0^{\lceil(\Omega\mathfrak{a},\Omega\mathfrak{b})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \iota \int_0^{\lceil(\mathfrak{a},\mathfrak{b})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \tag{1}$$

holds for all $\mathfrak{a},\mathfrak{b} \in \mathfrak{E}$ with $(\mathfrak{a},\mathfrak{b}) \in E(\mathfrak{G})$.

The first result of this paper is the following theorem.

Theorem 2.3 Let $(\mathcal{X},\lceil)$ be complete endowed with graph $\mathfrak{G}$ where its edges has the property of transitivity, $\mathfrak{E}_0 \neq \emptyset$ and $\Omega : \mathfrak{E} \to \mathfrak{F}$ be an integral Banach graph type contraction satisfying the following conditions:

(*i*) $\Omega(\mathfrak{E}_0) \subseteq \mathfrak{F}_0$ and the pair $(\mathfrak{E},\mathfrak{F})$ have the P-property;
(*ii*) there exist $\mathfrak{a}_0,\mathfrak{a}_1 \in \mathfrak{E}_0$ such that $(\mathfrak{a}_0,\mathfrak{a}_1) \in E(\mathfrak{G})$ and $\lceil(\mathfrak{a}_1,\Omega\mathfrak{a}_0) = \lceil(\mathfrak{E},\mathfrak{F})$.

Then Ω has a bpp in $\mathcal{X}$ if one of the following conditions holds:

1) Ω is orbitally $\mathfrak{G}$-continuous on $\mathfrak{E}$.
2) $(\mathcal{X},\lceil)$ satisfies the following property:
(∗) If $\mathfrak{a}_n \to \mathfrak{a}$ and $(\mathfrak{a}_n,\mathfrak{a}_{n+1}) \in E(\mathfrak{G})$ for all $n \geqslant 1$, then there is a subsequence $\{\mathfrak{a}_{n_k}\}$ of $\{\mathfrak{a}_n\}$ provided that $(\mathfrak{a}_{n_k},\mathfrak{a}) \in E(\mathfrak{G})$ for all $k \geqslant 1$.

Further, if for any two bpp(s) $u,v \in \mathfrak{E}$ we have $(u,v) \in E(\mathfrak{G})$, then Ω has a unique bpp.

Proof. By (*ii*), there are $\mathfrak{a}_0,\mathfrak{a}_1 \in \mathfrak{E}_0$ so that $(\mathfrak{a}_0,\mathfrak{a}_1) \in E(\mathfrak{G})$ and $\lceil(\mathfrak{a}_1,\Omega\mathfrak{a}_0) = \lceil(\mathfrak{E},\mathfrak{F})$. Due to $\Omega(\mathfrak{E}_0) \subseteq \mathfrak{F}_0$, there exists $\mathfrak{a}_2 \in E_0$ so that $\lceil(\mathfrak{a}_2,\Omega\mathfrak{a}_1) = \lceil(\mathfrak{E},\mathfrak{F})$ and because of order proximality of Ω, we have $(\mathfrak{a}_1,\mathfrak{a}_2) \in E(\mathfrak{G})$. Continuing this process, we have a sequence

$\{\mathfrak{a}_n\}$ in $\mathfrak{E}_0$ provided that

$$(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \in E(\mathfrak{G}) \quad \text{and} \quad \lceil(\mathfrak{a}_{n+1}, \Omega\mathfrak{a}_n) = \lceil(\mathfrak{E}, \mathfrak{F}) \qquad n = 0, 1, \ldots . \tag{2}$$

What's more, using the P-property of $(\mathfrak{E}, \mathfrak{F})$, we get for all $n \in \mathbb{N}$ that

$$\left.\begin{array}{l}\lceil(\mathfrak{a}_n, \Omega\mathfrak{a}_{n-1}) = \lceil(\mathfrak{E}, \mathfrak{F}) \\ \lceil(\mathfrak{a}_{n+1}, \Omega\mathfrak{a}_n) = \lceil(\mathfrak{E}, \mathfrak{F})\end{array}\right\} \Longrightarrow \lceil(\mathfrak{a}_n, \mathfrak{a}_{n+1}) = \lceil(\Omega\mathfrak{a}_{n-1}, \Omega\mathfrak{a}_n). \tag{3}$$

Now, letting $n \in \mathbb{N}$ and using $(\mathfrak{a}_{n-1}, \mathfrak{a}_n) \in E(\mathfrak{G})$, (1) and (3), we get

$$\begin{aligned}\int_0^{\lceil(\mathfrak{a}_n, \mathfrak{a}_{n+1})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} &= \int_0^{\lceil(\Omega\mathfrak{a}_{n-1}, \Omega\mathfrak{a}_n)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\ &\leqslant \iota \int_0^{\lceil(\mathfrak{a}_{n-1}, \mathfrak{a}_n)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \ldots \leqslant \iota^n \int_0^{\lceil(\mathfrak{a}_0, \mathfrak{a}_1)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}.\end{aligned} \tag{4}$$

Since $\int_0^{\lceil(\mathfrak{a}_0, \mathfrak{a}_1)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}$ is finite due to (Ψ_2) and $\iota \in (0, 1)$, we conclude that

$$\int_0^{\lceil(\mathfrak{a}_n, \mathfrak{a}_{n+1})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \to 0.$$

Using Lemma 1.3, $\lceil(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \to 0$ as $n \to +\infty$.

Now, we demonstrate that $\{\mathfrak{a}_n\}$ is a Cauchy sequence in $\mathfrak{E}_0 \subseteq \mathfrak{E}$. To contrary, assume that sequence $\{\mathfrak{a}_n\}$ isn't Cauchy, then there exist $\varepsilon > 0$ and positive integers m_k and n_k with $m_k > n_k \geqslant k$ and $\lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{n_k}) \geqslant \varepsilon$ for $k = 1, 2, \cdots$. Keeping n_k fixed for k large enough, say $k \geqslant k_0$, we can assume without loss of generality that m_k is the smallest integer greater than n_k with $\lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{n_k}) \geqslant \varepsilon$, i.e. $\lceil(\mathfrak{a}_{m_k-1}, \mathfrak{a}_{n_k}) < \varepsilon$ for all $k \geqslant k_0$. Consequently, we get

$$\begin{aligned}\varepsilon &\leqslant \lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{n_k}) \\ &\leqslant \lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{m_k-1}) + \lceil(\mathfrak{a}_{m_k-1}, \mathfrak{a}_{n_k}) \\ &< \lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{m_k-1}) + \varepsilon.\end{aligned}$$

Now, if $k \to +\infty$, then $\lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{n_k}) \to \varepsilon$ in that $\lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{m_k-1}) \to 0$. On the other hand, we have

$$\lceil(\mathfrak{a}_{m_k+1}, \mathfrak{a}_{n_k+1}) \leqslant \lceil(\mathfrak{a}_{m_k+1}, \mathfrak{a}_{m_k}) + \lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{n_k}) + \lceil(\mathfrak{a}_{n_k}, \mathfrak{a}_{n_k+1})$$

for all $k \geqslant 1$. Letting $k \to +\infty$ and applying (4), we gain

$$\limsup_{k \to +\infty} \lceil(\mathfrak{a}_{m_k+1}, \mathfrak{a}_{n_k+1}) \leqslant \varepsilon.$$

What's more, for all $k \geqslant 1$,

$$\lceil(\mathfrak{a}_{m_k+1}, \mathfrak{a}_{n_k+1}) \geqslant \lceil(\mathfrak{a}_{m_k}, \mathfrak{a}_{n_k}) - \lceil(\mathfrak{a}_{m_k+1}, \mathfrak{a}_{m_k}) - \lceil(\mathfrak{a}_{n_k}, \mathfrak{a}_{n_k+1})$$

holds. In like manner, we get

$$\liminf_{k\to+\infty} \lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \geqslant \varepsilon.$$

Hence, $\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \to \varepsilon$. By passing to two subsequences with the same choice function if necessary and without loss of generality, one can show that both $\{\lceil(\Omega\mathfrak{a}_{m_k},\Omega\mathfrak{a}_{n_k})\}$ and $\{\lceil(\Omega\mathfrak{a}_{m_k+1},\Omega\mathfrak{a}_{n_k+1})\}$ are monotone. As we know, edges of graph $\mathfrak{G}$ have the property of transitivity and $(\mathfrak{a}_{n_k},\mathfrak{a}_{m_k}) \in E(\mathfrak{G})$ for all $k \geqslant 1$. Now, using Lemma 1.3, we gain

$$\begin{aligned}\int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} &= \lim_{k\to+\infty} \int_0^{\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\ &= \lim_{k\to+\infty} \int_0^{\lceil(\Omega\mathfrak{a}_{m_k},\Omega\mathfrak{a}_{n_k})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\ &\leqslant \iota \lim_{k\to+\infty} \int_0^{\lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\ &= \iota \int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t},\end{aligned}$$

where $\psi \in \Psi$ and $\iota \in (0,1)$ are as in (ii). Then $\int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} = 0$, which is a contradiction. Therefore, $\{\mathfrak{a}_n\}$ is Cauchy in $\mathfrak{E}_0 \subseteq \mathfrak{E}$. Due to the completeness of $(\mathcal{X},\lceil)$, there exists $a^* \in \mathcal{X}$ provided that $\mathfrak{a}_n \to a^*$. In addition, it follows from the closeness of $\mathfrak{E}$ that $\mathfrak{a}^* \in \mathfrak{E}$.

We next show that $\mathfrak{a}^*$ is a bpp for Ω. First let Ω be $\mathfrak{G}$-continuous on $\mathfrak{E}$. Since $\mathfrak{a}_n \to \mathfrak{a}^*$ and by transitivity of edges $\mathfrak{G}$, $(\mathfrak{a}_n,\mathfrak{a}_{n+1}) \in E(\mathfrak{G})$ for $n = 0,1,\ldots$, we conclude that $\Omega\mathfrak{a}_n \to \Omega\mathfrak{a}^*$. Additionally, the joint continuity of the metric $\lceil$ implies that $\lceil(\mathfrak{a}_n,\Omega\mathfrak{a}_n) \to \lceil(\mathfrak{a}^*,\Omega\mathfrak{a}^*)$. On the other hand, by using (2), $\{\lceil(\mathfrak{a}_n,\Omega\mathfrak{a}_n)\}$ is a constant sequence converging to $\lceil(\mathfrak{E},\mathfrak{F})$. Thus, it follows from the uniqueness of the limit that $\lceil(\mathfrak{a}^*,\Omega^*\mathfrak{a}) = \lceil(\mathfrak{E},\mathfrak{F})$; that is, $\mathfrak{a}^*$ is a bpp for Ω.

Second, assume that $(*)$ holds. Then there exists a strictly increasing sequence $\{n_k\}$ of positive integers provided that $(\mathfrak{a}_{n_k},a^*) \in E(\mathfrak{G})$ for all $k \geqslant 1$. Hence, from the contractive condition (1), we gain

$$\int_0^{\lceil(\Omega\mathfrak{a}_{n_k},\Omega\mathfrak{a}^*)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \iota \int_0^{\lceil(\mathfrak{a}_{n_k},\mathfrak{a}^*)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \int_0^{\lceil(\mathfrak{a}_{n_k},\mathfrak{a}^*)} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \to 0$$

as $k \to +\infty$, i.e. $\Omega\mathfrak{a}_{n_k} \to \Omega\mathfrak{a}^*$. Again, the joint continuity of the metric $\lceil$ implies that $d(\mathfrak{a}_{n_k},\Omega\mathfrak{a}_{n_k}) \to d(\mathfrak{a}^*,\Omega\mathfrak{a}^*)$ and similarly, $\mathfrak{a}^*$ is a bpp for Ω.

Ultimately, for uniqueness, assume that $\mathfrak{a}^{**} \in \mathfrak{E}$ is another bpp for Ω provided that $(\mathfrak{a}^*,\mathfrak{a}^{**}) \in E(\mathfrak{G})$. Since the pair $(\mathfrak{E},\mathfrak{F})$ have the P-property, it follows that

$$\left.\begin{aligned}\lceil(\mathfrak{a}^*,\Omega\mathfrak{a}^*) = \lceil(\mathfrak{E},\mathfrak{F})\\ \lceil(\mathfrak{a}^{**},\Omega\mathfrak{a}^{**}) = \lceil(\mathfrak{E},\mathfrak{F})\end{aligned}\right\} \Longrightarrow \lceil(\mathfrak{a}^*,\mathfrak{a}^{**}) = \lceil(\Omega\mathfrak{a}^*,\Omega\mathfrak{a}^{**}).$$

Hence, by (1), we have

$$\int_0^{\lceil(\mathfrak{a}^*,\mathfrak{a}^{**})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} = \int_0^{\lceil(\Omega\mathfrak{a}^*,\Omega\mathfrak{a}^{**})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \iota \int_0^{\lceil(\mathfrak{a}^*,\mathfrak{a}^{**})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t},$$

which is a contradiction, except for $\int_0^{\lceil(\mathfrak{a}^*,\mathfrak{a}^{**})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}=0$. Hence, $\lceil(\mathfrak{a}^*,\mathfrak{a}^{**})=0$. Thus, $\mathfrak{a}^*=\mathfrak{a}^{**}$. ■

The second result of this paper is the following theorem.

Definition 2.4 A mapping $\Omega:\mathfrak{E}\to\mathfrak{F}$ is named an integral Kannan graph type contraction if

(*i*) Ω is graph proximal;
(*ii*) there exists $\psi\in\Psi$ and $\iota,\varsigma\in(0,\frac{1}{2})$ provided that

$$\int_0^{\lceil(\Omega\mathfrak{a},\Omega\mathfrak{b})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}\leqslant\iota\int_0^{\lceil(\mathfrak{a},\Omega\mathfrak{a})-\lceil(\mathfrak{E},\mathfrak{F})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}+\varsigma\int_0^{\lceil(\mathfrak{b},\Omega\mathfrak{b})-\lceil(\mathfrak{E},\mathfrak{F})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \tag{5}$$

holds for all $\mathfrak{a},\mathfrak{b}\in\mathcal{X}$ with $(\mathfrak{a},\mathfrak{b})\in E(\mathfrak{G})$.

Theorem 2.5 Let $(\mathcal{X},\lceil)$ be complete endowed with graph $\mathfrak{G}$ where its edges has the property of transitivity, $\mathfrak{E}_0\neq\emptyset$ and $\Omega:\mathfrak{E}\to\mathfrak{F}$ be an integral Kannan type contraction satisfying the following conditions:

(*i*) $\Omega(\mathfrak{E}_0)\subseteq\mathfrak{F}_0$ and the pair $(\mathfrak{E},\mathfrak{F})$ have the P-property;
(*ii*) there exist $\mathfrak{a}_0,\mathfrak{a}_1\in\mathfrak{E}_0$ such that $(\mathfrak{a}_0,\mathfrak{a}_1)\in E(\mathfrak{G})$ and $\lceil(\mathfrak{a}_1,\Omega\mathfrak{a}_0)=\lceil(\mathfrak{E},\mathfrak{F})$.
(*iii*) Ω is orbitally $\mathfrak{G}$-continuous on $\mathfrak{E}$.

Then Ω has a bpp in $\mathcal{X}$. Further, if for any two bpp(s) $u,v\in\mathfrak{E}$ we have $(u,v)\in E(\mathfrak{G})$, then Ω has a unique bpp.

Proof. By (*ii*), there are $\mathfrak{a}_0,\mathfrak{a}_1\in E_0$ so that $(\mathfrak{a}_0,\mathfrak{a}_1)\in E(\mathfrak{G})$ and $\lceil(\mathfrak{a}_1,\Omega\mathfrak{a}_0)=\lceil(\mathfrak{E},\mathfrak{F})$. Due to $\Omega(\mathfrak{E}_0)\subseteq\mathfrak{F}_0$, there exists $\mathfrak{a}_2\in\mathfrak{E}_0$ so that $\lceil(\mathfrak{a}_2,\Omega\mathfrak{a}_1)=\lceil(\mathfrak{E},\mathfrak{F})$ and because of order proximality of Ω, we have $(\mathfrak{a}_1,\mathfrak{a}_2)\in E(\mathfrak{G})$. Continuing this process, we have a sequence $\{\mathfrak{a}_n\}$ in $\mathfrak{E}_0$ provided that

$$(\mathfrak{a}_n,\mathfrak{a}_{n+1})\in E(\mathfrak{G})\quad\text{and}\quad\lceil(\mathfrak{a}_{n+1},\Omega\mathfrak{a}_n)=\lceil(\mathfrak{E},\mathfrak{F})\qquad n=0,1,\ldots. \tag{6}$$

Since the pair $(\mathfrak{E},\mathfrak{F})$ have the P-property, it follows for all $n\in\mathbb{N}$ that

$$\left.\begin{array}{l}\lceil(\mathfrak{a}_n,\Omega\mathfrak{a}_{n-1})=\lceil(\mathfrak{E},\mathfrak{F})\\ \lceil(\mathfrak{a}_{n+1},\Omega\mathfrak{a}_n)=\lceil(\mathfrak{E},\mathfrak{F})\end{array}\right\}\Longrightarrow\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})=\lceil(\Omega\mathfrak{a}_{n-1},\Omega\mathfrak{a}_n). \tag{7}$$

Now, considering $n\in\mathbb{N}$ and using $(\mathfrak{a}_{n-1},\mathfrak{a}_n)\in E(\mathfrak{G})$, (5) and (6), we get

$$\begin{aligned}\int_0^{\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}&=\int_0^{\lceil(\Omega\mathfrak{a}_{n-1},\Omega\mathfrak{a}_n)}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}\\&\leqslant\iota\int_0^{\lceil(\mathfrak{a}_{n-1},\Omega\mathfrak{a}_{n-1})-\lceil(\mathfrak{E},\mathfrak{F})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}+\varsigma\int_0^{\lceil(\mathfrak{a}_n,\Omega\mathfrak{a}_n)-\lceil(\mathfrak{E},\mathfrak{F})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}\end{aligned} \tag{8}$$

Moreover, since

$$\lceil(\mathfrak{a}_{n-1},\Omega\mathfrak{a}_{n-1})-\lceil(\mathfrak{E},\mathfrak{F})\leqslant\lceil(\mathfrak{a}_{n-1},\mathfrak{a}_n)+\underbrace{\lceil(\mathfrak{a}_n,\Omega\mathfrak{a}_{n-1})-\lceil(\mathfrak{E},\mathfrak{F})}_{=0}=\lceil(\mathfrak{a}_{n-1},\mathfrak{a}_n)$$

$$\lceil(\mathfrak{a}_n,\Omega\mathfrak{a}_n)-\lceil(\mathfrak{E},\mathfrak{F})\leqslant\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})+\underbrace{\lceil(\mathfrak{a}_{n+1},\Omega\mathfrak{a}_n)-\lceil(\mathfrak{E},\mathfrak{F})}_{=0}=\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1}).$$

Then, since integral is nondecreasing and by (8), we gain

$$\int_0^{\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \iota\int_0^{\lceil(\mathfrak{a}_{n-1},\mathfrak{a}_n)}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} + \varsigma\int_0^{\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}.$$

Hence,

$$\int_0^{\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \frac{\iota}{1-\varsigma}\int_0^{\lceil(\mathfrak{a}_{n-1},\mathfrak{a}_n)}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \leqslant \ldots \leqslant (\frac{\iota}{1-\varsigma})^n\int_0^{\lceil(\mathfrak{a}_0,\mathfrak{a}_1)}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}.$$

Since, $\int_0^{\lceil(\mathfrak{a}_0,\mathfrak{a}_1)}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t}$ is finite due to (Ψ_2) and $\frac{\iota}{1-\varsigma} < 1$, we get

$$\int_0^{\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1})}\psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \to 0.$$

Using Lemma 1.3, we have $\lceil(\mathfrak{a}_n,\mathfrak{a}_{n+1}) \to 0$ as $n \to +\infty$.

Now, we demonstrate that $\{\mathfrak{a}_n\}$ is a Cauchy sequence in $\mathfrak{E}_0 \subseteq \mathfrak{E}$. To contrary, assume that sequence $\{\mathfrak{a}_n\}$ isn't Cauchy, then there exist $\varepsilon > 0$ and positive integers m_k and n_k with $m_k > n_k \geqslant k$ and $\lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k}) \geqslant \varepsilon$ for $k = 1, 2, \cdots$. Keeping n_k fixed for k large enough, say $k \geqslant k_0$, we can assume without loss of generality that m_k is the smallest integer greater than n_k with $\lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k}) \geqslant \varepsilon$, i.e. $\lceil(\mathfrak{a}_{m_k-1},\mathfrak{a}_{n_k}) < \varepsilon$ for all $k \geqslant k_0$. Consequently, we get

$$\begin{aligned}\varepsilon &\leqslant \lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k})\\ &\leqslant \lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{m_k-1}) + \lceil(\mathfrak{a}_{m_k-1},\mathfrak{a}_{n_k})\\ &< \lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{m_k-1}) + \varepsilon.\end{aligned}$$

Now, if $k \to ++$, then $\lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k}) \to \varepsilon$ in that $\lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{m_k-1}) \to 0$. On the other hand, we have

$$\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \leqslant \lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{m_k}) + \lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k}) + \lceil(\mathfrak{a}_{n_k},\mathfrak{a}_{n_k+1})$$

for all $k \geqslant 1$. Letting $k \to +\infty$ and applying (8), we gain

$$\limsup_{k\to+\infty}\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \leqslant \varepsilon.$$

Further, for all $k \geqslant 1$,

$$\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \geqslant \lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{n_k}) - \lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{m_k}) - \lceil(\mathfrak{a}_{n_k},\mathfrak{a}_{n_k+1})$$

hold. In like manner, we get

$$\liminf_{k\to+\infty}\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \geqslant \varepsilon.$$

Hence, $\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1}) \to \varepsilon$. By passing to two subsequences with the same choice function if necessary and without loss of generality, one can show that both $\{\lceil(\Omega\mathfrak{a}_{m_k},\Omega\mathfrak{a}_{n_k})\}$

and $\{\lceil(\Omega\mathfrak{a}_{m_k+1}, \Omega\mathfrak{a}_{n_k+1})\}$ are monotone. As we know, edges of graph $\mathfrak{G}$ have the property of transitivity and $(\mathfrak{a}_{n_k}, \mathfrak{a}_{m_k}) \in E(\mathfrak{G})$ for all $k \geqslant 1$. Now, using Lemma 1.3, we obtain

$$\begin{aligned}
\int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} &= \lim_{k\to+\infty} \int_0^{\lceil(\mathfrak{a}_{m_k+1},\mathfrak{a}_{n_k+1})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\
&= \lim_{k\to+\infty} \int_0^{\lceil(\Omega\mathfrak{a}_{m_k},\Omega\mathfrak{a}_{n_k})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\
&\leqslant \iota \int_0^{\lceil(\mathfrak{a}_{m_k},\Omega\mathfrak{a}_{m_k})-\lceil(\mathfrak{E},\mathfrak{F})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} + \varsigma \int_0^{\lceil(\mathfrak{a}_{n_k},\Omega\mathfrak{a}_{n_k})-\lceil(\mathfrak{E},\mathfrak{F})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\
&\leqslant \iota \int_0^{\lceil(\mathfrak{a}_{m_k},\mathfrak{a}_{m_k+1})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} + \varsigma \int_0^{\lceil(\mathfrak{a}_{n_k},\mathfrak{a}_{n_k+1})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \to 0,
\end{aligned}$$

where $\psi \in \Psi$ and $\iota, \varsigma \in (0, \frac{1}{2})$ are as in (ii). Hence, $\int_0^{\varepsilon} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} = 0$, which is a contradiction. Thus, $\{\mathfrak{a}_n\}$ is a Cauchy sequence in $\mathfrak{E}_0 \subseteq \mathfrak{E}$. Due to the completeness of $(\mathcal{X}, \lceil)$, there exists $a^* \in \mathcal{X}$ provided that $\mathfrak{a}_n \to a^*$. In addition, it follows from the closeness of $\mathfrak{E}$ that $\mathfrak{a}^* \in \mathcal{X}$.

Next, we present that $\mathfrak{a}^*$ is a bpp for Ω. Since Ω is $\mathfrak{G}$-continuous on $\mathfrak{E}$ and $\mathfrak{a}_n \to \mathfrak{a}^*$ and by transitivity of edges $\mathfrak{G}$, $(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \in E(\mathfrak{G})$ for $n = 0, 1, \ldots$, then $\Omega\mathfrak{a}_n \to \Omega\mathfrak{a}^*$. Also, the joint continuity of metric $\lceil$ implies that $\lceil(\mathfrak{a}_n, \Omega\mathfrak{a}_n) \to \lceil(\mathfrak{a}^*, \Omega\mathfrak{a}^*)$. On the other hand, using (2), the sequence $\{\lceil(\mathfrak{a}_n, \Omega\mathfrak{a}_n)\}$ is a constant sequence converging to $\lceil(\mathfrak{E}, \mathfrak{F})$. Thus, it follows from the uniqueness of the limit that $\lceil(\mathfrak{a}^*, \Omega^*\mathfrak{a}) = \lceil(\mathfrak{E}, \mathfrak{F})$, i.e. $\mathfrak{a}^*$ is a bpp for Ω.

Ultimately, for uniqueness, suppose that $\mathfrak{a}^{**} \in E$ is another bpp for Ω provided that $(\mathfrak{a}^*, \mathfrak{a}^{**}) \in E(G)$. Since the pair $(\mathfrak{E}, \mathfrak{F})$ have the P-property, we conclude that

$$\left.\begin{array}{r} \lceil(\mathfrak{a}^*, \Omega\mathfrak{a}^*) = \lceil(\mathfrak{E}, \mathfrak{F}) \\ \lceil(\mathfrak{a}^{**}, \Omega\mathfrak{a}^{**}) = \lceil(\mathfrak{E}, \mathfrak{F}) \end{array}\right\} \Longrightarrow \lceil(\mathfrak{a}^*, a^{**}) = \lceil(\Omega\mathfrak{a}^*, \Omega\mathfrak{a}^{**}).$$

Using (5), we obtain

$$\begin{aligned}
\int_0^{\lceil(\mathfrak{a}^*,\mathfrak{a}^{**})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} &= \int_0^{\lceil(\Omega\mathfrak{a}^*,\Omega\mathfrak{a}^{**})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\
&\leqslant \iota \int_0^{\lceil(\mathfrak{a}^*,\Omega\mathfrak{a}^*)-\lceil(\mathfrak{E},\mathfrak{F})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} + \varsigma \int_0^{\lceil(\mathfrak{a}^{**},\Omega\mathfrak{a}^{**})-\lceil(\mathfrak{E},\mathfrak{F})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} \\
&= 0,
\end{aligned}$$

which induces that $\int_0^{\lceil(\mathfrak{a}^*,\mathfrak{a}^{**})} \psi(\mathfrak{t})\mathfrak{d}\mathfrak{t} = 0$, i.e. $\lceil(\mathfrak{a}^*, \mathfrak{a}^{**}) = 0$, which implies that $a^* = a^{**}$. Here, the proof ends. ■

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