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Grothendieck topologies and applications

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Abstract. Following [6], we define Grothendieck topologies on a small category and describe sheaves for these Grothendieck topologies. This generalizes, in a natural way, the theory of sheaves on a topological space.

Keywords: Presheaves, sheaves, Grothendieck topology, topos, Zariski topology, Etale topology, Nisnevich topology.

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1. Motivation

In 1998, Morel and Veovodsky [10] constructed the $\mathbb{A}^1$-homotopic category of k-schemes, where k is a perfect field using the Nisnevich site. This site is built in the category of k-schemes with a Grothendieck topology, (see details below). The richness of the $\mathbb{A}^1$-homotopic category of k-schemes allowed Voevodsky to prove Milnor's conjecture [9], Riou [11] to conceive a motivic analogue of Atiyah's theorem linking the ring of representations to the ring of K-theory of its classifying space for the linear group and Azi-Hamraoui to extend this motivic analogue to the special group [1]. Since 2014, in using Grothendieck topology in order to develop the non-commutative approach to the Riemann hypothesis, Connes and Consani [2–4] have constructed the arithmetic site and have proved that the completed Riemann zeta function is obtained as the Hasse Weil zeta function. We can also investigate the Mod 2 Steenrod algebra [8, 12].

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2. Introduction

To define a cohomology of a topological space X, as it is described in [5] we first define a category of presheaves over X of sets or A-modules where A is a commutative unitary ring, then its full sub-category of sheaves by constructing the sheaf associated to a presheaf. We will briefly recall this study in Section 3. In section 4, following [6], we define a topology, called Grothendieck topology, on a category C by means of covering sieves. Such categories are called sites. We then turn to the category of presheaves and define its full sub-category of sheaves called topos thanks to Yoneda's lemma.

Section 5 is devoted to the study of five examples: Considering the category $Ouv(X)$ where X is a topological space, we define a Grothendieck topology and its topos in order to recover the results of section 3. In the second example, we consider the category BG of G-sets where G is a group and define a Grothendieck topology and characterize its topos. In the last three examples, we work with the category of k-schemes where k is a field, we define three Grothendieck topologies, called Zariski, étale, Nisnevich and we study their corresponding toposes.

3. Sheaf on a topological space

Definition 3.1 Let X be a topological space and U be an open set of X and $V \subset U$ with V an open set. A presheaf of sets $\mathcal{F}$ over X is given by

(1) a subset $\mathcal{F}(U)$.
(2) a restriction morphism $\rho_{U|V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ that satisfies the following properties:
   - $\rho_{U|U} = Id_{\mathcal{F}(U)}$.
   - For each inclusion of open subsets $W \subset V \subset U$, we have $\rho_{V|W} \circ \rho_{U|V} = \rho_{U|W}$.

***Example* 3.2** We define the presheaf of continuous functions on $\mathbb{R}$ as follows: For any open U of $\mathbb{R}$, $\mathcal{F}(U) = C(U, \mathbb{R})$ the set of continuous functions on U. If U and V are open subsets such that $V \subset U$, a restriction morphism is given by $\rho_{U|V} : F(U) \to F(V)$ where $\rho_{U|V}(f) = f \circ i$.

Definition 3.3 A presheaf $\mathcal{F}$ is a sheaf if and only if for any family of open sets $\{U_i\}_{i\in I}$ of X, there is a bijection

$$\begin{array}{ccc} \mathcal{F}(\bigcup_{i\in I} U_i) & \longrightarrow & \{(s_i)_{i\in I}\} \\ s & \longmapsto & s_i \end{array}$$

such that $s_i \in \mathcal{F}(U_i)$ and for all $i, j \in I$ $\;s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$.

***Example* 3.4** The presheaf of continuous functions is a sheaf.

Not every presheaf is a sheaf, in this case certain additional conditions are imposed to force the presheaf to become a sheaf, hence the interest in the notion of a sheaf associated to a presheaf.

Proposition 3.5 Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \longrightarrow \mathcal{F}^+$ verifying for every morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ where $\mathcal{G}$ is a sheaf, there is a unique morphism $\psi : \mathcal{F}^+ \longrightarrow \mathcal{G}$ such that $\varphi = \psi \circ \theta$. The pair $(\mathcal{F}^+, \theta)$ is unique up to isomorphism.

Proof. See [7]. ■

Definition 3.6 The pair $(\mathcal{F}^+, \theta)$ which existence has been shown in the previous proposition is called the sheaf associated to the presheaf $\mathcal{F}$.

4. Grothendieck Topology

4.1 *Sieves*

Along this section $\mathcal{C}$ refers to a small category (in other terms the class of objects of $\mathcal{C}$ is a set). We denote by $PR\mathcal{F}_\mathcal{C}$ the category of presheaves of sets over $\mathcal{C}$. The objects of $PR\mathcal{F}_\mathcal{C}$ are presheaves of sets and for two presheaves $\mathcal{F}$ and $\mathcal{G}$, a morphism of presheaves from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation of $\mathcal{F}$ to $\mathcal{G}$.

***Example* 4.1** For each object X of $\mathcal{C}$, the functor $h_X = Hom_\mathcal{C}(-, X)$ is a presheaf called the presehaf represented by X.

The category $PR\mathcal{F}_\mathcal{C}$ can be equipped with a relation of order in the following way: for all presheaves $\mathcal{F}$ and $\mathcal{E}$ over $\mathcal{C}$, we will say that $\mathcal{E}$ is a subpresheaf of $\mathcal{F}$ if for every object X of $\mathcal{C}$, $\mathcal{E}(X)$ is a subset of $\mathcal{F}(X)$.

Definition 4.2 A presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ is said representable if there exists an object X of $\mathcal{C}$ such that the functor $h_X = Hom_\mathcal{C}(-, X)$ is isomorphic to $\mathcal{F}$.

Lemma 4.3 (Yoneda, see [6]) For every $X \in \mathcal{C}$ and every presheaf $\mathcal{F} \in PR\mathcal{F}_\mathcal{C}$, the following morphism is a bijection:

$$\begin{array}{ccc} Hom_{PR\mathcal{F}_\mathcal{C}}(h_X, \mathcal{F}) & \longrightarrow & \mathcal{F}(X) \\ \phi & \longmapsto & \phi_X(id_X) \end{array}.$$

Definition 4.4 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. The functor $\mathcal{F}$ is said faithful (resp. full) if the map: $Hom_\mathcal{C}(X, Y) \longrightarrow Hom_\mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$ is injective (resp. surjective). A functor that is both faithful and full is said to be fully faithful.

It follows from Yoneda's lemma that the functor $\mathcal{C} \longrightarrow PR\mathcal{F}_\mathcal{C}$ which to each object X associates the presheaf $Hom_\mathcal{C}(-, X)$ is fully faithful.

Definition 4.5 A sieve $\mathcal{S}$ on X is a subpresheaf of the presheaf represented by X.

Proposition 4.6 Studying a sieve on X is equivalent to consider a class $\mathcal{S}'$ of morphisms from the category $\mathcal{C}$ satisfying:

(i) each map f of $\mathcal{S}'$ has the target X,
(ii) for every map f of $\mathcal{S}'$ and every map g of $\mathcal{C}$, the composition $f \circ g$ is an element of $\mathcal{S}'$.

***Example* 4.7**

- For every object $X \in \mathcal{C}$, the presheaf $Hom(-, X)$ is a sieve called the trivial sieve.
- Consider the category $Ouv(X)$ where the objects are the open subsets of X and the morphisms are the inclusions of open subsets. In $Ouv(X)$ a sieve $\mathcal{S}$ on an open $V \subset U$ is given by
  - $\mathcal{S}(V) \subset Hom_{Ouv(X)}(V, U) = \{$ the inclusion of V in U$\}$.
  - $\mathcal{S}(V)$ is thus reduced to one element.

Definition 4.8 Let $\left(U_i \xrightarrow{f_i} X\right)_{i\in I}$ be a collection of morphisms of $\mathcal{C}$. The sieve generated by f_i is the set of maps:

$$\langle f_i\rangle = \{f : Y \longrightarrow X \text{ such that there exists } i \in I \ \ h_i : Y \longrightarrow U_i \text{ where } f = f_i \circ h_i\}\,.$$

Definition 4.9 Let $\mathcal{S}$ be a sieve on X and $f : Y \longrightarrow X$ a morphism of $\mathcal{C}$. The pullback $\mathcal{S} \underset{Hom(-,X)}{\times} Hom(-, Y)$ exists in the category of presheaves $PR\mathcal{F}_{\mathcal{C}}$, we denote it $\mathcal{S}^f$. $\mathcal{S}^f \subset Hom(-, Y)$ is a sieve on Y and it is defined for every object Z of the category $\mathcal{C}$ by $\mathcal{S}^f(Z) = \{g : Z \longrightarrow Y \text{ such that } f \circ g \in \mathcal{S}(Z)\}$.

Proposition 4.10 If the morphism $f : Y \longrightarrow X$ of Definition 4.9 belongs to $\mathcal{S}$, then $\mathcal{S}^f = Hom(-, Y)$.

Proof. See [6]. ■

Proposition 4.11 Suppose that the category $\mathcal{C}$ admits pullbacks. Let $\left(U_i \xrightarrow{f_i} X\right)_{i\in I}$ be a collection of morphisms of $\mathcal{C}$ and Yan object of C, $f : Y \longrightarrow X$ a morphism and $\mathcal{S}$ a sieve on X. If $S = \langle U_i \longrightarrow X,\ i \in I\rangle$, then $S^f = \langle U_i \times_X Y \longrightarrow Y,\ i \in I\rangle$.

Proof. See [6]. ■

4.2 *Sites*

4.2.1 Topologies

Definition 4.12 A Grothendieck topology T on the category C is the assignment, for every object X of C, of a collection $T(X)$ of sieves on X satisfying the following axioms:

(a) Stability under base change: For every object X of $\mathcal{C}$, every sieve $\mathcal{S} \in T(X)$ and every morphism $f : Y \longrightarrow X$ where $Y \in \mathcal{C}$, the sieve $\mathcal{S}^f \in T(Y)$.
(b) Local character: If $\mathcal{S}$ and $\mathcal{S}'$ are two sieves of X, if $\mathcal{S} \in T(X)$ and if for every $Y \in \mathcal{C}$ and every morphism $Hom(-, Y) \longrightarrow \mathcal{S}$ the sieve $\mathcal{S}'^f \in T(Y)$, then $\mathcal{S}' \in T(X)$.
(c) Identity: For every object X of $\mathcal{C}$, the trivial sieve $Hom(-, X) \in T(X)$. The sieves belonging to $T(X)$ are said T-covering. Each category $\mathcal{C}$ equipped with a Grothendieck topology T is called a site, it is denoted $(\mathcal{C}, T)$. The category $\mathcal{C}$ is called the underlying category of that site.

The inclusion relation of presheaves induces a preorder relation on the sieves and consequently on the covering ones. We will say for two sieves $\mathcal{S}$ and $\mathcal{S}'$ on X that $\mathcal{S}$ is finer than $\mathcal{S}'$ if $\mathcal{S} \subset \mathcal{S}'$.

Definition 4.13 Let $(\mathcal{C}, T)$ be a site and X an object of $\mathcal{C}$. A collection of morphisms $(f_i : U_i \longrightarrow X)_{i\in I}$ of $\mathcal{C}$ is said to be T-covering if the sieve generated by the family $(f_i)_{i\in I}$ is a T-covering sieve on X.

Lemma 4.14 If $(\mathcal{C}, T)$ is a site, then for each object X of $\mathcal{C}$ and every sieves $\mathcal{S}$ and $\mathcal{S}'$ on X, the following assertions are satisfied:

(i) If $\mathcal{S}$ and $\mathcal{S}'$ are T-covering, then $\mathcal{S} \cap \mathcal{S}'$ is T-covering.
(ii) If $\mathcal{S}$ is T-covering and $\mathcal{S}$ is finer than $\mathcal{S}'$, then $\mathcal{S}'$ is T-covering.
(iii) The ordered set $T(X)$ is filtered.

Proof.

(i) Let $f : Y \longrightarrow X$ be a morphism of $\mathcal{S}$. It follows from Proposition 4.10 and Definition 4.12 that the sieve $(\mathcal{S} \cap \mathcal{S}')^f = Hom(-, Y)$ is T-covering for Y according to (c). We deduce from (b) that $\mathcal{S} \cap \mathcal{S}' \in T(X)$.

(ii) Similar to (i), let $f : Y \longrightarrow X$ be a map of $\mathcal{S}$. Since $\mathcal{S}$ is finer than $\mathcal{S}'$ we have once again $(\mathcal{S}')^f = Hom(-, Y)$ and we conclude with (b).

(iii) It follows from the fact that $T(X)$ is non-empty according to (c) and the assertions (i) and (ii).

■

Definition 4.15 If T_1 and T_2 are two topologies on the category $\mathcal{C}$, we will say that T_1 is finer than T_2 if for each object X of $\mathcal{C}$ the sieve $T_1(X) \subset T_2(X)$.

***Example* 4.16**

- The topology defined on $\mathcal{C}$ by $T(X) = Hom(-, X)$ is the least fine of all topologies on $\mathcal{C}$, we call it the trivial topology.
- The topology defined on $\mathcal{C}$ by $T(X) = \{\text{All the sieves on } X\}$ is the finest of all topologies on $\mathcal{C}$, we call it discrete topology.

Proposition 4.17 If $(T_i)_{i \in I}$ is a family of topologies on $\mathcal{C}$, then the topology T given by $T(X) = \bigcap_{i \in I} T_i(X)$ is

(1) a topology;
(2) the infimum of T_i for the order defined on the topologies.

Proof. See [6]. ■

The family $(T_i)_{i \in I}$ also admits a supremum, it is the intersection topology of topologies finer than each of the T_i.

4.3 *Pretopologies*

Definition 4.18 We call Grothendieck pretopology on $\mathcal{C}$ the data consisting of: for all $X \in Ob(\mathcal{C})$, $Cov(X)$ is a set of family of morphisms $(f_i : U_i \longrightarrow X)_{i \in I}$. That collection satisfies the following properties:
1. Existence of pullbacks: For every $(f_i : U_i \longrightarrow X)_{i \in I} \in Cov(X)$ the morphisms f_i are quadrable, which means that for all $Y \longrightarrow X$ in the category $\mathcal{C}$ the pullback $U_i \times_X Y$ exists.
2. Stability under base change: for all $X \in Ob(\mathcal{C})$, for all $(f_i : U_i \longrightarrow X)_{i \in I} \in Cov(X)$ and for all $Y \longrightarrow X$ in the category $\mathcal{C}$, $(f_i : U_i \times_X Y \longrightarrow Y)_{i \in I} \in Cov(Y)$.
3. Stability under composition: for all $X \in Ob(\mathcal{C})$, for all $(f_i : U_i \longrightarrow X)_{i \in I} \in Cov(X)$ and for all $i \in I$, let us consider $(g_{j_i} : V_{j_i} \longrightarrow U_i)_{j_i \in J_i} \in Cov(U_i)$. Then the family $(f_i \circ g_{j_i} : V_{j_i} \longrightarrow X)_{j_i \in J_i, i \in I} \in Cov(X)$.
4. Identity: for all $X \in Ob(\mathcal{C})$, we have $(id_X : X \longrightarrow X) \in Cov(X)$.

Definition 4.19 Let Cov be a pretopology on $\mathcal{C}$ and X be an object of $\mathcal{C}$ and $\mathcal{S}$ a sieve on X. We say that $\mathcal{S}$ is elementary if there exists a covering $Cov(X) = (f_i : U_i \longrightarrow X)_{i \in I}$ such that $\mathcal{S} = \langle U_i \longrightarrow X, i \in I \rangle$.

4.4 *Topos*

Definition 4.20 Let $\mathcal{C}$ be a category equipped with a Grothendieck topology and $\mathcal{F}$ a presheaf on $\mathcal{C}$. We will say that $\mathcal{F}$ is a separated presheaf (resp. a sheaf) of sets on the site $(\mathcal{C}, T)$ if for every object X of $\mathcal{C}$ and every covering sieve $\mathcal{S}$ on X, the map $Hom_{PR\mathcal{F}_\mathcal{C}}\left(Hom_\mathcal{C}(-,X),\mathcal{F}\right) \longrightarrow Hom_{PR\mathcal{F}_\mathcal{C}}\left(\mathcal{S},\mathcal{F}\right)$ is injective (resp. bijective). We denote by $\mathcal{F}_\mathcal{C}$ the subcategory of $PR\mathcal{F}_\mathcal{C}$ formed by sheaves over $\mathcal{C}$ and we call it the associated topos to the site $(\mathcal{C},T)$.

Proposition 4.21 If the topology on $\mathcal{C}$ is generated by a pretopology defined by a family of coverings $(f_i : U_i \longrightarrow X)_{i\in I}$ for every object X, then a presheaf on $\mathcal{C}$ is a sheaf if and only if

$$\mathcal{F}(X)\xrightarrow{\sim}\left\{(s_i)_{i\in I}\text{ such that } s_i\in\mathcal{F}(U_i)\text{ and for all } i,j\in I,\ s_{i|U_i\times_X U_j}=s_{j|U_i\times_X U_j}\right\}$$

is an isomorphism.

Proof. See [6] ■

5. Examples of sites

5.1 *The site Ouv(X)*

5.1.1 The topology

Proposition 5.1 For every topological space X, we get a site by equipping the category $Ouv(X)$ with the covering family $(U_i \longrightarrow U)_{i\in I} \in Cov(U)$ if and only if $U = \bigcup_{i\in I} U_i$.

Proof. 1. Existence of pullbacks: Let $U, \{U_i\}_{i\in I} \in Ouv(X)$ such that $f : U_i \longrightarrow U$ and $g : U_j \longrightarrow U$.

$$\begin{aligned} U_i \times_U U_j &= \{(x,y)\in U_i\times U_j \ \ / \ f(x)=g(y)\} \\ &= \{(x,y)\in U_i\times U_j \ \ / \ x=y\} \\ &= U_i\cap U_j \in Ouv(X). \end{aligned}$$

Moreover, for all $W \in Ouv(X)$, $W \xrightarrow{a} U_i$ and $W \xrightarrow{b} U_j$ such that $f\circ a = g\circ b$, there is a unique morphism $u : W \longrightarrow U_i \cap U_j$ and for every $w \in W$, we have

$$(f\circ a)(w) = (g\circ b)(w) \Longrightarrow a(w) = b(w).$$

Since $a(w) \in U$ and $b(w) \in V$, then $(a(w), b(w)) = u(w)$.
2. Stability under base change: Let $(U_i \longrightarrow U)_{i\in I} \in Cov(U)$. Then $U = \bigcup_{i\in I} U_i \Longrightarrow V\cap U = V\cap\left(\bigcup_{i\in I} U_i\right)$. Since $V \subset U$, we get $V = V\cap\left(\bigcup_{i\in I} U_i\right)$. Thus, $V = \bigcup_{i\in I}(V\cap U_i)$. We conclude that $(V\cap U_i \longrightarrow V)_{i\in I} \in Cov(V)$.

3. Stability under composition:

$$(g_{j_i} : V_{j_i} \longrightarrow U_i)_{i \in I, j_i \in J_i} \in Cov(U_i)$$
$$\iff U_i = \bigcup_{i \in I, j_i \in J_i} V_{j_i}$$
$$\iff U = \bigcup_{i \in I} \left(\bigcup_{i \in I, j_i \in J_i} V_{j_i} \right) = \bigcup_{i \in I, j_i \in J_i} V_{j_i}.$$

As a result, $(f_i \circ g_{j_i} : V_{j_i} \longrightarrow U) \in Cov(U)$.
4. Identity: It is clear that for all $U \in Ouv(X)$, $id_U \in Cov(U)$.
Conclusion: $(Ouv(X), Cov(U))_{U \in Ouv(X)}$ is a site. This is how the notion of site on a category generalizes the notion of topology on a set. ■

5.1.2 Topos on $Ouv(X)$

A presheaf $\mathcal{F}$ on the site $Ouv(X)$ is a sheaf if and only if

$$\mathcal{F}(X) \xrightarrow{\sim} \left\{ (s_i)_{i \in I} \text{ such that } s_i \in \mathcal{F}(U_i) \text{ for all } i, j \in I \quad s_{i|U_i \times_X U_j} = s_{j|U_i \times_X U_j} \right\}.$$

is an isomorphism. Since $U_i \times_X U_j = U_i \cap U_j$, we find the definition of a sheaf in the case of a topological space. The category of toposes on $Ouv(X)$ and the one of sheaves X are equivalent.

5.2 *The site BG where G is a group*

Definition 5.2 Let G be a group and X a set. We say that X is a left G-set if there exists a map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g.x \end{aligned}$$

such that

(1) for all $g_1, g_2 \in G$; $(g_1.g_2)(x) = g_1.(g_2.x)$.
(2) for all $x \in X$; there exists $e \in G$ such that $ex = x$.

Definition 5.3 We denote BG the category where

- $Ob(BG)$={G-sets}.
- $Hom_{BG}(E_1, E_2) = \{f : E_1 \longrightarrow E_2\}$ such that for all $x \in E_1$ and for all $g \in G$, $f(g.x) = g.f(x)$.

5.2.1 A pretopology on BG

Proposition 5.4 The following covering family is a Grothendieck pretopology on BG:

$$(U_i \longrightarrow X)_{i \in I} \in Cov(X) \iff \coprod_{i \in I} U_i \longrightarrow X \text{ surjective.}$$

Proof.

(1) Existence of pullbacks: Let U_i and U_j be two G-sets:

$$\begin{array}{ccc} U_i \times U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow f \\ U_j & \underset{g}{\longrightarrow} & X \end{array}$$

$U_i \underset{X}{\times} U_j = \{(a,b) \in U_i \times U_j \text{ such that } f(a) = g(b)\}$
$a \in U_i \Longrightarrow$ there exist $x \in U_i$ and $g_1 \in G$ so that $a = g_1.x$
$b \in U_j \Longrightarrow$ there exist $y \in U_j$ and $g_2 \in G$ so that $b = g_2.y$
$U_i \underset{X}{\times} U_j = \{(a,b) \in U_i \times U_j \text{ such that } g_1.f(x) = g_2.g(y) \quad g_1, g_2 \in G\}$
$U_i \underset{X}{\times} U_j = \{(a,b) \in U_i \times U_j \text{ such that } f(x) = g_1^{-1}.g_2.g(y) \quad g_1, g_2 \in G\}$
It is easy to check that $U_i \underset{X}{\times} U_j$ is a G-set since

$$\begin{gathered} G \times U_i \underset{X}{\times} U_j \to U_i \underset{X}{\times} U_j \\ (h, \alpha) \mapsto h.\alpha \end{gathered}$$

$\alpha \in U_i \underset{X}{\times} U_j \Longrightarrow \alpha = (x,y) \in U_i \times U_j$ such that $f(x) = g_1^{-1}.g_2.g(y)$
$h.\alpha = h.(x,y) = (h.x, h.y) \in U_i \times U_j$ such that $f(h.x) = g_1^{-1}.g_2.g(h.y)$
$\Longrightarrow h.\alpha = h.(x,y) = (h.x, h.y) \in U_i \times U_j$ such that $h.f(x) = h.g_1^{-1}.g_2.g(y)$.
Then $U_i \underset{X}{\times} U_j$ is an object of the category BG.

(2) Stability under composition: Suppose that $(f_i)_{i\in I} \in Cov(U)$ and $(g_{j_i} : V_{j_i} \longrightarrow U_i)_{j_i \in J_i} \in Cov(U_i)$.

$$f_i \in Cov(U) \iff \pi_{f_i} : \coprod_{i\in I} U_i \longrightarrow X \text{ surjective}$$

$$g_{j_i} \in Cov(U_i) \iff \pi_{g_j} : \coprod_{j_i \in J_i} V_{j_i} \longrightarrow U_i \text{ surjective}$$

Let us consider the following composition:

$$\coprod_{j_i \in J_i} V_{j_i} \xrightarrow{\pi_{g_j}} U_i \xrightarrow{\pi} \coprod_{i\in I} U_i \xrightarrow{\pi_{f_i}} X$$

Consequently, $\pi_{f_i} \circ \pi \circ \pi_{g_j} : \coprod_{j_i \in J_i} V_{j_i} \longrightarrow X$ is surjective as being the composition of two surjective maps. Thus, the family $(f_i \circ g_{j_i} : V_{j_i} \longrightarrow X)_{j_i \in J_i, i \in I} \in Cov(X)$.

(3) Stability under base change:
$(U_i \longrightarrow X)_{i\in I} \in Cov(X) \iff \coprod_{i\in I} U_i \longrightarrow X$ surjective. And for all $Y \longrightarrow X$, the map $\coprod_{i\in I}(U_i \underset{X}{\times} Y) \longrightarrow Y$ is surjective. Then $(U_i \underset{X}{\times} Y \longrightarrow Y)_{i\in I} \in Cov(Y)$.

(4) Identity: Since $X \coprod X = X$, we get $id_X : X \longrightarrow X \in Cov(X)$.

■

5.2.2 *Topos on the site BG*

Let U be a G-set. We can construct a presheaf of sets on U as follows:

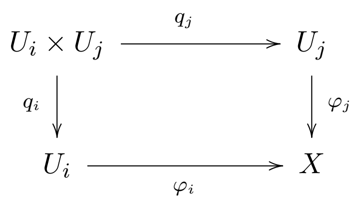
$$\begin{array}{rcl} \mathcal{F}_U : BG^{Opp} & \longrightarrow & Ens \\ V & \longmapsto & Hom_{BG}(V,U) \end{array}$$

To show that the presheaf $\mathcal{F}_U$ defined on BG is a sheaf we must prove that the map φ_i^* is bijective where

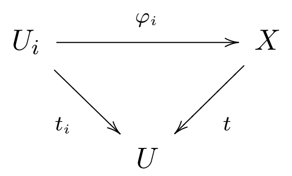
$$\mathcal{F}_U(X) \xrightarrow{\varphi_i^*} \mathcal{A}(X) = \left\{ s_i \in \mathcal{F}_U(U_i) \text{ such that } s_{i|U_i\times_X U_j} = s_{j|\, U_i \times_X U_j} \right\}$$

(i) Injectivity: Let us consider the morphism $Hom(X,U) \xrightarrow{\varphi_i^*} Hom(U_i,U)$ which associates to the map f the morphism $\varphi_i^*(f) = f\circ\varphi_i = \mathcal{F}_U(\varphi_i)$. Let $h,g \in \mathcal{F}_U(X) = Hom(X,U)$ such that $\varphi_i^*(h) = \varphi_i^*(g)$ and let us verify that $h = g$. $\varphi_i^*(h) = \varphi_i^*(g)$ if and only if for all $(\varphi_i : U_i \longrightarrow X)_{i\in I} \in Cov(X)$, $h\circ\varphi_i = g\circ\varphi_i$. Let $x \in X$, then there exists $i \in I$ such that $x = \varphi_i(u_i)$. $h(x) = h\circ\varphi_i(u_i) = (h\circ\varphi_i)(u_i) = (g\circ\varphi_i)(u_i) = g(x)$. This implies that φ_i^* is injective.

(ii) Surjectivity: Consider the diagram:



Let $(t_i)_{i\in I} \in Hom(U_i,U)$ such that $q_i^*(t_i) = q_j^*(t_j) \iff t_i\circ q_i = t_j\circ q_j$. We have to construct $t \in Hom(X,U)$ such that $t_i = \varphi_i^*(t) = t\circ\varphi_i$. Since $\coprod_{i\in I} U_i \longrightarrow X$ is surjective, then for all $x \in X$, there exists $i \in I$ such that $x = \varphi_i(u_i)$. According to the following diagram:



The needed map t is given by $(t\circ\varphi_i)(u_i) = t(\varphi_i(u_i) = t(x) = t_i(u_i)$. Finally, φ_i^* is surjective.

Conclusion: for all $U \in Ob(BG)$, the functor $Hom(-,U)$ is a sheaf of sets on BG.

5.3 *Grothendieck topologies on the category of schemes*

Let Sch/k be the category of separated and finite type schemes over a field k [7].

5.3.1 *Zariski topology*

Definition 5.5 An open subscheme of a scheme X is a scheme U whose the topological space is an open subset of X and whose the structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_{X|U}$ of the sheaf to X. An open immersion is a morphism $f : X \longrightarrow Y$ that induces an isomorphism from X to an open subscheme of Y.

Definition 5.6 Let X be a scheme. We say that a family of morphisms $(f_i : U_i \longrightarrow X)_{i\in I}$ is a Zariski covering if f_i is an open immersion for each $i \in I$. The corresponding site of this pretopology is denoted by $(X)_{Zar}$.

5.3.2 Étale topology

Definition 5.7 Let k be a field and X, Y be two schemes on k. A morphism of schemes $f : X \longrightarrow Y$ is étale if f is locally of finite presentation, flat and if for each point $y \in Y$, the $k(y)$-scheme X_y is étale. We say that a scheme X is étale if the morphism $X \longrightarrow Spec(k)$ is étale.

Definition 5.8 Let X be a scheme in Sch/k. We say that a family of morphisms of the form $\left(U_i \xrightarrow{f_i} X\right)_{i\in I}$ is an étale covering if:

(1) the morphisms f_i are étale,
(2) the morphism $\coprod_{i\in I} U_i \longrightarrow X$ is surjective.

The topology generated by the pretopology of étale coverings is called étale topology on Sch/k. We denote $(Sch/k)_{Et}$ the corresponding site.

5.3.3 Characterization of the topos of Sch/k

The aim of this part is to describe the structure of any étale topos associated to a given field.

Proposition 5.9 Let k be a field, $\bar{k}$ its separable closure and $k_{ét}$ the category of k-étale algebras. Sheaves $\mathcal{F}$ on $k_{ét}$ are the discrete G-sets X where $G = Gal(\bar{k}/k)$ and we have the following equivalence of categories:

$$\begin{array}{ccc} \mathcal{F}_{k_{ét}} & \xrightarrow{i} & discrete\ G\text{-sets} \\ \mathcal{F} & \longmapsto & i(\mathcal{F}) = \lim\limits_{L/k \text{ finite, } L\subset\bar{k}} \mathcal{F}(L). \end{array}$$

Proof. If $\mathcal{F}$ is a sheaf on k, we take

$$i(\mathcal{F}) = \lim_{L/k \text{ finite, } L\subset\bar{k}} \mathcal{F}(L).$$

Then $i(\mathcal{F})$ is a set on which G operates. Conversely, let E be a discrete G-set.Consider the presheaf defined on the subcategory of $\hat{k}_{ét}$ formed by the separable finite extensions of k included in $\bar{k}$ by

$$\mathcal{F}_E(L) = E^{Gal(\bar{k}/L)}.$$

Since the inclusion presheaf of this subcategory in $\hat{k}_{ét}$ is an equivalence of categories, we can extend $\mathcal{F}_E$ to $\hat{k}_{ét}$ in a substantially unique way and then to $k_{ét}$ in a unique way into a functor which commutes with disjoint sums. Thus, $\mathcal{F}_E$ is a sheaf. ■

5.3.4 Nisnevich topology

Definition 5.10 Let X be a scheme. We say that a family of morphisms $(f_i : U_i \longrightarrow X)_{i\in I}$ is a Nisnevich covering if

(1) the morphism $\coprod_{i\in I} U_i \longrightarrow X$ is surjective.
(2) the morphism f_i is étale for each $i \in I$.
(3) for each point $x \in X$, there is an index $i \in I$ and a point $u \in U_i$ such that $f_i(u) = x$ and the morphism f_i induced on the residual fields $k(x) \xrightarrow{\sim} k(u)$ is an isomorphism.

The topology generated by this pretopology is called Nisnevich topology on Sch/k and the underlying site is $(Sch/k)_{Nis}$.

5.3.5 Characterization of the topos of Sch/k

Proposition 5.11 Let k be a field. A presheaf on Sch/k is a Nisnevich sheaf if the following two conditions are satisfied:
i) $\mathcal{F}(\varnothing)$ is a singleton.
ii) for all $E, F \in Ob(Sch/k)$, the map $\mathcal{F}(E \times F) \longrightarrow \mathcal{F}(E) \times \mathcal{F}(F)$ is a bijection.

Proof. See [11]. ■

Remark 1 *Note that the étale topology is finer than Nisnevich topology which is finer than Zariski topology because every Nisnevich morphism is an étale morphism and every étale morphism is a Zariski morphism.*

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