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b-metric spaces with a graph and best proximity points for some contractions

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Abstract. In this paper, we introduce a new type of graph contraction using a special class of functions and give a best proximity point theorem for this contraction in complete metric spaces endowed with a graph. Then we support our main theorem by a non-trivial example and give some consequences of it for usual graphs.

Keywords: Best proximity point, $\mathcal{G}$-continuous mapping, fixed point.

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1. Introduction and preliminaries

Let $(\mathcal{X}, \rho)$ be a metric space, $\mathcal{A}$ and B be two non-empty subsets of $\mathcal{X}$, $\rho(\mathcal{A}, \mathcal{B}) = \inf\{\rho(a, b); a \in \mathcal{A},\ b \in \mathcal{B}\}$ and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a non-self mapping. The best proximity point of $\mathcal{T}$ are the set all points $\mathcal{A}$ in $\mathcal{X}$ provided that $\rho(a, \mathcal{T}a) = \rho(a, b)$. The main purpose of best proximity point theory is to prepare sufficient conditions that assure the existence of such points. Hence, many authors were studied the existence and uniqueness of these points in metric spaces and partially ordered metric spaces in [8–11] and references therein. In 2008, Jachymski [6] applied graphs in metric fixed point theory and generalized the Banach contraction principle. In the sequel, many authors proved some fixed point theorems and best proximity point results in various metric spaces endowed with a graph (for example, see [4, 5] and references therein). On the other hand, the concept of b-metric spaces was studied by Bakhtin [1] and Czerwik [3].

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Definition 1.1 Let $\mathcal{A}$ be a nonempty set and $s \geqslant 1$ be a real number. Suppose that the mapping $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfies in the following conditions:

(d_1) $\rho(a, b) = 0$ if and only if $a = b$;
(d_2) $\rho(a, b) = \rho(y, x)$ for all $a, b \in \mathcal{X}$;
(d_3) $\rho(a, c) \leqslant s[\rho(a, b) + \rho(b, c)]$ for all $a, b, c \in \mathcal{X}$.

Then ρ is called a b-metric and $(\mathcal{X}, \rho)$ is called a b-metric space.

Obviously, for $s = 1$, a b-metric space is a metric space. Also, for notions such as convergent and Cauchy sequences, completeness, continuity and etc in b-metric spaces, see [1, 3].

In an arbitrary (not necessarily simple) graph $\mathcal{G}$, a link is an edge of $\mathcal{G}$ with distinct ends and a loop is an edge of $\mathcal{G}$ with identical ends. Two or more links of $\mathcal{G}$ with the same pairs of ends are called parallel edges of $\mathcal{G}$. Let $(\mathcal{X}, \rho)$ be a metric space and G be a directed graph with vertex set $V(\mathcal{G}) = \mathcal{X}$ such that the edge set $E(\mathcal{G})$ contains all loops, that is, $(a, a) \in E(\mathcal{G})$ for all $a \in \mathcal{X}$. Assume further that $\mathcal{G}$ has no parallel edges. Under these hypotheses, the graph $\mathcal{G}$ can be easily denoted by the ordered pair $(V(\mathcal{G}), E(\mathcal{G}))$ and it is said that the metric space $(\mathcal{X}, \rho)$ is endowed with the graph $\mathcal{G}$.For more details on graphs, see [2].

Now, consider a pair $(\mathcal{A}, \mathcal{B})$ of nonempty subsets of $(\mathcal{X}, \rho)$. Then we will apply the following notations in the sequel.

$$\mathcal{A}_0 = \{a \in \mathcal{A} : \ \rho(a, b) = \rho(\mathcal{A}, \mathcal{B}) \text{ for some } b \in \mathcal{B}\},$$
$$\mathcal{B}_0 = \{b \in \mathcal{B} : \ \rho(a, b) = \rho(\mathcal{A}, \mathcal{B}) \text{ for some } a \in \mathcal{A}\}.$$

The purpose of this paper is to prove the existence and uniqueness of best proximity points for Reich-type [7] contractive mappings in b-metric spaces endowed with a graph. Our results are the extensions of some best proximity point theorems given in terms of metric spaces, partially ordered metric spaces and b-metric spaces to b-metric spaces equipped with a graph $\mathcal{G}$.

2. Main results

Suppose that $(\mathcal{X}, \rho)$ is a b-metric space with parameter $s \geqslant 1$ endowed with a graph $\mathcal{G}$ and $(\mathcal{A}, \mathcal{B})$ is a pair of nonempty closed subsets of $\mathcal{A}$ with $\mathcal{A}_0 \neq \emptyset$ unless otherwise stated.

Definition 2.1 An element $a \in \mathcal{A}$ is said to be a best proximity point for $\mathcal{T}$ if $\rho(a, \mathcal{T}a) = \rho(a, b)$.

Note that if a is a best proximity point for $\mathcal{T}$, then we have $a \in \mathcal{A}_0$ and $\mathcal{T}a \in \mathcal{B}_0$.

Definition 2.2 The pair $(\mathcal{A}, \mathcal{B})$ is said to be have the P-property if

$$\left.\begin{array}{l}\rho(a_1, b_1) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(a_2, b_2) = \rho(\mathcal{A}, \mathcal{B})\end{array}\right\} \Longrightarrow \rho(a_1, a_2) = \rho(b_1, b_2)$$

for all $a_1, a_2 \in \mathcal{A}_0$ and $b_1, b_2 \in \mathcal{B}_0$.

In Definition 2.1 and Definition 2.2, set $s = 1$. Then we have the same definitions in metric spaces introduced in [8, 10]. Also, we can rewrite the notion of $\mathcal{G}$-continuity introduced by Jachymski [6] in b-metric spaces endowed with a graph as follows:

Definition 2.3 A mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is said to be $\mathcal{G}$-continuous on $\mathcal{A}$ if $a_n \to a$ in $\mathcal{A}$ implies $\mathcal{T}a_n \to \mathcal{T}a$ in $\mathcal{B}$ for all sequences $\{a_n\}$ in $\mathcal{A}$ with $(a_n, a_{n+1}) \in E(\mathcal{G})$ for $n = 1, 2, \cdots$.

Also, by using the idea of Sadiq Basha [10], we consider the concept of a $\mathcal{G}$-proximal mapping in a b-metric space endowed with a graph as follows:

Definition 2.4 A non-self mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ is $\mathcal{G}$-proximal if

$$\left.\begin{array}{r}(b_1, b_2) \in E(\mathcal{G}) \\ \rho(a_1, \mathcal{T}b_1) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(a_2, \mathcal{T}b_2) = \rho(\mathcal{A}, \mathcal{B})\end{array}\right\} \Longrightarrow (a_1, a_2) \in E(\mathcal{G})$$

for all $a_1, a_2, b_1, b_2 \in \mathcal{A}$.

The following theorem is the main result of this paper.

Theorem 2.5 Let $(\mathcal{X}, \rho)$ be a complete b-metric with parameter $s \geqslant 1$ endowed with a graph $\mathcal{G}$ and ρ be a continuous mapping in two variables. Suppose that there exists $\mathcal{G}$-continuous mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ so that the following conditions hold:

(*i*) $\mathcal{T}$ is a $\mathcal{G}$-proximal with $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and the pair $(\mathcal{A}, \mathcal{B})$ satisfies the P-property;
(*ii*) there exist $a_0, a_1 \in \mathcal{A}_0$ so that $(a_0, a_1) \in E(\mathcal{G})$ and $\rho(a_1, \mathcal{T}a_0) = \rho(\mathcal{A}, \mathcal{B})$;
(*iii*) there exist three functions $\alpha, \beta, \gamma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ with

$$\sup\left\{s(\alpha(a, b) + \beta(a, b)) + s^2\gamma(a, b) : a, b \in \mathcal{X}\right\} < 1$$

such that

$$\begin{aligned}\rho(\mathcal{T}a, \mathcal{T}b) \leqslant\ & \alpha(a, b)\rho(a, b) + \beta(a, b)\rho(a, \mathcal{T}a) + \gamma(a, b)\rho(y, \mathcal{T}b) \\ & - s(\beta(a, b) + \gamma(a, b))\rho(\mathcal{A}, \mathcal{B}) \qquad (1)\end{aligned}$$

for all $a, b \in \mathcal{A}$ with $(\mathcal{A}, \mathcal{B}) \in E(\mathcal{G})$.

Then $\mathcal{T}$ has a best proximity point in $\mathcal{A}$. Also, if we have $(u, v) \in E(\mathcal{G})$ for any two best proximity points $u, v \in \mathcal{A}$, then $\mathcal{T}$ has a unique best proximity point in $\mathcal{A}$.

Proof. From $a_1 \in \mathcal{A}_0$ and $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$, there exists an $a_2 \in \mathcal{A}$ such that $\rho(a_2, \mathcal{T}a_1) = \rho(\mathcal{A}, \mathcal{B})$. In particular, $a_2 \in \mathcal{A}_0$. Since $\rho(a_1, Ta_0) = \rho(\mathcal{A}, \mathcal{B})$, $(a_0, a_1) \in E(\mathcal{G})$ and $\mathcal{T}$ is $\mathcal{G}$-proximal, then $(a_1, a_2) \in E(\mathcal{G})$. Continuing this process, we obtain a sequence $\{a_n\}$ in $\mathcal{A}_0$ such that

$$\rho(a_{n+1}, \mathcal{T}a_n) = \rho(\mathcal{A}, \mathcal{B}) \ \text{ and } \ (a_n, a_{n+1}) \in E(\mathcal{G}) \qquad n = 0, 1, \cdots. \qquad (2)$$

Since the pair $(\mathcal{A}, \mathcal{B})$ satisfies the P-property, it follows for all $n \in \mathbb{N}$ that

$$\left.\begin{array}{r}\rho(a_n, \mathcal{T}a_{n-1}) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(a_{n+1}, \mathcal{T}a_n) = \rho(\mathcal{A}, \mathcal{B})\end{array}\right\} \Longrightarrow \rho(a_n, a_{n+1}) = \rho(\mathcal{T}a_{n-1}, \mathcal{T}a_n). \qquad (3)$$

Using (1)-(3) and triangle inequality, since $(a_n, a_{n+1}) \in E(\mathcal{G})$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
\rho(a_n, a_{n+1}) &= \rho(\mathcal{T}a_{n-1}, \mathcal{T}a_n) \\
&\leqslant \alpha(a_{n-1}, a_n)\rho(a_{n-1}, a_n) + \beta(a_{n-1}, a_n)\rho(a_{n-1}, \mathcal{T}a_{n-1}) \\
&\quad + \gamma(a_{n-1}, a_n)\rho(a_n, \mathcal{T}a_n) - s(\beta(a_{n-1}, a_n) + \gamma(a_{n-1}, a_n))\rho(\mathcal{A}, \mathcal{B}) \\
&\leqslant \alpha(a_{n-1}, a_n)\rho(a_{n-1}, a_n) + \beta(a_{n-1}, a_n)s[\rho(a_{n-1}, a_n) + \rho(a_n, \mathcal{T}a_{n-1})] \\
&\quad + \gamma(a_{n-1}, a_n)s[\rho(a_n, a_{n+1}) + \rho(a_{n+1}, \mathcal{T}a_n)] \\
&\quad - s(\beta(a_{n-1}, a_n) + \gamma(a_{n-1}, a_n))\rho(\mathcal{A}, \mathcal{B}) \\
&= \alpha(a_{n-1}, a_n)\rho(a_{n-1}, a_n) + s\beta(a_{n-1}, a_n)\rho(a_{n-1}, a_n) \\
&\quad + s\beta(a_{n-1}, a_n)[\rho(a_n, \mathcal{T}a_{n-1}) - \rho(\mathcal{A}, \mathcal{B})] + s\gamma(a_{n-1}, a_n)\rho(a_n, a_{n+1}) \\
&\quad + s\gamma(a_{n-1}, a_n)[\rho(a_{n+1}, \mathcal{T}a_n) - \rho(\mathcal{A}, \mathcal{B})] \\
&= \alpha(a_{n-1}, a_n)\rho(a_{n-1}, a_n) + s\beta(a_{n-1}, a_n)\rho(a_{n-1}, a_n) + s\gamma(a_{n-1}, a_n)\rho(a_n, a_{n+1}),
\end{aligned}$$

which induces that

$$\rho(a_n, a_{n+1}) \leqslant \frac{\alpha(a_{n-1}, a_n) + s\beta(a_{n-1}, a_n)}{1 - s\gamma(a_{n-1}, a_n)}\rho(a_{n-1}, a_n)$$

for all $n \in \mathbb{N}$. Thus, we have

$$\rho(a_n, a_{n+1}) \leqslant \lambda^n \rho(a_0, a_1) \tag{4}$$

for each $n \in \mathbb{N}$ and each a_i for $i = 1, 2, \cdots$, where

$$0 \leqslant \lambda_i = \frac{\alpha(a_{i+1}, a_i) + s\beta(a_{i+1}, a_i)}{1 - s\gamma(a_{i+1}, a_i)} < \frac{1}{s}$$

We put $\lambda = \sup_{i \in \mathbb{N}} \lambda_i$. Now, let $m, n \in \mathbb{N}$ with $m \geqslant n \geqslant 1$. Using (4) and the triangle inequality, we have

$$\rho(a_n, a_m) \leqslant \frac{s\lambda^n}{1 - s\lambda}\rho(a_0, a_1) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\{a_n\}$ is Cauchy sequence in complete b-space $(\mathcal{X}, \rho)$ and there exists $a^* \in \mathcal{X}$ depending on a_0 and a_1 so that $a_n \to a^*$. Moreover, $a^* \in \mathcal{A}$ (since $\mathcal{A}$ is closed).

Now, we show that a^* is a best proximity point for $\mathcal{T}$. Since $a_n \to a^*$ and $(a_n, a_{n+1}) \in E(\mathcal{G})$ for $n = 0, 1, \cdots$ and $\mathcal{T}$ is $\mathcal{G}$-continuous on $\mathcal{A}$, we get $\mathcal{T}a_n \to \mathcal{T}a^*$. Also, the joint continuity of the metric function ρ implies that $\rho(a_n, \mathcal{T}a_n) \to \rho(a^*, \mathcal{T}a^*)$. On the other hand, (2) shows that the sequence $\{\rho(a_n, \mathcal{T}a_n)\}$ is a constant sequence converging to $\rho(\mathcal{A}, \mathcal{B})$. Since the limit of a sequence is unique, we have $\rho(a^*, \mathcal{T}a^*) = \rho(\mathcal{A}, \mathcal{B})$. Hence, a^* is a best proximity point for $\mathcal{T}$. Moreover, $a^* \in \mathcal{A}_0$ and $\mathcal{T}a^* \in \mathcal{B}_0$.

To show uniqueness, suppose that a^{**} is another best proximity point of $\mathcal{T}$ so that $(a^*, a^{**}) \in E(\mathcal{G})$. Since the pair $(\mathcal{A}, \mathcal{B})$ satisfies the P-property, we get

$$\left.\begin{array}{r} \rho(a^*, \mathcal{T}a^*) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(a^{**}, \mathcal{T}a^{**}) = \rho(\mathcal{A}, \mathcal{B}) \end{array}\right\} \Longrightarrow \rho(a^*, a^{**}) = \rho(\mathcal{T}a^*, \mathcal{T}a^{**})$$

for $a^*, a^{**} \in \mathcal{A}_0$ and $\mathcal{T}a^*, \mathcal{T}a^{**} \in \mathcal{B}_0$. Therefore, by (1), we have

$$\begin{aligned}
\rho(a^*, a^{**}) &= \rho(\mathcal{T}a^*, \mathcal{T}a^{**}) \\
&\leqslant \alpha(a^*, a^{**})\rho(a^*, a^{**}) + \beta(a^*, a^{**})\rho(a^*, \mathcal{T}a^*) + \gamma(a^*, a^{**})\rho(a^{**}, \mathcal{T}a^{**}) \\
&\quad - s(\beta(a^*, a^{**}) + \gamma(a^*, a^{**}))\rho(\mathcal{A}, \mathcal{B}) \\
&\leqslant \alpha(a^*, a^{**})\rho(a^*, a^{**}) + \beta(a^*, a^{**})[\rho(a^*, \mathcal{T}a^*) - \rho(\mathcal{A}, \mathcal{B})] \\
&\quad + \gamma(a^*, a^{**})[\rho(a^{**}, \mathcal{T}a^{**}) - \rho(\mathcal{A}, \mathcal{B})] - (s-1)(\beta(a^*, a^{**}) \\
&\quad + \gamma(a^*, a^{**}))\rho(\mathcal{A}, \mathcal{B}) \\
&\leqslant \alpha(a^*, a^{**})\rho(a^*, a^{**}).
\end{aligned}$$

Since

$$\alpha(a^*, a^{**}) \leqslant s\alpha(a^*, a^{**})$$

and

$$s(\alpha(a^*, a^{**}) + \gamma(a^*, a^{**})) + s^2\beta(a^*, a^{**}) < 1,$$

we have $\alpha(a^*, a^{**}) < 1$. Thus, we get $\rho(a^*, a^{**}) = 0$; that is, $a^* = a^{**}$. The proof ends. ■

***Example* 2.6** Let $\mathcal{X} = \mathbb{R}^2$ and $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be defined by

$$\rho\big((a_1, b_1), (a_2, b_2)\big) = (a_1 - a_2)^2 + (b_1 - b_2)^2$$

for $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$. Then $(\mathcal{X}, \rho)$ is a b-metric space with parameter $s = 2$. Let

$$\mathcal{A} = \big\{(a, 1) : x \in [0, 1]\big\} \quad \text{and} \quad \mathcal{B} = \big\{(b, 0) : y \in [0, 1]\big\}.$$

Clearly, $\rho(\mathcal{A}, \mathcal{B}) = 1$, $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{B} = \mathcal{B}_0$. In particular, $\mathcal{A}_0$ is nonempty. Also, define $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ as follow:

$$\mathcal{T}(a, 1) = \begin{cases} (0, 0) & 0 \leqslant a < 1, \\ (\frac{2}{3}, 0) & a = 1 \end{cases} \qquad \big(a \in [0, 1]\big).$$

Consider the functions $\alpha, \beta, \gamma : \mathcal{X} \times \mathcal{X} \to [0, 1)$ for all $(a, b) \in \mathcal{X} \times \mathcal{X}$ by

$$\alpha(a, b) = \frac{a}{6 + 6a^2}, \ \beta(a, b) = \frac{b}{7 + 7b^2} \ \text{ and } \ \gamma(a, b) = \frac{1}{12}.$$

Observe that for all $(a, b) \in \mathcal{X} \times \mathcal{X}$, we have

$$\sup\big\{2(\alpha(a, b) + \beta(a, b)) + 4\gamma(a, b) : a, b \in \mathcal{X}\big\} < 1.$$

Also, for elements $(1,1)$ and $(\frac{1}{2},1)$, we have

$$\begin{aligned}
\rho\big(\mathcal{T}(1,1),\mathcal{T}(\frac{1}{2},1)\big) &= \rho\big((\frac{2}{3},0),(0,0)\big)\\
&= \frac{4}{9}\\
&> \frac{4}{9}\Big(2\alpha\big((1,1),(\frac{1}{2},1)\big)+2\gamma\big((1,1),(\frac{1}{2},1)\big)+4\beta\big((1,1),(\frac{1}{2},1)\big)\Big)\\
&> \frac{\alpha\big((1,1),(\frac{1}{2},1)\big)}{4}+\frac{10\beta\big((1,1),(\frac{1}{2},1)\big)}{9}+\frac{5\gamma\big((1,1),(\frac{1}{2},1)\big)}{4}\\
&\quad -2\Big(\beta\big((1,1),(\frac{1}{2},1)\big)+\gamma\big((1,1),(\frac{1}{2},1)\big)\Big)\\
&= \alpha\big((1,1),(\frac{1}{2},1)\big)\rho\big((1,1),(\frac{1}{2},1)\big)+\beta\big((1,1),(\frac{1}{2},1)\big)\rho\big((1,1),(\frac{2}{3},0)\big)\\
&\quad +\gamma\big((1,1),(\frac{1}{2},1)\big)\rho\big((\frac{1}{2},1),(0,0)\big)\\
&\quad -2\Big(\beta\big((1,1),(\frac{1}{2},1)\big)+\gamma\big((1,1),(\frac{1}{2},1)\big)\Big)\\
&= \alpha\big((1,1),(\frac{1}{2},1)\big)\rho\big((1,1),(\frac{1}{2},1)\big)+\beta\big((1,1),(\frac{1}{2},1)\big)\rho\big((1,1),\mathcal{T}(1,1)\big)\\
&\quad +\gamma\big((1,1),(\frac{1}{2},1)\big)\rho\big((\frac{1}{2},1),\mathcal{T}(\frac{1}{2},1)\big)-2\Big(\beta\big((1,1),(\frac{1}{2},1)\big)\\
&\quad +\gamma\big((1,1),(\frac{1}{2},1)\big)\Big)\rho(\mathcal{A},\mathcal{B}).
\end{aligned}$$

So $\mathcal{T}$ does not satisfy the usual version (non-graph version) of (1). Now, define a graph $\mathcal{G}_4$ by $V(\mathcal{G}_4)=\mathbb{R}^2$ and

$$E(\mathcal{G}_4)=\big\{\big((a_1,a_2),(a_1,a_2)\big):(a_1,a_2)\in\mathbb{R}^2\big\}\cup\big\{\big((0,1),(1,1)\big),\big((1,1),(0,1)\big)\big\},$$

Suppose that $(\mathbb{R}^2,\rho)$ is endowed with $\mathcal{G}_4$. Clearly, the pair $(\mathcal{A},\mathcal{B})$ have the P-property, $\mathcal{T}$ is $\mathcal{G}_4$-proximal, $\mathcal{T}(\mathcal{A}_0)\subseteq\mathcal{B}_0$ and $\mathcal{T}$ is a $\mathcal{G}_4$-continuous. Now, take

$$\alpha(a,b)=\frac{17}{36}\frac{a}{1+a^2},\ \ \beta(a,b)=0\ \text{ and }\ \gamma(a,b)=0$$

for all $(a,b)\in\mathcal{X}\times\mathcal{X}$, so

$$2\alpha(a,b)+2\gamma(a,b)+4\beta(a,b)<1.$$

Assume that $a\in[0,1]$. Then we have

$$\begin{aligned}
\rho\big(\mathcal{T}(a,1),\mathcal{T}(a,1)\big) &= 0\\
&\leqslant \alpha\big((a,1),\mathcal{T}(a,1)\big)\rho\big((a,1),(a,1)\big)+\beta\big((a,1),\mathcal{T}(a,1)\big)\rho\big((a,1),\mathcal{T}(a,1)\big)\\
&\quad +\gamma\big((a,1),\mathcal{T}(a,1)\big)\rho\big((a,1),\mathcal{T}(a,1)\big)-2\Big(\beta\big((a,1),\mathcal{T}(a,1)\big)\\
&\quad +\gamma\big((a,1),\mathcal{T}(a,1)\big)\Big)\rho(\mathcal{A},\mathcal{B}).
\end{aligned}$$

Also,

$$\begin{aligned}
\rho\big(\mathcal{T}(0,1),\mathcal{T}(1,1)\big) &= \rho\big((0,0),(\frac{2}{3},0)\big)\\
&= \frac{4}{9}\\
&\leqslant \alpha\big((0,1),(1,1)\big)\rho\big((0,1),(1,1)\big)+\beta\big((0,1),(1,1)\big)\rho\big((0,1),\mathcal{T}(0,1)\big)\\
&\quad+\gamma\big((0,1),(1,1)\big)\rho\big((1,1),\mathcal{T}(1,1)\big)-2\Big(\beta\big((0,1),(1,1)\big)\\
&\quad+\gamma\big((0,1),(1,1)\big)\Big)\rho(\mathcal{A},\mathcal{B}).
\end{aligned}$$

Thus, $\mathcal{T}$ satisfies in (1). Moreover, all hypotheses of Theorem 2.5 is satisfied. Therefore, $\mathcal{T}$ has a best proximity point $a^* = (0,1)$.

Now, let $a^{**} = (a,1) \in \mathcal{A}$ with $a \in (0,1]$ be another best proximity point of $\mathcal{T}$. If $x \in (0,1)$, then

$$\rho\big((a,1),\mathcal{T}(a,1)\big) = \rho\big((a,1),(0,0)\big) = a^2+1 > \rho(\mathcal{A},\mathcal{B}).$$

Otherwise, if $x = 1$, then

$$\rho\big((1,1),\mathcal{T}(1,1)\big) = \rho\big((1,1),(\frac{2}{3},0)\big) = \frac{10}{9} > \rho(\mathcal{A},\mathcal{B}),$$

which is a contradiction. Hence, $(0,1)$ is the unique best proximity point of $\mathcal{T}$.

Several consequences of Theorem 2.5 follow for particular choices of the graph. First, consider the b-metric space $(\mathcal{X},\rho)$ endowed with the complete graph $\mathcal{G}_0$. If we set $\mathcal{G} = \mathcal{G}_0$ in Theorem 2.5, then it is clear that $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ is a $\mathcal{G}_0$-proximally on $\mathcal{A}$.

Corollary 2.7 Let $(\mathcal{X},\rho)$ be a complete b-metric with parameter $s \geqslant 1$ and ρ be a continuous mapping in two variables. Assume that there exists continuous mapping $\mathcal{T}:\mathcal{A}\to\mathcal{B}$ so that the following conditions hold:

(*i*) $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{T}\mathcal{B}_0$ and $(\mathcal{A},\mathcal{B})$ have the P-property;
(*ii*) there exist $a_0, a_1 \in \mathcal{A}_0$ so that $\rho(a_1,\mathcal{T}a_0) = \rho(\mathcal{A},\mathcal{B})$;
(*iii*) there exist three functions $\alpha,\beta,\gamma:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ with

$$\sup\Big\{s(\alpha(a,b)+\beta(a,b))+s^2\gamma(a,b) : a,b\in\mathcal{X}\Big\} < 1$$

so that

$$\begin{aligned}
\rho(\mathcal{T}a,\mathcal{T}b) \leqslant{}& \alpha(a,b)\rho(a,b)+\beta(a,b)\rho(a,\mathcal{T}a)+\gamma(a,b)\rho(b,\mathcal{T}b)\\
&- s(\beta(a,b)+\gamma(a,b))\rho(\mathcal{A},\mathcal{B})
\end{aligned}$$

for all $a,b\in\mathcal{A}$.

Then $\mathcal{T}$ has a unique best proximity point in $\mathcal{X}$.

Now, suppose that $(\mathcal{X},\preceq)$ is a poset. Consider on the poset $\mathcal{X}$ the graph $\mathcal{G}_1$ given by $V(\mathcal{G}_1)=\mathcal{X}$ and $E(\mathcal{G}_1)=\big\{(a,b)\in\mathcal{X}\times\mathcal{X} : a\preceq b\big\}$. If we set $\mathcal{G}=\mathcal{G}_1$ in Theorem 2.5, then we obtain following best proximity point result.

Corollary 2.8 Let $(\mathcal{X}, \preceq)$ be a poset, $(\mathcal{X}, \rho)$ be a complete b-metric with parameter $s \geqslant 1$ and endowed with a graph $\mathcal{G}_1$ and ρ be a continuous mapping in two variables. Assume that there exists $\mathcal{G}_1$-continuous mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ so that the following conditions hold:

(i) $\mathcal{T}$ is a $\mathcal{G}_1$-proximal with $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and $(\mathcal{A}, \mathcal{B})$ have the P-property;
(ii) there exist $a_0, a_1 \in \mathcal{A}_0$ so that $a_0 \preceq a_1$ and $\rho(a_1, \mathcal{T}a_0) = \rho(\mathcal{A}, \mathcal{B})$;
(iii) there exist three functions $\alpha, \beta, \gamma : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ with

$$\sup\left\{s(\alpha(a,b) + \beta(a,b)) + s^2\gamma(a,b) : a, b \in \mathcal{X}\right\} < 1$$

so that

$$\begin{aligned}\rho(\mathcal{T}a, \mathcal{T}b) \leqslant\ & \alpha(a,b)\rho(a,b) + \beta(a,b)\rho(a,\mathcal{T}a) + \gamma(a,b)\rho(b,\mathcal{T}b) \\ & - s(\beta(a,b) + \gamma(a,b))\rho(\mathcal{A}, \mathcal{B})\end{aligned}$$

for all $a, b \in \mathcal{A}$ with $a \preceq b$.

Then $\mathcal{T}$ has a best proximity point in $\mathcal{A}$. Also, if $u \preceq v$ for any two best proximity point $u, v \in \mathcal{A}$, then $\mathcal{T}$ has a unique best proximity point in $\mathcal{A}$.

For our next consequence, suppose again that $(\mathcal{X}, \preceq)$ is a poset and consider the graph $\mathcal{G}_2$ defined by $V(\mathcal{G}_2) = \mathcal{X}$ and $E(\mathcal{G}_2) = \left\{(a,b) \in \mathcal{X} \times \mathcal{X} : a \preceq b \ \vee \ b \preceq a\right\}$. Then an ordered pair $(a,b) \in \mathcal{X} \times \mathcal{X}$ is an edge of $\mathcal{G}_2$ if and only if a and b are comparable elements of $(\mathcal{X}, \preceq)$.

Corollary 2.9 Let $(\mathcal{X}, \preceq)$ be a poset, $(\mathcal{X}, \rho)$ be a complete b-metric space with parameter $s \geqslant 1$ and endowed with a graph $\mathcal{G}_2$, and ρ be a continuous mapping in two variables. Assume that there exists $\mathcal{G}_2$-continuous mapping $\mathcal{T} : A \to B$ so that the following conditions hold:

(i) $\mathcal{T}$ is a $\mathcal{G}_2$-proximal with $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and $(\mathcal{A}, \mathcal{B})$ have the P-property;
(ii) there exist comparable elements $a_0, a_1 \in \mathcal{A}_0$ so that $\rho(a_1, \mathcal{T}a_0) = \rho(\mathcal{A}, \mathcal{B})$;
(iii) there exist three functions $\alpha, \beta, \gamma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ with

$$\sup\left\{s(\alpha(a,b) + \beta(a,b)) + s^2\gamma(a,b) : a, b \in \mathcal{X}\right\} < 1$$

so that

$$\begin{aligned}\rho(\mathcal{T}a, \mathcal{T}b) \leqslant\ & \alpha(a,b)\rho(a,b) + \beta(a,b)\rho(a,\mathcal{T}a) + \gamma(a,b)\rho(b,\mathcal{T}b) \\ & - s(\beta(a,b) + \gamma(a,b))\rho(\mathcal{A}, \mathcal{B})\end{aligned}$$

for all comparable $a, b \in \mathcal{A}$.

Then $\mathcal{T}$ has a best proximity point in $\mathcal{A}$. Also, if each two best proximity point are comparable, then $\mathcal{T}$ has a unique best proximity point in $\mathcal{A}$.

Let $\varepsilon > 0$ be a fixed. Recall that two elements $a, b \in \mathcal{X}$ are said to be ε-close if $\rho(a,b) < \varepsilon$. Finally, let a number $\varepsilon > 0$ be a fixed and consider the graph $\mathcal{G}_\varepsilon$ given by $V(\mathcal{G}_\varepsilon) = \mathcal{X}$ and $E(\mathcal{G}_\varepsilon) = \left\{(a,b) \in \mathcal{X} \times \mathcal{X} : \ \rho(a,b) < \varepsilon\right\}$.

Corollary 2.10 Let $\varepsilon > 0$ be a fixed, $(\mathcal{X}, \rho)$ be a complete b-metric space with parameter $s \geqslant 1$ and endowed with a graph $\mathcal{G}_\varepsilon$, and ρ be a continuous mapping in two variables.

Assume that there exists $\mathcal{G}_\varepsilon$-continuous mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ so that the following conditions hold:

(i) $\mathcal{T}$ is a $\mathcal{G}_\varepsilon$-proximal with $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and $(\mathcal{A}, \mathcal{B})$ satisfies the P-property;
(ii) there exist ε-close elements $a_0, a_1 \in \mathcal{A}_0$ such that $\rho(a_1, \mathcal{T}a_0) = \rho(\mathcal{A}, \mathcal{B})$;
(iii) there exist three functions $\alpha, \beta, \gamma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ with

$$\sup\left\{s(\alpha(a,b) + \beta(a,b)) + s^2\gamma(a,b) : a, b \in \mathcal{X}\right\} < 1$$

such that

$$\rho(\mathcal{T}a, \mathcal{T}b) \leqslant \alpha(a,b)\rho(a,b) + \beta(a,b)\rho(a,\mathcal{T}a) + \gamma(a,b)\rho(b,\mathcal{T}b) - s(\beta(a,b) + \gamma(a,b))\rho(\mathcal{A},\mathcal{B})$$

for all ε-close elements $a, b \in \mathcal{A}$.

Then $\mathcal{T}$ has a best proximity point in $\mathcal{A}$. Also, if each two best proximity point are ε-close, then $\mathcal{T}$ has a unique best proximity point in $\mathcal{A}$.

In Theorem 2.5 and his corollaries, set $\beta = \gamma = 0$. Then we obtain best proximity point result for Banach-type contraction in complete b-metric spaces endowed with the graph $\mathcal{G}$.

Corollary 2.11 Let $(\mathcal{X}, \rho)$ be a complete b-metric with parameter $s \geqslant 1$ endowed with a graph $\mathcal{G}$ and ρ be a continuous mapping in two variables. Assume that there exists $\mathcal{G}$-continuous mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ such that the following conditions are held:

(i) $\mathcal{T}$ is a $\mathcal{G}$-proximal so that $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and $(\mathcal{A}, \mathcal{B})$ have the P-property;
(ii) there exist a_0, a_1 in $\mathcal{A}_0$ so that $(a_0, a_1) \in E(\mathcal{G})$ and $\rho(a_1, \mathcal{T}a_0) = \rho(\mathcal{A}, \mathcal{B})$;
(iii) there exists function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \frac{1}{s})$ so that $\rho(\mathcal{T}a, \mathcal{T}b) \leqslant \alpha(a,b)\rho(a,b)$ for all $a, b \in A$ with $(a, b) \in E(\mathcal{G})$.

Then $\mathcal{T}$ has a best proximity point in $\mathcal{X}$. Also, if $(u, v) \in E(\mathcal{G})$ for any two best proximity point $u, v \in \mathcal{A}$, then $\mathcal{T}$ has a unique best proximity point in $\mathcal{X}$.

In Theorem 2.5 and his corollaries, set $\alpha = 0$ and $\beta = \gamma = K$. Then we obtain best proximity point result for Kanan-type contraction in complete b-metric spaces endowed with the graph $\mathcal{G}$.

Corollary 2.12 Let $(\mathcal{X}, \rho)$ be a complete b-metric with parameter $s \geqslant 1$ endowed with a graph $\mathcal{G}$ and ρ be a continuous mapping in two variables. Suppose that there exists $\mathcal{G}$-continuous mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ such that the following conditions are held:

(i) $\mathcal{T}$ is a $\mathcal{G}$-proximal so that $\mathcal{T}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ and $(\mathcal{A}, \mathcal{B})$ have the P-property;
(ii) there exist elements a_0 and a_1 in $\mathcal{A}_0$ such that $(a_0, a_1) \in E(\mathcal{G})$ and $\rho(a_1, \mathcal{T}a_0) = \rho(\mathcal{A}, \mathcal{B})$;
(iii) there exists $K \in [0, \frac{1}{s^2+s})$ such that

$$\rho(\mathcal{T}a, \mathcal{T}b) \leqslant K[\rho(a, \mathcal{T}a) + \rho(b, \mathcal{T}b)] - 2Ks\rho(\mathcal{A}, \mathcal{B})$$

for all $a, b \in A$ with $(a, b) \in E(\mathcal{G})$.

Then $\mathcal{T}$ has a best proximity point in $\mathcal{X}$. Also, if $(u, v) \in E(\mathcal{G})$ for any two best proximity point $u, v \in \mathcal{A}$, then $\mathcal{T}$ has a unique best proximity point in $\mathcal{X}$.

In Theorem 2.5 and its corollaries, set $s = 1$. Then we have the same assertions in the framework of complete metric spaces endowed with a graph $\mathcal{G}$.

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